FUZZY ORDERED SETS AND DUALITY FOR FINITE FUZZY DISTRIBUTIVE LATTICES

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Abstract. The starting point of this paper is given by Priestley's papers, where a theory of representation of distributive lattices is presented. The purpose of this paper is to develop a representation theory of fuzzy distributive lattices in the finite case. In this way, some results of Priestley's papers are extended. In the main theorem, we show that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices. Several examples are also presented.

1. Introduction

The study of fuzzy relations was started by Zadeh [17] in 1971. In that celebrated paper the author introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings. The concept of fuzzy order was introduced by generalizing the notion of reflexivity, antisymmetry and transitivity, there by facilitating the derivation of known results in various areas and stimulating the discovery of new ones. Fuzzy orderings have broad utility. They can be applied, for example, when expressing our preferences with a set of alternatives.

Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In [16], Venugopalan introduced a definition of fuzzy ordered set (foset) (P, μ) and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. The notion of a multichain in a fuzzy ordered set is defined in [1]. In [14], Šešelja and Tepavčević presented a survey on representations of ordered structures by fuzzy sets. An order relation and a ranking method for type-2 fuzzy values are proposed in [10]. See also[3, 6, 8, 9, 13].

In a series of papers, Priestley [11, 12] gave a theory of representation of distributive lattices. In this paper, we extend some results of [11, 12], more precisely we give a representation theory of fuzzy distributive lattices in the finite case.

This paper is organized as follows: In the next section, basic definitions and notions are presented. In the third section, we give and prove the main result using a definition of fuzzy ordering admitting the minimum t-norm. The result can be generalized to any t-norm as introduced in [3]. Using the previous result of Priestly [12], this extension is obtained in a natural way.

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2. **Preliminaries**

There are two types of relations which often arise in mathematics: order relations and equivalence relations. An order relation is a generalization of both set inclusion and the order relation on the real line. In this section, we recall some definitions and concepts that we shall need in the sequel.

Let X be a non-empty set. A fuzzy set R on $X \times X$ (i.e., $X \times X \to [0.1]$ mapping) is called a *fuzzy binary relation* on X. A fuzzy binary relation R on X is called

(1) *reflexive*, if $R(x, x) = 1$, for all $x \in X$,
(2) *antisymmetric*, if $R(x, y) \wedge R(y, x) = 0$ whenever $x \neq y$, for all $x, y \in X$,
(3) *transitive*, if $R(x, y) \wedge R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

A reflexive and transitive fuzzy relation is called a *fuzzy preordering*. Moreover, a fuzzy preordering which is antisymmetric, is called a *fuzzy ordering relation*.

A set equipped with a fuzzy order relation is called a *fuzzy ordered set* (*foset*). Let R be a fuzzy binary relation on a set X. The *domain* of R is the fuzzy set on X, denoted by Dom$R$, whose membership function is defined by:

$$\mathrm{Dom}R(x) = \vee_{y\neq x} \{R(x, y) \mid y \in X\}.$$

Similarly, the *range* of R is denoted by Ran$R$ and is defined by:

$$\mathrm{Ran}R(y) = \vee_{x\neq y} \{R(x, y) \mid x \in X\}.$$

The *height* of R is denoted by $h(R)$ and is defined by:

$$h(R) = \vee_{\{(x,y)|x\neq y\}} \{R(x, y)\}.$$

Let X be a foset and $x \in X$. The fuzzy set $(\downarrow x)$ on X is defined by:

$$(\downarrow x)(y) = R(y, x), \text{for all } y \in X.$$

On the other hand $(\uparrow x)$ denotes the fuzzy set on X which is given by:

$$(\uparrow x)(y) = R(x, y), \text{ for all } y \in X.$$

If A is a subset of X, then we define

$$\uparrow A = \cup_{x\in A} (\uparrow x) \text{ and } \downarrow A = \cup_{x\in A} (\downarrow x).$$

Now, we recall the definition of lower and upper bounds, respectively, from [16, 17]. Let A be a subset of a foset X. The *upper bound* $U(A)$ of A is the fuzzy set on X defined as follows:

$$U(A)(y) = \begin{cases} 0 & \text{if } R(x, y) = 0 \text{ for all } x \in A, \\ \wedge_{x\in A} R(x, y) & \text{otherwise.} \end{cases}$$

The *lower bound* $L(A)$ of A is the fuzzy set on X defined as follows:

$$L(A)(y) = \begin{cases} 0 & \text{if } R(y, x) = 0 \text{ for all } x \in A, \\ \wedge_{x\in A} R(y, x) & \text{otherwise.} \end{cases}$$

When $U(A)(y) \succ 0$ we write $y \in U(A)$. Similarly, for $L(A)$.

