Chaos: The Mathematics Behind the Butterfly Effect

James Manning
Advisor: Jan Holly
Colby College Mathematics
Spring, 2017

1. Introduction

A butterfly flaps its wings, and a hurricane hits somewhere many miles away. Can these two events possibly be related? This is an adage known to many but understood by few. That fact is based on the difficulty of the mathematics behind the adage. Now, it must be stated that, in fact, the flapping of a butterfly's wings is not actually known to be the reason for any natural disasters, but the idea of it does get at the driving force of Chaos Theory. The common theme among the two is *sensitive dependence on initial conditions.*

This is an idea that will be revisited later in the paper, because we must first cover the concepts necessary to frame chaos. This paper will explore one, two, and three dimensional systems, maps, bifurcations, limit cycles, attractors, and strange attractors before looking into the mechanics of chaos. Once chaos is introduced, we will look in depth at the Lorenz Equations.

2. One Dimensional Systems

We begin our study by looking at nonlinear systems in one dimension. One of the important features of these is the nonlinearity. Nonlinearity in an equation evokes behavior that is not easily predicted due to the disproportionate nature of inputs and outputs. Also, the term "*system*" is often a misnomer because it often evokes the idea of a system of equations. This will be the case as we move our focus off of one dimension, but for now we do not want to think of a system of equations. In this case, the type of system we want to consider is a *first-order system* of a single equation. This is an equation of the form

$$\dot{x} = f(x)$$

The dot above the x in this equation represents differentiation, and will be used throughout this paper. In this case and in most others, the differentiation is done with respect to a time variable, t. Often, the $f(x)$ will be a function that one can easily interpret, but sometimes these functions are difficult to conceptualize. For our purposes, let us consider something simple, $f(x) = \cos(x)$. In order to analyze this in a meaningful way, we can attempt to find an implicit solution to this differential equation.

$$dt = \frac{dx}{\cos(x)}$$

$$t = \int \sec(x)dx$$

$$t = \ln|\tan(x) + \sec(x)| + c$$

where c is the integration constant. To find the solution, we must now find c. To do this, we consider an initial condition for x, namely $x = x_0$. Setting $t = 0$, we find that $c = -\ln|\tan(x_0) + \sec(x_0)|$. As follows, our implicit solution is then

$$t = \ln\left|\frac{\tan(x) + \sec(x)}{\tan(x_0) + \sec(x_0)}\right|$$

We use this example to introduce the idea of *fixed points*. Fixed points are aptly named in that they are points in a system that remain fixed—they exhibit no change as t increases. In this system, we can think of fixed points as values of x at which the derivative is equal to 0, i.e. the *change* in x with respect to time is 0. If the change in x with respect to time is 0, then x does not move from its starting position. Hence, the fixed points for this system are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{n\pi}{2}$ for odd $n \in \mathbb{Z}$.

Fixed points comprise a major part of nonlinear dynamics. As such, it is important that we understand how they work. There are three types of fixed points; stable, unstable, and half-stable. Stability is determined by identifying the behavior of the flow at a given value of x.

Stable Fixed Points

It is important to make the distinction between stable fixed points and attracting fixed points. Both of these classifications are technically considered to be stable, but they exhibit slightly different behavior. An attracting fixed point is one near which the flow is inwards (that is, toward the fixed point) on both the right and the left of the point. This is the case when the sign of the derivative $\dot{x}$ changes from positive to negative. A stable fixed point that is not attracting is generally seen when the derivative is 0 for more than one point consecutively. For an easy-to-see example, consider $\dot{x} = 0$. This is a horizontal line on the x-axis, which we now know is a line entirely comprised of fixed points. Since they are consecutive, all of these fixed point are called stable fixed points.

Unstable Fixed Points

Unstable fixed points are, as one might guess, the opposite of stable fixed points. On both the right and left of the point, the flow is outwards (away from the fixed point). This happens when the sign of the derivative $\dot{x}$ changes from negative to positive.

Half-Stable Fixed Points

Half-stable fixed points occur when the sign of the derivative $\dot{x}$ does not change, but its value reaches 0 at a single point. This means that the flow is inwards on one side of the fixed point and outwards on the other.

An example of a function $f(x)$ for which the system $\dot{x} = f(x)$ would exhibit a half-stable fixed point is $f(x) = x^3$. At $x = 0$, the derivative changes from positive on the left to 0, and then back to positive on the right. This means that on either side of $x = 0$ the flow is in the same direction. In this case, the fixed point would be stable on the left and unstable on the right. It is worth noting that the negative of that function, $f(x) = -x^3$, also exhibits a half-stable fixed point. This is the opposite case as the last, meaning that it is unstable on the left and stable on the right. These two cases show the different ways in which a half-stable fixed point can occur, but in both cases they are considered the same.

We have mentioned the idea of the flow at a certain point. A way of interpreting this flow is by imagining how a particle will move over time. In this context we can think of a particle at a fixed point not being moved at all, because there is no flow at a fixed point. It is useful to interpret fixed points in this context because, not only does it tell us about the behavior at a fixed point, but it also tells us what happens *near* a fixed point.

Consider a particle placed just to the left of a fixed point. Depending on the stability of the point, we can easily find out what happens to this particle. If the fixed point is attracting, the flow is inwards, so the particle moves toward it. Once it reaches the fixed point, it stays fixed. If the fixed point is unstable, the flow is outwards, and so the particle will follow the flow away from the fixed point.

Now, if the fixed point is half-stable, the behavior depends on which side of the fixed point exhibits which type of behavior. This is easiest to think about if we consider again the two cases of $f(x) = x^3$ and $f(x) = -x^3$. In the first case, the left half is the stable half, so the particle would move toward the fixed point. In the second case, the

left half is the unstable half, and so the particle would move away from the fixed point.

