Summary of the IPAM Long Program

Mathematics of Turbulence: Recent Progress and Open Questions

Introduction

Turbulence is perhaps the primary paradigm of complex nonlinear multi-scale dynamics. It is ubiquitous in fluid flows and plays a major role in problems ranging from the determination of drag coefficients and heat and mass transfer rates in engineering applications, to important dynamical processes in environmental science, ocean and atmosphere dynamics, geophysics, and astrophysics. Understanding turbulent mixing and transport of heat, mass, and momentum remains an important open challenge for 21st century physics and mathematics.

Advances in computation and experimentation over the past two decades have yielded increasingly detailed "full field" realizations of turbulent flows, and these have been successfully exploited to advance both theoretical and applied aims. From the ever-increasing unwieldiness of these data sets alone, it is clear that future progress on the turbulence problem will increasingly depend upon on the quality of guidance provided by the underlying theoretical foundations, as well as the continued development of mathematical frameworks for analysis, prediction, optimization, and control. With these broad aims in mind, the Institute for Pure and Applied Mathematics hosted a Long Program on the Mathematics of Turbulence from September 8, 2014 through December 12, 2014. In addition to a series of tutorials presented during the first week, this long program hosted four workshops: 1) Mathematical Analysis of Turbulence, 2) Turbulent Transport and Mixing, 3) Geophysical and Astrophysical Turbulence, and 4) Turbulence in Engineering Applications.

Although each of the workshops focused on topics specific to their respective areas, as the workshop progressed a number of emergent, recurring, and cross-cutting themes became apparent. These largely pertain to the increasing toolbox of mathematical methods and theoretical concepts that are being leveraged to advance both fundamental and applied research on the mathematical properties of turbulence, as well as the prediction and control of this complex physical phenomenon. Primary among these topics are those associated with the fundamental nature of the Navier-Stokes and Euler equations and their solutions in the turbulent regime, adaptations of the mathematics of control and optimization theory for prediction and data assimilation, the development of rigorous bounds for transport properties in scalar mixing and momentum transport, and a variety of methods used in the development of reduced order models that are gaining an increasingly well-founded basis in the underlying partial differential equations. The following sections of this summary provide brief descriptions of the research directions currently being actively pursued within this broadly based research community, along with commentary pertaining to their future potential to advance mathematical approaches to the problem of fluid turbulence.

Analysis of Navier-Stokes and Euler Equation Solutions

There have been significant advancements in our understanding of properties of solutions to the Euler and Navier-Stokes equations in the last decade, but key questions remain open. The theory of wild solutions for the Euler equations has provided a new tool to attack Onsager's conjecture. The issue remaining to be settled is whether energy dissipation can be used as a selection principle for physically meaningful solutions of the inviscid fluid equations, and in particular whether solutions arising as a limit of solutions to the Navier-Stokes equations in the vanishing viscosity limit can dissipate energy and hence represent turbulent flows. In the context of 2D turbulence it is known that enstrophy-bounded solutions of the Euler equations cannot dissipate enstrophy. However, uniqueness of such solutions is wide open, as is uniqueness of solutions with initial vorticities that are only measures, such as point vortices and vortex rings.

An exciting new direction of investigation is the use of stochastic analysis to study deterministic fluid models. Properties conserved by regular inviscid flows, such as circulation, can be conserved in an averaged sense in the presence of dissipation provided past trajectories are properly endowed with a probability measure. Such an approach can be useful in studying the behavior of flows at very high Reynolds numbers away from boundaries. It remains to be seen whether these ideas can be successfully applied to study the limit of vanishing viscosity in the presence of walls.

When noise is added to flow trajectories, it is possible that in the limit of zero noise the resulting system retains randomness, exhibiting spontaneous stochasticity as in the Kraichnan model. Spontaneous stochasticity has also been shown to occur for Burger's equation. Randomness leads to irregular transport properties, a possible mechanism for anomalous dissipation. At the same time stochasticity can enforce uniqueness of fluid trajectories, at least in the probabilistic sense, which can be used to estimate the size of attractors.

There is also interest in adding noise to the fluid equations (which then become stochastic PDEs), e.g., to represent the unavoidable "molecular noise" remaining in the hydrodynamic limit of the microscopic molecular dynamics. An important question here is whether the effects of the noise persist in the high Reynolds-number turbulence limit. If so, the non-uniqueness of weak solutions to the Cauchy problem for Euler would have a physical meaning as intrinsic unpredictability of turbulent flows.