Let E be a (crisp) subset of a non empty foset X.

(1) An element z of X is the *supremum* of E (written $z = \sup E$) if $z \in U(E)$ and $y \in U(E)$ implies that $y \in U(z)$.
(2) An element z of X is the *infimum* of E (written $z = \inf E$) if $z \in L(E)$ and $y \in L(E)$ implies that $y \in L(z)$.

It is known that: If E is a subset of a foset X and if $\sup E$ (resp. $\inf E$) exists, then it is unique.

Let (X, R) be a fuzzy ordered set. A subset E of X is called *decreasing* if x belongs to E and $R(y, x) > 0$ (y is a lower bound of x) then y belongs to E (an increasing set is defined in a similar way) [16].

A *fuzzy ordered space* is a triplet (X, τ, R), where X is a non empty set, τ is a topology on X and R is a fuzzy order on X.

A *fuzzy lattice* is a fuzzy order (A, R), where A is a non-empty crisp set, such that any two elements have a supremum and an infimum, it is denoted by $(A, \vee, \wedge, R)$, where the symbols $\vee$ and $\wedge$ stand for supremum and infimum, respectively. For $a, b \in A$, $a \vee b$ is the supremum of a and b with respect to the fuzzy order R, and $a \wedge b$ is the infimum of a and b with respect to the fuzzy order R. A fuzzy lattice A is called *complete* if every subset of A have a supremum and an infimum.

A fuzzy lattice A is called *fuzzy distributive lattice* (shortly, *F-D-lattice*) if for every $x, y, z \in A$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ or $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

A fuzzy ordered space (X, τ, R) is called *totally order disconnected* if for $x, y \in X$ and $R(x, y) = 0$, there exists an increasing $\tau-$clopen U and a decreasing $\tau-$clopen V such that $U \cap V = \emptyset$ with $x \in U$ and $y \in V$. We recall that a *clopen set* in a topological space is a set which is both open and closed. A fuzzy ordered space (X, τ, R) is called a *fuzzy Priestley space* if it is compact and totally order disconnected.

3. **Priestley Duality for Finite Fuzzy Distributive Lattices**

Throughout this section, all fuzzy distributive lattices (F-D-lattices) are finite and homomorphisms preserve first (0) and last (1) elements. If $(A, \vee, \wedge, R)$ is a F-D-lattice, then its dual space is defined by: $T(A) = (X, \tau, R_1)$, where X is the set of $0-1$ homomorphisms from A onto $\{0, 1\}$, τ is the product topology induced by $\{0, 1\}^A$ and R_1 is the fuzzy order on X. Indeed, R_1 is defined on R, see Lemma 3.1.

If $\delta = (X, \tau, r)$ is a finite fuzzy Priestley space, then its dual is defined by: $(L(\delta), \vee, \wedge, r_1)$, where

$$L(\delta) = \{Y \subseteq X \mid Y \text{ is increasing and } \tau\text{-clopen}\}$$

and r_1 is a fuzzy order adequately chosen.

Lemma 3.1. *If $(A, \vee, \wedge, R)$ is an F-D-lattice, then there exists two fuzzy orders R_1, R_2 such that:*

(1) *$T(A) = (X, \tau, R_1)$ is a fuzzy Priestley space,*
(2) *$(L(T(A)), \vee, \wedge, R_2)$ is an F-D-lattice.*

Proof. (1) Let R_1 be such that:

$$R_1(f,g) = \begin{cases} R\left(\wedge g^{-1}(1),\ \wedge f^{-1}(1)\right) & \text{if } f^{-1}(1) \subset g^{-1}(1) \\ 0 & \text{otherwise,} \end{cases}$$

where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R. We show that R_1 is a fuzzy order. We have $R_1(f,f) = R\left(\wedge f^{-1}(1), \wedge f^{-1}(1)\right) = R(a,a) = 1$ for all $f \in X$ (R-reflexivity).