It is important to keep in mind that, no matter what first-order equation we look at, any given trajectory will always exhibit one of three types of behavior. If the initial position of a trajectory is at a fixed point, it will not move. Otherwise, it will approach a fixed point or it will continue flowing to infinity. (Strogatz 1995, 28). It is this simple fact that rules out any chance for a solution to a one-dimensional, autonomous equation to oscillate. The only points at which the direction of the flow changes are stable and unstable fixed points, but by definition, no trajectory can ever actually cross these points.

Bifurcations

Now that we have an understanding of fixed points and their role in one dimensional systems, we can begin to look at **bifurcations**. A bifurcation is a type of change in the actual dynamics of a system. Much like fixed points, there are various types of bifurcations. Actually, it turns out that bifurcations are entirely dependent on fixed points! This is true for one dimension, and we will have to amend the previous statement as we explore higher dimensions, but for now we keep our focus on fixed points.

Saddle-Node Bifurcations

One of the more intuitive examples of a bifurcation is a saddle-node bifurcation. It is a bifurcation that occurs when two fixed points coalesce and mutually annihilate. This sounds like a mouthful, but an example of a saddle-node bifurcation will make this definition clear.

Consider the first-order equation
$$\dot{x} = x^3 - 27x + r$$
where r is a parameter that we manually shift to give rise to a bifurcation. If we set $r = 0$, we see that $\dot{x}$ has a minimum at $(-3, 54)$ and a maximum at $(3, -54)$. See Figure 1 for a graph of this function.

Next, we increase r to $r = 54$, since the value of $\dot{x}$ is -54 at the minimum. As $r \to 54$, the stable fixed point and the right unstable fixed point approach one another, and then coalesce into one half-stable fixed point at $x = 3$. See Figure 2 for a representation of this.

It follows that the left half of the half-stable fixed point is the stable half since, of the two fixed points that coalesced, the left-most one was
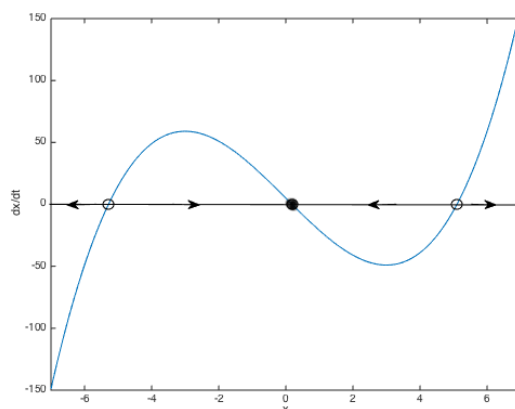
FIGURE 1. A graph of $\dot{x} = x^3 - 27x$. Notice that there are three fixed points, two of which are unstable and one that is stable.
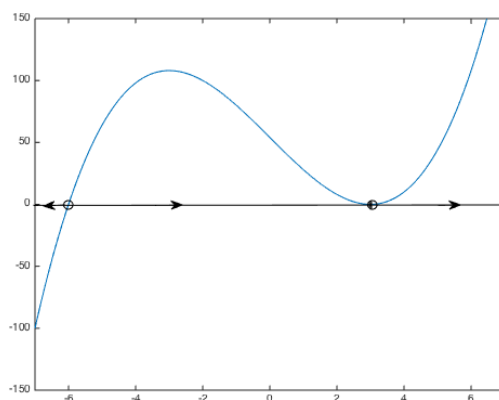

FIGURE 2. A graph of $\dot{x} = x^3 - 27x + 54$. It appears as though one fixed point has disappeared. What has actually happened is that the two right-most fixed points have combined into one for this specific r-value.

stable and the right-most one was unstable.

Now, what happens as we increase r beyond $r = 54$? For this we should first look at the graphical representation of it. To illustrate the point clearly, let's choose $r = 70$ since 70 is well beyond this special point of $r = 54$. See Figure 3 for the graphical representation of this.
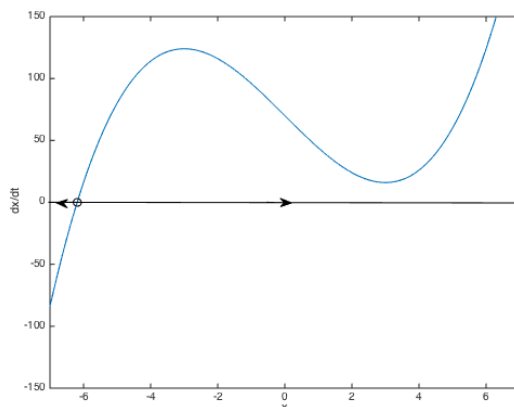
FIGURE 3. A graph of $\dot{x} = x^3 - 27x + 70$. Notice that there is only one fixed point. Two of the original three have seemed to completely disappear.

So, what happened? The system went from having one fixed point, then to two, and lastly to one. The two points that came together annihilated! This behavior is what we were referring to when we first defined a saddle-node bifurcation. That special value of r, $r = 54$, is called the *bifurcation value*.

If these events were to occur in the opposite order (that is, if two fixed points were created and move away from one another as a parameter is varied), we would still refer to it as a saddle-node bifurcation.

Pitchfork Bifurcation

The next type of bifurcation we will look at is strikingly similar to the saddle-node bifurcation in that it is a situation in which fixed points can appear or disappear. Again, this type of bifurcation is easiest to understand with graphical representations, so let us consider an example. In this case, instead of letting the parameter have degree 0, we put the parameter into the first degree x-term. That is, we let

$\dot{x} = x^3 - rx$

where r is the aforementioned parameter.