Methods for Control and Data Assimilation

There are many situations in engineering where it is desired for a flow to be controlled to achieve some specific objective, where this control is achieved by means of a combination of sensing——measuring the flow——and actuation—suitably perturbing the flow to achieve a certain goal. There are also many situations where data is collected measuring a flow of interest——the classic example being the atmosphere——where this data is "assimilated" into a model of the flow to produce a prediction or forecast——e.g., will it rain tomorrow? It is becoming increasingly clear that these two important classes of problem are deeply related both in their fundamental character and the diverse mathematical tools that are relevant to their solution.

In both situations the "data" are inevitably sparse and inherently uncertain. The appropriate management of this uncertainty is attracting much attention in terms of techniques including Bayesian statistics and stochastic calculus. Furthermore, both situations are inherently time-limited and the models used to produce the forecasts or to predict the (hopefully desired) effects of actuators are often substantially reduced and simplified compared to the "real" underlying system. Capturing the key characteristics of the real system in a robust and efficient manner is obviously an enormously interesting and challenging problem. Various workshops and attendant discussions demonstrated that a key open question is to what extent the enormously powerful tools of "linear" analysis in all its forms can be deployed on under-constrained, uncertain, but undoubtedly nonlinear real-world systems.

Turbulent Transport: Analysis & Bounds, and Computation, Optimization & Control

Transport and mixing are elemental properties of turbulent flows and their quantitative characterizations present ongoing challenges of significant importance both for basic physics and for myriad applications in the applied sciences and engineering. The derivation of reliable predictions and rigorous limits on these processes from the fundamental equations of motion is an established and lively area of mathematical research that has enjoyed substantial success in recent decades. New approaches involving novel applications of optimization and control technologies have recently emerged that are leading the field into new and promising directions.

Control has long been utilized in engineering fluid dynamics to do what its name suggests; to manipulate flows to achieve desired goals. Recently, however, the theoretical technology has also found novel application in exploring the ultimate limits on certain aspects of the dynamics. For example, optimal control theory and related optimization methods are now used to study extreme behaviour, and, in particular, regarding transport and mixing features and sensitivity to instabilities, producing new physical and mathematical insights into complex flow characteristics. These ideas are central to modern turbulence research in a variety of ways ranging from research into the transition from laminar to turbulent flow, to producing informed conjectures on the ultimate transport properties of fully developed turbulent flows.

Turbulence in Engineering Applications

One of the distinguishing features of turbulence is the presence of a broad range of scales in space and time yielding a characteristically complex fluid motion. Despite this complexity, however, there is recent recognition and consensus that the bulk of the transport of momentum and energy rests in——and can be faithfully described by——a limited number of coherent structures, a finding that is confirmed by both numerical simulations and experiments. Equally acknowledged is the fact that the modelled flow (based on these structures) has to be sustained by a feedback mechanism accounting for the neglected or truncated dynamics.

A variety of promising approaches now incorporate a closed-loop, reduced-order, linear model based on a few coherent structures and a deterministic or stochastic feedback unit. Beyond their common overall layout the models distinguished themselves by the type of structures used in the linear model and by the type of feedback. Models for wall-bounded shear flows in generic geometries, e.g., use frequency-response modes from an input-output formulation of the governing equations linearized about a turbulent mean flow, and selected nonlinear terms provide feedback and re-initialization of these structures. Similarly, asymptotic solutions based on a hierarchical perturbation approach also attempt to extract a low-dimensional model from the full governing equations with closure accomplished by selected nonlinearities

Alternative to nonlinearities, the feedback loop can also be modelled by a stochastic process. In this case, an evolution equation for a second-order statistic (variance) is used to determine the driving term of a perturbation equation that in turn couples, via the mean flow, to the equation for the variance, thus yielding a fully coupled system of equations.

A dynamical-systems approach has also emerged that focuses on fixed points and edge states as critical entities that govern the dynamics of phase-space trajectories. This type of analysis and optimal-path calculations by variational principles have made important contributions to our understanding of turbulence, in particular the transition from laminar to turbulent fluid motion

A promising direction has become apparent at the workshop: the design of lean models that act as minimal representations of turbulent fluid motion. They rely on the concept of dynamic sparsity where only a few structures are responsible for the bulk of the momentum and energy transport. These models contain linearized governing equations and a nonlinear or stochastic feedback loop to sustain the dynamic process and capture the principal features of turbulence. These self-sustaining closed-loop systems, together with advanced phase-space techniques, are expected to give new insight into the physical mechanisms of turbulent and transitional fluid motion and to provide the foundation for reduced-order modelling to guide numerical and experimental investigations into more complex turbulent flows.