For all $f, g \in X$ such that $f \neq g$

$$\begin{aligned} R(f,g) \wedge R(g,f) &= R\left(\wedge g^{-1}(1),\ \wedge f^{-1}(1)\right) \wedge R\left(\wedge f^{-1}(1), \wedge g^{-1}(1))\right) \\ &= R(a,b) \wedge R(b,a) = 0 \end{aligned}$$

(because $a \neq b$ otherwise, $f = g$) (R-antisymmetry).

Now, for all $f, g, h \in X$, we show that

$$R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h).$$

The only case for investigating is

$$f^{-1}(1) \subset g^{-1}(1) \text{ and } g^{-1}(1) \subset h^{-1}(1).$$

By the transitivity of R, for every a, b, c in A, we have $R(a,b) \wedge R(b,c) \leq R(a,c)$. This yields

$$R\left(\wedge g^{-1}(1),\ \wedge f^{-1}(1)\right) \wedge R\left(\wedge h^{-1}(1),\ \wedge g^{-1}(1)\right) \leq R\left(\wedge h^{-1}(1),\ \wedge f^{-1}(1)\right).$$

The last inequality is true for every $b \in A$. Then for all $f, g, h \in X$, $R_1(f,g) \wedge R_1(g,h) \leq R_1(f,h)$ holds, and R_1 is transitive. So, R_1 is a fuzzy order and by [11], $T(A) = (X, \tau, R_1)$ is a Priestley space.

(2) Let $m_0 = \wedge_x \wedge_y \{r(x,y) \mid x \neq y \text{ and } r(x,y) \succ 0\}$. We define R_2 by:

$$R_2(H,D) = \begin{cases} 1 & \text{if } H = D \\ R\left(\wedge \bigcap_{f \in H} f^{-1}(1),\ \wedge \bigcap_{g \in D} g^{-1}(1)\right) & \text{if } H \subset D \text{ and } H \neq \emptyset \\ m_0 & \text{if } H = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R. First, we show that R_2 is a fuzzy order. We have $R_2(A,A) = 1$ (R-reflexivity) and $R_2(A,B) \wedge R_2(B,A)) = 0$ whenever $A \neq B$, i.e., R_2 is antisymmetric. In order to show the transitivity, we use the following truth table, where the proposition D is

$$R_2(A,B) \wedge R_2(B,C) \leq R_2(A,C).$$

$A \sqsubseteq C$	$A \sqsubseteq B$	$B \sqsubseteq C$	D
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	Impossible case
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Then R_2 is transitive.

Finally, the upper and lower bounds of A and B are denoted by $A \vee B$ and $A \wedge B$, respectively, and they are equal to $A \cup B$ and $A \cap B$, respectively. This shows that $(L(\delta), \vee, \wedge, R_2)$ is an F-D-lattice. □

Note that in order to see the role of the topology in the proof of Lemma 3.1, it is sufficient to see that the dual of the Priestley space, $L(T(A))$ is defined by this topology, i.e., $L(T(A)) = \{Y \subset X \ : \ Y \text{ is increasing and } \tau\text{−clopen}\}$.

Lemma 3.2. *If* $\delta = (X, \tau, r)$ *is a fuzzy finite Priestley space, then there exists two fuzzy orders* r_1 *and* r_2 *such that:*

(1) $(L(\delta), \vee, \wedge, r_1)$ *is a F-D-lattice,*
(2) $(T(L(\delta)), \tau, r_2)$ *is a fuzzy Priestley space.*

Proof. (1) If $h(r) = 0$, then X is an antichain and we can write r_1 as follows:

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B, \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that r_1 is a fuzzy order and $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for every A and B from $L(\delta)$, where $(L(\delta), \vee, \wedge, r_1)$ is a fuzzy distributive lattice.

If $h(r) \neq 0$, then X is not an antichain, we choose

$$m_0 = \wedge_x \wedge_y \{\mu_r(x, y) \mid x \neq y \text{ and } \mu_r(x, y) \succ 0\}.$$

Then $m_0 \neq 0$ and we can take r_1 such that

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B, \\ Max\left(m_0, \vee_{\substack{a \in A, b \in B \\ a \neq b}} r(a, b)\right) & \text{if } A \subset B, \\ 0 & \text{othewise.} \end{cases}$$