As one can see, as r is varied, the heights of the maximum and the minimum change. Let us first consider when r is negative (choose r=50 to best demonstrate the point).
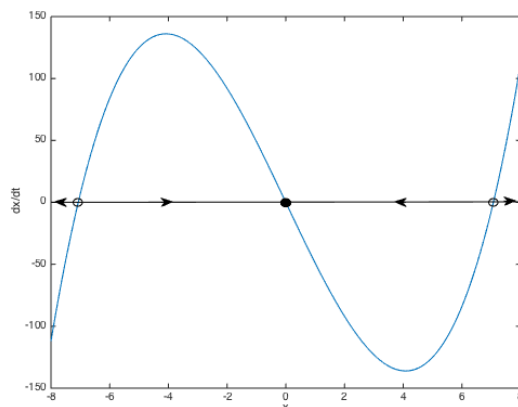
FIGURE 4. A graph of $\dot{x} = x^3 - 50x$. Starting with a negative r-value, there are three fixed points. The outer two are unstable while in inner one is stable.

Now we vary r from a positive number to 0. This reduces the equation to be just $\dot{x} = x^3$. This indicates that the cubic function loses much of its concavity on either side of the origin, and so it only intersect the x-axis at one point. As such, it is easy to conclude that two of the original three fixed points must have disappeared. Hence, by definition, we have reached our *bifurcation value*.
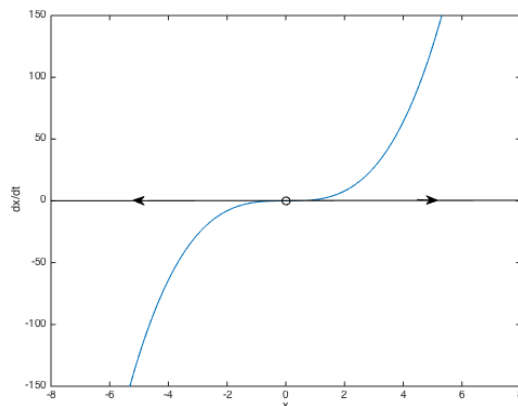

FIGURE 5. A graph of $\dot{x} = x^3$. This is just a simple cubic function, so it only crosses the x-axis at one place, indicating that there is only one fixed point for the equation.

Note that the fixed point at $(0,0)$ remains where it is, but changes stability. It moves from being an attracting fixed point to an unstable fixed point. Again, if we change the order of events, the bifurcation is still considered a pitchfork bifurcation. In fact, if we plot the location of the fixed point (its x-value) as a function of r, it produces a diagram called a *bifurcation diagram*. This diagram is the reason for which this type of bifurcation is named a "pitchfork bifurcation". To show this, we can simultaneously show that varying r in the opposite direction also produces the same type of bifurcation. So, if we vary r from a negative value to a positive value, the bifurcation diagram is as follows:
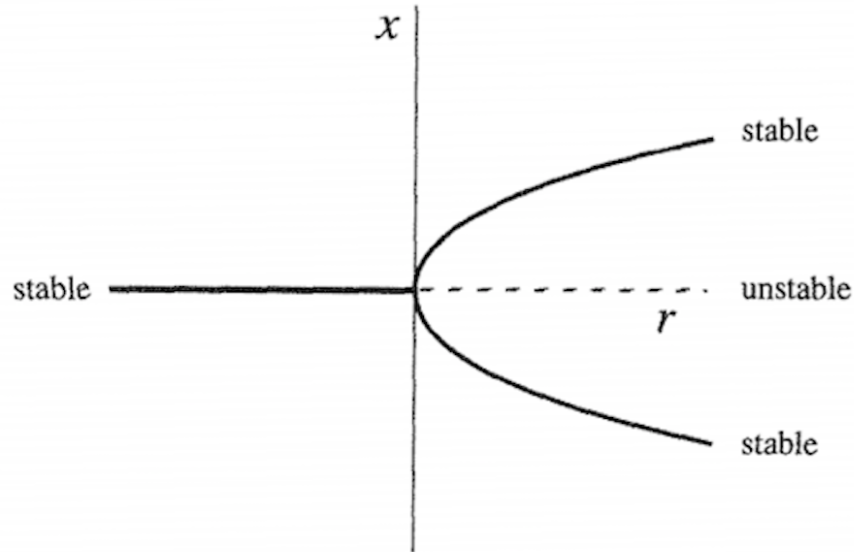

FIGURE 6. This is a bifurcation diagram for the first-order equation $\dot{x} = x^3 - rx$. The solid line indicates a stable or attracting fixed point, and the dashed line indicates an unstable fixed point. (Strogatz 1995, 56)

Bifurcations are an extremely important part of nonlinear dynamics, and will play a large role in the study of chaos. More types of bifurcations exist, but it is not particularly important to go over them in great detail.

3. Two Dimensional Systems

Now, we can use the more common definition of the word system to describe a system of equations in two dimensions. By this, we mean a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, and d are parameters (Strogatz 1995, 123). This equation is linear, meaning that any linear combination of a solution to the system is also a solution. An important feature of these systems is the **vector field** that is associated with each system. In one dimension, the vector field was what we called the flow on the line. It describes the direction that a trajectory would take given an initial condition at a coordinate (x, y).