Similar to the previous lemma, r_1 is a fuzzy order and we can assume that $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for every A and B from $L(\delta)$, where $(L(\delta), \vee, \wedge, r_1)$ is a fuzzy distributive lattice. For the second assertion, let

$$r_2(f,g) = \begin{cases} 1 & \text{if } f = g, \\ r\left(\underset{A\in f^{-1}(1)}{\wedge\wedge} A\, , \underset{B\in g^{-1}(1)}{\wedge\wedge} B \right) & \text{if } f^{-1}(1) \subset g^{-1}(1), \\ 0 & \text{otherwise,} \end{cases}$$

where the first infimum $\wedge$ is in the sense of the fuzzy relation r and the second infimum $\wedge$ is in the sense of the fuzzy relation r_1. Note that r_2 is well defined: $A_1 = \wedge_{A\in f^{-1}(1)} A$, where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation r_1, it exists because $L(\delta)$ is a lattice and $a = \wedge A_1$, where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation r, is unique , otherwise A_1 cannot be the minimal element of $f^{-1}(1)$ [11]. Furthermore, $(T(L(\delta)), \tau, r_2)$ is a fuzzy Priestley space. □

The following theorem, shows that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices.

In [11], Priestley remarked that the basis can be characterized by the fact that they are increasing according to inclusion of prime filters from A by taking the sets $\{F_a : a \in A\}$ as basis, where F_A is the set of all lattice homomorphisms from A onto the chain $\{0, 1\}$, non-identical nulls (taking 1 in a).

Theorem 3.3. (1) *Let A be an F-D-lattice. The map $F_A : A \longrightarrow L(T(A))$ defined by $F_A(a) = \{f \in X \mid f(a) = 1\}$ is a fuzzy lattice isomorphism.*

(2) *If $\delta = (X, \tau, r)$ is a finite Priestley space, then the map $G_\delta : \delta \longrightarrow T(L(\delta))$ defined by*

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y, \end{cases}$$

for all $Y \in L(\delta\ )$ is an isomorphism of fuzzy Priestley space, i.e., a bijection and increasing map.

(3) *If $f : A_2 \longrightarrow A_2$ is a fuzzy lattice homomorphism, then the map $T(f) : T(A_2) \longrightarrow T(A_2)$ defined by $T(f)(g) = g \circ f$ is a homomorphism of fuzzy Priestley space, i.e., a continuous and increasing map.*

(4) *If $h : \delta_1 \longrightarrow \delta_2$ is a homomorphism of fuzzy Priestley space, then the map $L(h) : L(\delta_2) \longrightarrow L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a fuzzy lattice homomorphism.*

(5) *If f and h are as in (3) and (4), then the following diagrams are commutative.*

$$\begin{array}{ccc} A_1 & \overset{f_1}{--------\rightarrow} & A_2 \\ F_{A_1}\Big\downarrow & & \Big\downarrow F_{A_2} \\ L(T(A_1)) & \underset{L(T(f))}{--------\rightarrow} & L(T(A_2)) \end{array}$$

and

$$\begin{array}{ccc} \delta_1 & \overset{h}{- - - - - - - - - \rightarrow} & \delta_2 \\ | & & | \\ G_{\delta_1}| & & |G_{\delta_2} \\ \downarrow & & \downarrow \\ T\left(L\left(\delta_2\right)\right) & \underset{T(L(h_2))}{- - - - - - - - - \rightarrow} & T\left(L\left(\delta_1\right)\right) \end{array}$$

Proof. (1) Let us show that the map $F_A\left(a\right) = \{f \in X \mid \ f\left(a\right) = 1\}$ is a fuzzy lattice isomorphism. We have $R\left(x, y\right) \leq R_2\left(F_A(x), F_A(y)\right)$, where

$$R_2(F_A(x), F_A(y)) = \begin{cases} 1 & \text{if } F_A(x) = F_A(y) \\ R\left(\wedge \bigcap\limits_{f \in F_A(x)} f^{-1}(1), \ \wedge \bigcap\limits_{g \in F_A(y)} g^{-1}(1) \right) & \text{if } F_A(x) \subset F_A(y) \\ 0 & \text{otherwise,} \end{cases}$$

and the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation R. Note that if $R\left(x, y\right) \succ 0$, then $F_A(x) \subset F_A(y)$ which implies that $R_2\left(F_A(x), F_A(y)\right) = R\left(x, y\right)$. This shows that $R\left(x, y\right) \leq R_2\left(F_A(x), F_A(y)\right)$ and then the map F_A is a fuzzy lattice isomorphism.

(2) According to [11], it suffices to show that $r\left(x, y\right) \leq r_2(G_\delta\left(x\right), G_\delta\left(y\right))$. If $x = y$, then

$$Z_0 = \bigcap_{A \in G_\delta^{-1}(x)(1)} A = \bigcap_{B \in G_\delta^{-1}(x)(1)} B = Z_1$$

and so $r\left(x, y\right) \leq r_2\left(G_\delta\left(x\right), G_\delta\left(y\right)\right)$. If $x \neq y$, then there are two cases as follows:

Case 1: if $r\left(x, y\right) = 0$, then we have $r\left(x, y\right) \leq r_2\left(G_\delta\left(x\right), G_\delta\left(y\right)\right)$;

Case 2: if $r\left(x, y\right) \succ 0$, then y belongs to each $\tau-$clopen which contains x, so, $Z_0 \subset Z_1$, and then we have

$$r_2\left(G_\delta\left(x\right), G_\delta\left(y\right)\right) = r_2\left(\wedge Z_0, \wedge Z_1\right) = r\left(x, y\right),$$

where the symbol $\wedge$ stands for an infimum with respect to the fuzzy relation r. The remaining assertions are obtained by the same reasoning. □

Remark 3.4. We know that the minimum t-norm T_M (Zadeh's norm) [4] dominates any other t-norm. Then according to Lemma 2.4 (3) in [3], T_M is stronger than any other t-norm. Consequently, the results can be extended to any other t-norm.

Example 3.5. Let $(A, \vee, \wedge, R)$ be a fuzzy distributive lattice, where $A = \{a, b, c, d, e, f\}$ and R is a fuzzy relation defined by:

R	a	b	c	d	e	f
a	1	0.1	0.3	0.3	0.5	0.7
b	0	1	0.2	0.2	0.4	0.6
c	0	0	1	0	0.2	0.3
d	0	0	0	1	0.2	0.3
e	0	0	0	0	1	0.2
f	0	0	0	0	0	1

Then its dual is:

$T(A) =$ The set of $0-1$ homomorphisms from A onto $\{0,1\} = \{f_1, f_2, f_3, f_4, \}$,

such that

A	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
a	0	0	0	0
b	0	0	0	1
c	0	1	0	1
d	0	0	1	1
e	0	1	1	1
f	1	1	1	1

and its bidual is:

$$L(T(A)) = \{\emptyset,\ \{f_4\},\ \{f_2, f_4\},\ \{f_3, f_4\},\ \{f_2, f_3, f_4\},\ X\},$$

where R_2 is given by:

R_2	$\emptyset$	$\{f_4\}$	$\{f_2, f_4\}$	$\{f_3, f_4\}$	$\{f_2, f_3, f_4\}$	X
$\emptyset$	1	0.1	0.3	0.3	0.5	0.7
$\{f_4\}$	0	1	0.2	0.2	0.4	0.7
$\{f_2, f_4\}$	0	0	1	0	0.2	0.6
$\{f_3, f_4\}$	0	0	0	1	0.2	0.3
$\{f_2, f_3, f_4\}$	0	0	0	0	1	0.3
X	0	0	0	0	0	1

Finally, $F_A : A \longrightarrow L(T(A))$ is given by:

A	$F_A(a_i)\ i = 1$ to 6
a	$\emptyset$
b	$\{f_4\}$
c	$\{f_2, f_4\}$
d	$\{f_3, f_4\}$
e	$\{f_2, f_3, f_4\}$
f	X

Example 3.6. Let (X, τ, r) be a Priestley space, where $X = \{x, y, z\}$ and r is given by:

r	x	y	z
x	1	0	0
y	0	1	0
z	0	0	1

Then $L(X) = \{\emptyset,\ \{x\},\ \{y\},\ \{z\},\ \{x, y\},\ \{x, z\},\ \{y, z\},\ X\}$ and r_1 is given by:

$$r_1(A, B) = \begin{cases} 1 & \text{if } A = B \\ 1 - \frac{cardA}{cardB} & \text{if } A \subset B \\ 0 & \text{otherwise.} \end{cases}$$

Then r_1 will be given by:

r_1	$\emptyset$	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	X
$\emptyset$	1	1	1	1	1	1	1	1
$\{x\}$	0	1	0	0	1/2	1/2	0	1/3
$\{y\}$	0	0	1	0	1/2	0	1/2	1/3
$\{z\}$	0	0	0	1	0	1/2	1/2	1/3
$\{x, y\}$	0	0	0	0	1	0	0	2/3
$\{x, z\}$	0	0	0	0	0	1	0	2/3
$\{y, z\}$	0	0	0	0	0	0	1	2/3
X	0	0	0	0	0	0	0	1

and the set of $0-1$ homomorphisms from $L(X)$ onto $\{0, 1\}$, i.e., $T(L(X))$ is equal to $\{f_1, f_2, f_3\}$