To actually see the more long-term behavior for a given trajectory, we look to the **phase portrait** for a system. This portrait is meant to be a representation of the general behavior of a system, but can also be used to analyze specific cases. Another tool used for a similar purpose is the direction field. It shows the direction of the flow at any point, but does not show the 'strength' of the flow. Shown in Figures 7 & 8 is an example of a direction field, and an example of a phase portrait overlaid onto the same direction field for the system

$$\dot{x} = 2x + 2y$$
$$\dot{y} = 5x - y$$

To find the solutions to systems like this, it is easiest to consider the system in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This matrix then has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$. Now, to find the eigenvectors, we consider vector $\mathbf{v} = (v_1, v_2)$ which satisfies

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 2 \\ 5 & -1 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
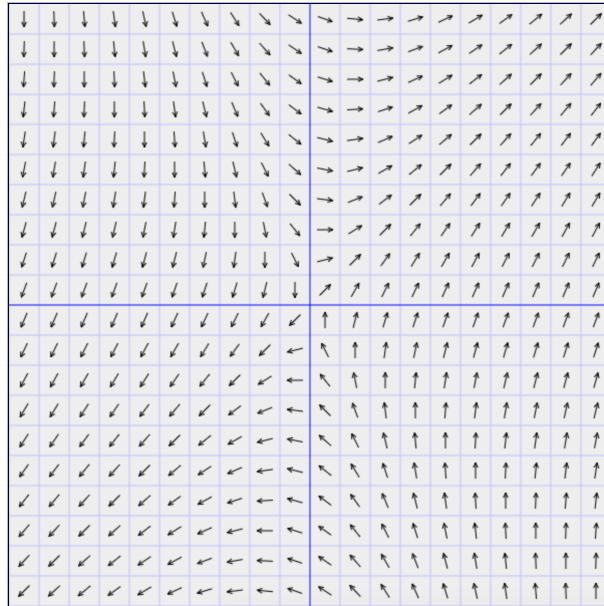
FIGURE 7. An example of a direction field. This figure was created using the phase plane plotter tool at http://comp.uark.edu/ aeb019/pplane.html

For λ_1, we find that

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

and for λ_2,

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence, our general solution to the system is

$$(1) \qquad \mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$
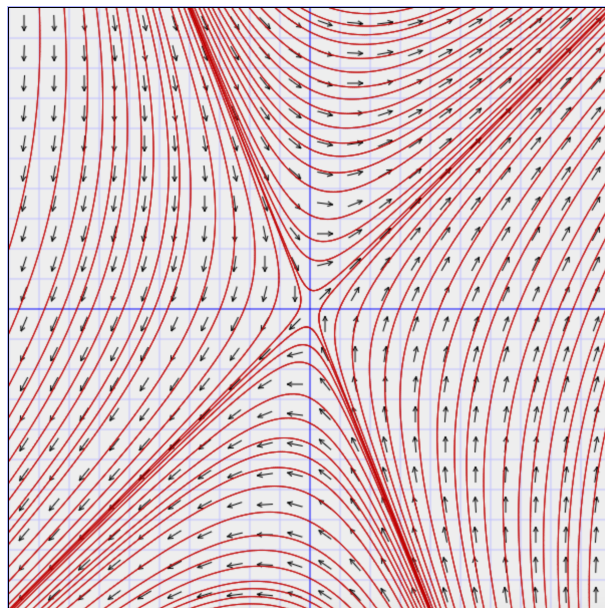
FIGURE 8. An example of a phase portrait. This figure was created using the phase plane plotter tool at http://comp.uark.edu/ aeb019/pplane.html

Inserting $(x_0, y_0) = (-3, 4)$ while $t = 0$, we find $c_1 = -1$ and $c_2 = -1$. Now, if we substitute this back into the general solution, we obtain

$$x(t) = -2e^{-3t} - e^{4t}$$

(2)
$$y(t) = 5e^{-3t} - e^{4t}$$

and thus, we have found a solution to the two-dimensional system.

In two dimensions, thanks to the additional degree of freedom, we see far more variety in the behavior of systems. For example, if we see that the eigenvalues of a system are complex, the fixed points can be *spirals* or *centers*. Much like in one dimension, there are different types of stability for some of these new types of fixed points. For example, spirals can be unstable or stable. This stability is again related to the direction of the flow in relation to the fixed point. A stable spiral is a spiral toward which trajectories move and an unstable spiral is a spiral away from which trajectories move. A center is a fixed point around which we see a family of *closed orbits*. A closed orbit is a trajectory that, starting at an initial condition, returns to itself and repeats.

For real eigenvalues, we see stable nodes, unstable nodes, and saddle points. Additionally, for particular trace and determinant value

combinations, we can see interesting fixed points such as stars and degenerate nodes. We call these *borderline cases* and they occur along the line $\tau^2 - 4\Delta = 0$ in the trace-determinant plane.

The Existence and Uniqueness Theorem

The existence and uniqueness theorem in two dimensions states that, for an initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x}(0) = \mathbf{x}_0$, if $\mathbf{f}$ is continuous and its derivatives are continuous, then the system will have a solution and the solution will be unique. This has some interesting consequences in the phase plane. Since a solution is unique, no two trajectories can intersect. So what does this mean for trajectories that are bounded within a space? If the space contains one or more fixed points, trajectories in the space may eventually approach one. Otherwise, if there is no fixed point, the **Poincaré-Bendixson Theorem** states that it will eventually approach a closed orbit.

The Poincaré-Bendixson Theorem is an important theorem in the field of nonlinear dynamics, and it is provides us with some interesting results that we will look at more later on. The theorem is as follows: If R is a closed, bounded subset of the plane with no fixed points, and $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R, and if there exists a trajectory C confined in R, then C is a closed orbit or spirals toward a closed orbit (Strogatz 1995, 203). One of the more poignant implications of this theorem is that any R satisfying the conditions will contain a closed orbit.

An important type of closed orbit that we will look at is a **limit cycle**. A limit cycle is a closed orbit that is isolated, meaning that the trajectories surrounding the closed orbit are not closed. They either approach the limit cycle or move away. An example of a closed orbit that is not a limit cycle is a closed orbit within a system that has a center fixed point. The trajectories surrounding the orbits are also closed, thus not fulfilling the criteria of a limit cycle.