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$
$\emptyset$	0	0	0
$\{x\}$	1	0	0
$\{y\}$	0	1	0
$\{z\}$	0	0	1
$\{x, y\}$	1	1	0
$\{x, z\}$	1	0	1
$\{y, z\}$	0	1	1
X	1	1	1

And r_2 will be given by:

r_2	f_1	f_2	f_3
f_1	1	0	0
f_2	0	1	0
f_3	0	0	1

and the isomorphism G_X is defined by $G_X : X \longrightarrow T(L(X))$, where

X	$G_X(X_i),\ X_i \in X$
x	f_1
y	f_2
z	f_3

.

Example 3.7. Let $(X,\ \tau,\ r)$ be a Priestley space, where $X = \{x,\ y,\ z,\ t\}$ and r is given by:

r	x	y	z	t
x	1	0	0	0.3
y	0	1	0	0.4
z	0	0	1	0.7
t	0	0	0	1

and $L(X) = \{\emptyset, \{t\}, \{x,t\}, \{y,t\}, \{z,t\}, \{x,y,t\}, \{x,z,t\}, \{y,z,t\}, X\}$, where r_1 is given by:

r_1	$\emptyset$	$\{t\}$	$\{x,t\}$	$\{y,t\}$	$\{z,t\}$	$\{x,y,t\}$	$\{x,z,t\}$	$\{y,z,t\}$	X
$\emptyset$	1	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$\{t\}$	0	1	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$\{x,t\}$	0	0	1	0	0	0.3	0.3	0.3	0.3
$\{y,t\}$	0	0	0	1	0.4	0	0.4	0.7	0.7
$\{z,t\}$	0	0	0	0	1	0	0.7	0.7	0.7
$\{x,y,t\}$	0	0	0	0	0	1	0	0	0.7
$\{x,z,t\}$	0	0	0	0	0	0	1	0	0.7
$\{y,z,t\}$	0	0	0	0	0	0	0	1	0.7
X	0	0	0	0	0	0	0	0	1

and $T(L(X)) = \{f_1,\ f_2,\ f_3, f_4\}$ such that

$L(X)$	$f_1(X_i)$	$f_2(X_i)$	$f_3(X_i)$	$f_4(X_i)$
$\emptyset$	0	0	0	0
$\{t\}$	0	0	0	1
$\{x,t\}$	1	0	0	1
$\{y,t\}$	0	1	0	1
$\{z,t\}$	0	0	1	1
$\{x,y,t\}$	1	1	0	1
$\{x,z,t\}$	1	0	1	1
$\{y,z,t\}$	0	1	1	1
X	1	1	1	1

The isomorphism G_X is defined as follows: $G_X : X \longrightarrow T(L(X))$

X	$G_X(X_i)$, $X_i \in X$
x	f_1
y	f_2
z	f_3
t	f_4

4. **Conclusions and Open Problems**

The Priestley duality comes from the classical Stone representation of distributive lattices.

Stone in [15], developed a representation theory for distributive lattices generalizing that for Boolean algebras. This he achieved by topologizing the set X of prime ideals of a closed distributive lattice A (with a first and last elements) by taking $\{I_a\colon\ a \in A\}$ as a base (where I_a denotes the set of prime ideals of A not containing a).

In 1970, H. A. Priestley developed a new duality for a closed distributive lattices by replacing $(I_a\colon\ a \in A)$ of prime ideals by $(F_a : a \in A)$ where F_a is the set of all $0-1$ lattice homomorphisms from A onto the chain $\{0,1\}$ and taking 1 in a.

The purpose of this paper is to establish a duality for closed finite fuzzy distributive lattices type Priestley extending the classical case. In this way, some results of [11, 12] are extended. The main theorem (Theorem 3.3) shows that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices.

Now, we give two open problems:

(1) Is it possible to obtain such representation for an infinite fuzzy distributive lattice?
(2) Is it possible to obtain such representation if we change the definition of fuzzy set? In other words, what happens if we replace the unite interval $[0,1]$ by any closed distributive lattice L?

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