Much like a fixed point, a limit cycle has a stability feature. The three possibilities for stability are stable, unstable, and half-stable. Also similar to fixed points, the stability is determined in the same fashion. A stable limit cycle is a limit cycle around which the neighboring trajectories spiral toward it—both on the inside of the cycle and the outside of the cycle. Unstable limit cycles are the opposite; trajectories surrounding the limit cycle spiral away from it. Half-stable limit cycles can occur in two ways; outer trajectories spiraling toward the cycle with inner trajectories spiraling away, or outer trajectories spiraling away from the cycle with inner trajectories spiraling toward.
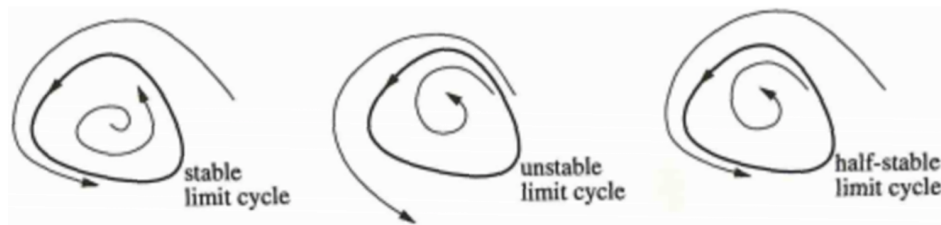
FIGURE 9. A schematic for the stability of limit cycles.
(Strogatz 1995, 196)

One of the techniques for using the Poincaré-Bendixson Theorem is
to construct a trapping region to find a confined orbit. This is a re-
gion of any shape on the boundaries of which the vector field is always
pointing inward. If the vector field is pointing inwards, that means that
any trajectory in the region and near the boundaries will move more
inwards, i.e. it will not leave the region. We can extend this to saying
that *all* trajectories in the region will not leave because, to get out of
the region, a trajectory must first approach a boundary, and then we
can apply the rule stated above. This method is useful because it is far
easier to construct a trapping region than it is to find a closed orbit,
and so we use this tool to satisfy all four conditions of the Poincaré-
Bendixson Theorem.

The Hopf Bifurcation

The Hopf bifurcation is the most subtle of bifurcations. There are
two main types; supercritical and subcritical. A supercritical Hopf bi-
furcation occurs when the stability of a spiral changes from stable to
unstable, and a limit cycle appears surrounding the unstable spiral. A
subcritical Hopf bifurcation occurs when an unstable limit cycle shrinks
around a stable fixed point, rendering it unstable, thus the trajectories
near the fixed point spiral away from it toward another attractor of
some form or toward infinity. These bifurcations are very important as
we move from two dimensions into three and even higher. See Figure
10 for a drawn interpretation of supercritical and subcritical Hopf bi-
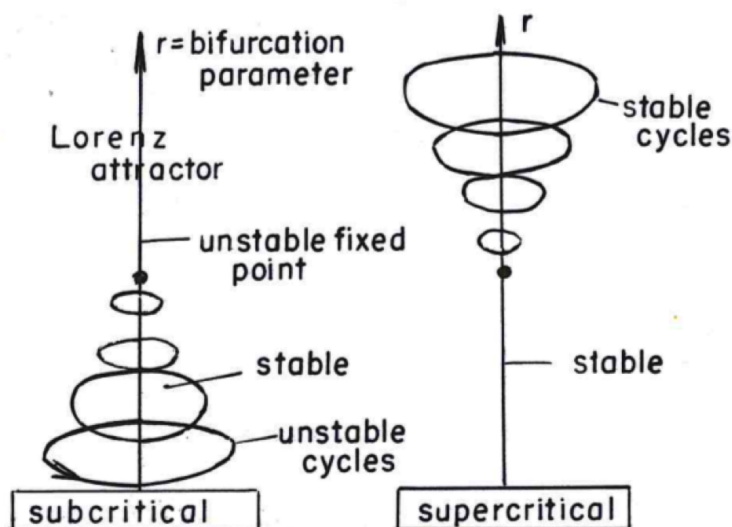furcations.

FIGURE 10. A diagram to show the process by which a Hopf bifurcation happens both in the subcritical and supercritical case. (Marsden and McCracken 1976, 148)

4. Three Dimensions and the Lorenz System

Now that we have worked through one and two dimensions, we have the necessary tools to move to three dimensions and look at chaos. We begin with the **Lorenz equations**. The system is as follows

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

where σ, r, $b > 0$ are parameters. This system was the first system in which chaos was observed. It is named after Ed Lorenz, who created the system of equations while studying convection rolls in the atmosphere.

We want to learn about Lorenz equations in the way that they were discovered, so we consider the versions of these parameters that relate to convection rolls.

σ is the **Prandtl number**, which is defined as follows:

$$\text{(3)} \qquad \sigma = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$$

(Coulson, J. M.; Richardson, J. F. 1999)
where ν is the momentum diffusivity, measured in m^2/s, and α is the thermal diffusivity, also measured in m^2/s. From this, we can easily see that this parameter is a dimensionless one.

r is the **Rayleigh number**, which is defined as follows:

$$\text{(4)} \qquad r = Gr_x Pr_x$$

where Pr_x is the Prandtl number discussed above, and Gr_x is the **Grashof number**, another dimensionless number that describes the approximate relation between the buoyancy of a fluid and the viscous force acting on said fluid (Turcotte, D.; Schubert, G. 2002, Bird, R. Byron, Warren E Stewart, and Edwin N Lightfoot. Transport Phenomena. New York: J. Wiley, 2002)

The third parameter, b does not have a name, but its significance in the convection problem is its relation to the height of the fluid layer in question (Strogatz 1995, 301).

The Lorenz equations describe a complex system, but this system exhibits a number of basic properties that are ubiquitously true across all instances of the system.

Nonlinearity

The first and most basic of these properties is its *nonlinearity*. The Lorenz sytem is, after all, a system of nonlinear differential equations. The nonlinear terms appear in the second and third equations in the system; xz in the second and xy in the third. The nonlinearity of this system makes it so that any change in input is not directly proportional to the change in output it is related to.

Symmetry

The Lorenz system has symmetry across a change in sign of the x and y variables. That is, if the point (x, y, z) is a solution to the system, so is $(-x, -y, z)$.

Volume Contraction

This is one of the more important properties of the system because it essentially says that solutions to the Lorenz system will always stay within a finite set. The property itself states that volumes in phase

space contract, i.e. that any given volume in phase space, over any length of time, will shrink to a smaller volume.

Fixed Points

The Lorenz system has two types of fixed points. First, for any given parameters, the origin is a fixed point. Next, for $r > 1$, a symmetric pair of fixed points is brought about. This pair of fixed points is described by $(x^*, y^*, z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. As $r \to 1$ from the right (i.e. as r decreases to 1), the two fixed points coalesce with the origin to form a pitchfork bifurcation. We call this symmetric pair C^+ and C^-.

Linear Stability of the Origin

The linearized system about the origin is defined as follows:
$\dot{x} = \sigma(y-x)$, $\dot{y} = rx - y$, $\dot{z} = -bz$.

The equation for z depends only on z, hence $z(t) \to 0$ exponentially fast. The behavior in the x and y directions is determined by the system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Global Stability of the Origin

For $r < 1$, every trajectory approaches the origin as $t \to \infty$ (this implies that the origin is globally stable). To show this, we construct a Lyapunov function (a smooth positive function that decreases along trajectories (Strogatz 1995, 315)).

Consider $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. This creates concentric ellipsoids around the origin. We want to show that $\dot{V} < 0$ along trajectories. To do this, we must calculate $\dot{V}$. By implicit differentiation, we find

$\frac{1}{2}\dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z}$
$= (yx - x^2) + (ryx - y^2 - xzy) + (zxy - bz^2)$
$= (r+1)xy - x^2 - y^2 - bz^2$

Now, we group terms to extract squares

$\frac{1}{2}\dot{V} = -[x - \frac{r+1}{2}y]^2 - [1 - (\frac{r+1}{2})^2]y^2 - bz^2$.

If $r < 1$ and $(x, y, z) \neq (0, 0, 0)$, the right hand side is strictly negative, since we have three squares with negative coefficients. Hence, if $\dot{V} = 0$, then $(x, y, z) = (0, 0, 0)$. Otherwise, $\dot{V} < 0$, as desired (Strogatz 1995, 315).

Now, we consider the stability of the fixed points C^+ and C^-. For $1 < r < r_H$, the symmetric pair are linearly stable.

The value $r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ is representative of the r value for which the pair lose stability in a Hopf bifurcation.

Of course, this then raises the question of what happens as we increase r to a value just past r_H. The bifurcation is subcritical, meaning that the limit cycles are unstable and disappear for $r > r_H$. Determining the nature of this bifurcation requires an extremely long calculation, which essentially shows that the third derivative of a displacement function is greater than zero, implying that the bifurcation is subcritical. The large calculation is compressed by Marsden and McCracken (1976) into $V'''(0) = (A_1 + A_2)\xi$, where $\xi = \frac{3\pi(\sigma-b-1)^2}{2\sigma b(\sigma+1)^3 \omega^2}\sqrt{\frac{2b(\sigma-b-1)}{\sigma(\sigma+1)}}$. In this case, A_1 and A_2 are the names Marsden and McCracken gave to extremely large terms. They wrote that, since $\xi > 0$, the orbits from the bifurcation are stable if $(A_1 + A_2) < 0$, and unstable if $(A_1 + A_2) > 0$. In the case they presented, for example, $\sigma = 10$, $b = \frac{8}{3}$, so $A_1 \approx 1.63 \times 10^9$, $A_2 \approx 0.361 \times 10^9$, so $A_1 + A_2 \approx 1.99 \times 10^{10}$. Hence, $A_1 + A_2 > 0$, so the Hopf bifurcation is subcritical.

Chaos on a Strange Attractor

Before we dive into chaos, it is important to first define *attractors* and *strange attractors*. Firstly, an attractor is defined as a closed set with the following properties:

(1) The set is *invariant*, meaning that any and all trajectories that begin in the set remain in the set forever.
(2) The set attracts an open set of initial conditions (Strogatz 1995, 324). This means that, for trajectories within a set S of which the attractor is a subset, those trajectories are attracted toward the attracting set. The requirement for S is that it is sufficiently small such that trajectories starting within it are sufficiently close to the attractor to be pulled toward it. The largest S is named the *basin of attraction to A* where A is the attracting set.
(3) The set is *minimal* meaning that there does not exist any proper subset of the attractor that satisfies the first two conditions.

The third condition follows intuitively because it is simply saying that the attracting set does not contain any smaller attracting sets. If one thinks about the attractor as a piece of paper, the smallest set that satisfies the first two conditions can be cut out, and what remains can be considered part or all of the open set of initial conditions S.

Now, defining a *strange attractor* is very simple; a *strange attractor*

is an attractor (a set satisfying the above conditions) that also exhibits sensitive dependence on initial conditions.

Next, it is important to actually define *chaos*. Chaos is defined as aperiodic behavior over time of a deterministic system that exhibits sensitive dependence on initial conditions (Strogatz 1995, 323). It is important to note that, for chaotic systems, trajectories cannot escape to infinity.

Now, equipped with this definition, we can look at an example of chaos in the Lorenz system. To do this, we consider the set of parameters used by Lorenz: $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. This r-value is significant because it is slightly past the r_H value $r_H = 24.74$ for the system involving $\sigma = 10$ and $b = \frac{8}{3}$. What comes out of plotting this system is something truly exceptional. The system creates a set of zero volume but has infinite surface area. Such a phenomenon is made possible by the fact that, in the Lorenz Attractor, there are infinitely many two dimensional (flat) layers. Since they are two dimensional, they have zero volume, but still have surface area. Hence, the entire set still has zero volume while having infinite surface area.

This set is an attractor. It is invariant - no trajectories within the set ever leave it, it is attracting—the distance between the set and nearby trajectories approaches 0 as $t \to 0$, and it is minimal—this set is the smallest set that satisfies condition one and condition two. In this case, we actually have a *strange* attractor—the Lorenz system certainly exhibits sensitive dependence on initial conditions.

To show this more mathematically, we consider two trajectories in the set that begin close to one another, one beginning at $\mathbf{x}(t)$ and the other beginning at $\mathbf{x}(t) + \delta_0$, where δ_0 is the initial separation of the trajectories. In observing the Lorenz system, one will find that

$$\| \delta(t) \| \sim \| \delta_0 \| e^{\lambda t}$$

where $\lambda \approx 0.9$ for this system. The exponential term implies that the separation increases exponentially quickly. Hence, trajectories that begin very close together separate rapidly. From here we can logically conclude that the system exhibits sensitive dependence on initial conditions.

Something important to note in the previous calculation is λ. In this instance, λ is called the **Lyapunov Exponent**. This exponent is very important to chaotic systems because it provides us with an avenue to calculating just how far we can accurately predict outputs in chaotic systems. The value of time for which prediction breaks down is called

the *time horizon*. The equation for this is

$$t_{\text{horizon}} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{\| \delta_0 \|}\right)$$

where a is a term representing a *tolerance* for error, meaning that he or she who is calculating the time horizon must decide at what distance of separation prediction breaks down. e.g. if $\delta(t) = 3 \times 10^{-10}$ is too large, he or she would input $a = 3 \times 10^{-10}$.

5. Chaos in One-Dimensional Maps

In this section, we shift our focus from three dimensions and higher to just one dimension. Everything up to this point has implied that chaos does not happen in one dimension, but this is not the case. If we consider a function defined by a map instead of a differential equation, we can actually find examples of chaos. Maps are discrete-time dynamical systems that define the point $n+1$ using the previous point, n. The general form of a map is as follows

$$(5) \qquad\qquad x_{n+1} = f(x_n)$$

The important aspect of maps that we need to take note of is the consistent discontinuity exhibited by maps. Since here time is discrete, the only points defined on the map tend to be a certain distance away from the previous point. This is the feature of maps that allows for chaos to exist! So what happens if a point maps back to itself?

Fixed Points for Maps

Since a point in a map is defined by the previous point, it is easy to see that if a point maps to itself, it will happen again and again, thereby causing the values of the map to remain at that point for all time.

Cobwebbing

Cobwebbing is a tool we use to get an idea of how a map behaves in general. The way that it works is, given a function $f(x)$ we draw a vertical line from the initial x-value, x_0, to the function, and once it intersects, connect that line to the diagonal $(y = x)$ with a horizontal line. Next, we draw a vertical line from the diagonal back to the function. The height of the first intersection is then defined as x_1 and the second is x_2. Thus, starting at $n = 0$ we have obtained x_n, x_{n+1} and

x_{n+2}.

The Logistic Map

The Logistic Map is defined by the equation

(6) $$x_{n+1} = rx_n(1 - x_n)$$

The graph of the map is unimodal, and has a maximum at $(\frac{1}{2}, \frac{r}{4})$. If we restrict r to $0 \leq r \leq 4$, then the interval $0 \leq x \leq 1$ maps onto itself (Strogatz 1995, 353). If we fix r, we can see that as the map iterates, the x_n's actually become periodic. For small values of r, the map exhibits a fixed point. As we increase r, the periodicity doubles, and we can see *period-2 cycles*. If we increase r even more, we find a *period-4 cycle*, and this trend, known as *period doubling*, continues as we increase r more and more until we reach r_∞. *After* this value, we see chaos!

Orbit Diagrams

Orbit diagrams have become a poster child for chaos. They plot x-values of attractors versus the parametric r-values and exhibit extreme complexity. However, this complexity is quite well ordered. Each point on the diagram represents an x-value of an attractor for a given r-value, meaning that, for a given r, the diagram exhibits every value of x that the map 'hits'. See Figure 11 for the orbit diagram for both the sine map and the logistic map.

These orbit diagrams show just how difficult it is to interpret something that is chaotic. However, the empty strips in the diagram represent *periodic windows*. These windows highlight values of r for which the maps exhibit periodicity, i.e. that the map 'hits' only a certain number of x-values. For example, a period-4 cycle would have only 4 x-values plotted for a given r value. So, even in chaos there is some shelter from the storm. What is also interesting is the apparent similarity between these two graphs. This is due to the fact that the sine map is also unimodal. The difference across these maps, though, is the horizontal scale. The sine map's orbit diagram goes from $r = 0$ to $r = 1$. This difference comes from the fact that the maximum for the sine graph occurs at r, versus $\frac{r}{4}$ (Strogatz 1995, 370).
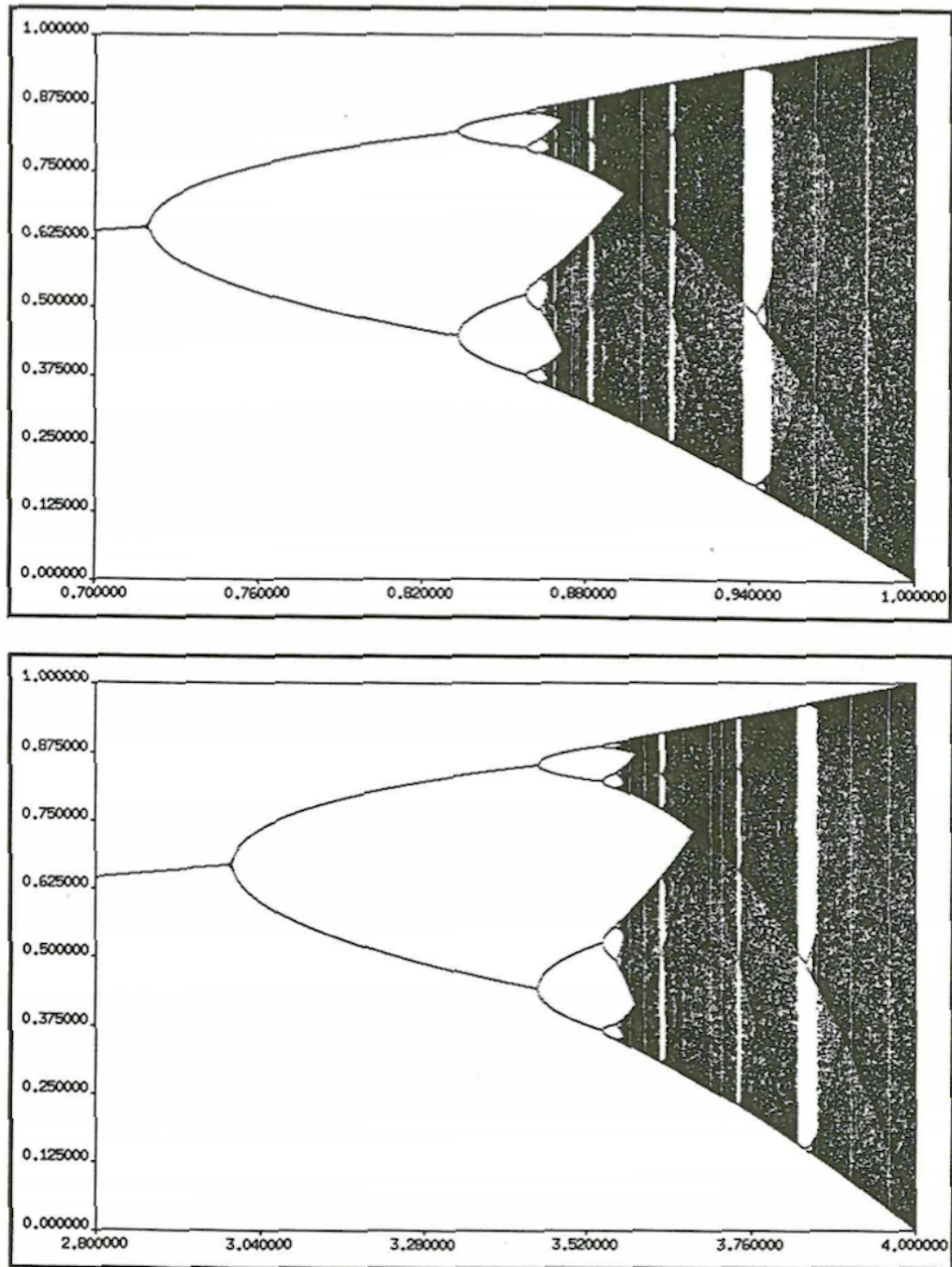
FIGURE 11. The orbit diagrams for the sine map (top) and the logistic map (bottom) (Strogatz 1995, 371).

6. Conclusions

We have now gone in-depth through nonlinear dynamics in one through three dimensions. In doing this, we gathered tools that help us to get a fundamental understanding of nonlinear systems and their idiosyncrasies. One dimensional systems taught us the basics of fixed points and bifurcations, and then we used two dimensions to learn about Hopf bifurcations and limit cycles. Next, in three dimensions, we were able to look at the Lorenz equations and an example of chaos, along with a strange attractor. Since we have these tools, we can now understand how chaos both arises and how it works. Despite the counterintuitive nature of one-dimensional chaos, it is still possible through the use of maps. All of these subjects form a very interesting and important field in the real world, as well.

Chaos is seen in natural systems like weather patterns and traffic patterns. Having this knowledge is important because it enables us to create better prediction tools for weather, or design roads in a more efficient way. Unfortunately, chaos has not always been the forefront of mathematical attention, and so it has not always been considered when implementing systems (in the colloquial sense) of this nature. This is still an open field and more remains to be done if we want to broaden our understanding both of the Lorenz equations and of chaotic systems in general.

7. Acknowledgments

I would like to thank Professor Jan Holly for her patience and support as I worked through this project. Without her encouragement and help, learning everything outlined in this paper would not have been possible. I would also like to thank Professor George Welch, not only for being a secondary reader to this honors thesis, but also for being a large reason why I decided to continue with the math major after beginning in his Calculus 121 class. Additionally, I want to thank all the professors that I have had, and even those that I have not had, for fostering an academic environment unlike any other. I will cherish all the memories I have gathered during my time at Colby, but none will stick out in the same way that those I have gathered from all of you will. So, as sincerely as I can mean it, thank you.

24

References

[1] Strogatz, Stephen H. (1995) *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering.* Addison-Wesley Publishing Company

[2] Marsden, J. E.; McCracken, M. (1976) *The Hopf Bifurcation and its Applications* Springer-Verlag Publishing Company.

[3] Coulson, J. M.; Richardson, J. F. (1999). Chemical Engineering Volume 1 (6th ed.). Elsevier.

[4] Bird, R. Byron, Warren E Stewart, and Edwin N Lightfoot. Transport Phenomena. New York: J. Wiley, 2002. Print. Pages 318, 359.