Complex Analysis Qual Sheet

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"Tricks and traps. Basically all complex analysis qualifying exams are collections of tricks and traps."

- Jim Agler

1 Useful facts

1. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

2. $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{1}{2i}(e^{iz} - e^{-iz})$

3. $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!} = \frac{1}{2}(e^{iz} + e^{-iz})$

4. If g is a branch of f^{-1} on G, then for $a \in G$, $g'(a) = \frac{1}{f'(g(a))}$

5. $|z \pm a|^2 = |z|^2 \pm 2\text{Re}\bar{a}z + |a|^2$

6. If f has a pole of order m at $z = a$ and $g(z) = (z-a)^m f(z)$, then
$$\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

7. The elementary factors are defined as
$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$
Note that elementary factors are entire and $E_p(z/a)$ has a simple zero at $z = a$.

8. The factorization of sin is given by
$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

9. If $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$, then
$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

2 Tricks

1. If $f(z)$ nonzero, try dividing by $f(z)$. Otherwise, if the region is simply connected, try writing $f(z) = e^{g(z)}$.

2. Remember that $|e^z| = e^{\text{Re}z}$ and $\arg e^z = \text{Im}z$. If you see a Rez anywhere, try manipulating to get e^z.

3. On a similar note, for a branch of the log, $\log re^{i\theta} = \log|r| + i\theta$.

4. Let $z = e^{i\theta}$.

5. To show something is analytic use Morera or find a primitive.

6. If f and g agree on a set that contains a limit point, subtract them to show they're equal.

7. Tait: "Expand by power series."

8. If you want to count zeros, either Argument Principle or Rouché.

9. Know these Möbius transformations:

   (a) To map the right half-plane to the unit disk (or back), $\dfrac{1-z}{1+z}$.

   (b) To map from the unit disk to the unit disk, remember $\varphi_a(z) = \dfrac{z-a}{1-\overline{a}z}$. This is a bijective map with inverse $\varphi_{-a}(z)$. Also, $\varphi_a(a) = 0$, $\varphi_a'(z) = \dfrac{1-|a|^2}{(1-\overline{a}z)^2}$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = \dfrac{1}{1-|a|^2}$.

10. If $f(z)$ is analytic, then $\overline{f(\overline{z})}$ is analytic (by Cauchy-Riemann). So if, for example, $f(z)$ is real on the real axis, then $f(z) = \overline{f(\overline{z})}$.

11. To prove that a function defined by an integral is analytic, try Morera and reversing the integral. (e.g. $\int_\epsilon^\infty e^{-t}t^{z-1}dt$ is analytic since $\int_T \int_\epsilon^\infty e^{-t}t^{z-1}dtdz = \int_\epsilon^\infty \int_T e^{-t}t^{z-1}dzdt = 0$.)

12. If given a point of f (say $f(0) = a$) and some condition on f' on a simply connected set, try $\int_{[0,z]} f' = f(z) - f(0)$.

13. To create a non-vanishing function, consider exponentiating.

3 Theorems

1. **Cauchy Integral Formula**: Let G be region and $f : G \to \mathbb{C}$ be analytic. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G with $\sum_{k=0}^m n(\gamma_k; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus (\cup_{k=1}^m \{\gamma_k\})$,
$$f^{(n)}(a) \cdot \sum_{k=0}^m n(\gamma_k; a) = \frac{n!}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{(z-a)^{n+1}} dz.$$

2. **Cauchy's Theorem**: Let G be a region and $f : G \to \mathbb{C}$ be analytic. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G with $\sum_{k=0}^{m} n(\gamma_k; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then

$$\sum_{k=1}^{m} \int_{\gamma_k} f(z)dz = 0$$

3. **Liouville's Theorem**: If f is a bounded entire function, then f is constant.

4. **Maximum Modulus Theorem**: Let G be a region and $f : G \to \mathbb{C}$ be analytic. If there exists an $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant on G.

5. **Morera's Theorem**: Let G be a region and $f : G \to \mathbb{C}$ be continuous. If $\int_T f = 0$ for every triangular path T in G, then f is analytic on G.

6. **Goursat's Lemma**: Let G be a region and let $f : G \to \mathbb{C}$. If f is differentiable, then f is analytic on G.

7. **Cauchy-Riemann Equations**: Let $f(x, y) = u(x, y) + iv(x, y)$ for real-valued functions u and v. Then f is analytic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

8. **Constant functions**: Let $f : G \to \mathbb{C}$ be analytic. Then the following are equivalent

   (i) $f(z) \equiv \alpha$;

   (ii) $\{z \in G \mid f(z) = \alpha\}$ has a limit point in G;

   (iii) there exists $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \geq 1$.

9. **Conformality**: Let $f : G \to \mathbb{C}$ be analytic. Then if $z \in G$ and $f'(z) \neq 0$, f is conformal at z.

10. **Roots of an analytic function**: Let $f : G \to \mathbb{C}$ be analytic. If $f(a) = 0$, then there exists a unique $m \geq 1$ and g analytic such that

$$f(z) = (z - a)^m g(z)$$

with $g(a) \neq 0$.

11. **Power series**: A function f is analytic on $B(a; R)$ if and only if there exists a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ where we compute

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}} dz.$$

The series converges absolutely on $B(a; R)$ and uniformly on $B(a; r)$ for $0 \leq r < R$.

12. **Cauchy's Estimate**: If f analytic on $B(a; R)$, and $|f(z)| \leq M$ for each $z \in B(a; R)$, then

$$\left| f^{(n)}(a) \right| \leq \frac{n! M}{R^n}.$$

13. **Winding Number**: To compute the index of a closed curve about a point a,

$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}.$$

14. **Open Mapping Theorem**: Let G be a region, f a non-constant analytic function. If U is an open subset of G, then $f(U)$ is open.

15. **Zero-Counting Theorem**: Let G be a region, $f : G \to \mathbb{C}$ analytic with roots $a_1, \ldots a_m$. If $\{\gamma\} \subseteq G$ and $a_k \notin \{\gamma\}$ for all k, and $\gamma \approx 0$ in G, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma; a_k)$$

Corollary: If $f(a) = \alpha$, then $f(z) - \alpha$ has a root at a. So if $f(a_k) = \alpha$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - \alpha} dz = \sum_{k=1}^{m} n(\gamma; a_k)$$

Corollary 2: If $\sigma = f \circ \gamma$ and $\alpha \notin \{\sigma\}$ and a_k are the points where $f(a_k) = \alpha$, then

$$n(\sigma; \alpha) = \sum_{k=1}^{m} n(\gamma; a_k) \text{ or}$$

$$n(f \circ \gamma; f(a)) = \sum_{k=1}^{m} n(\gamma; a_k)$$

16. **Roots of analytic functions**: Suppose f is analytic on $B(a; R)$ and let $f(a) = \alpha$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there exist $\epsilon > 0$ and $\delta > 0$ such that if $0 < |\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a, \epsilon)$.

17. **Existence of Logarithm**: Let $f(z)$ be analytic and $f(z) \neq 0$ on G, a simply connected region. Then there is analytic function $g(z)$ on G such that $f(z) = e^{g(z)}$ for all $z \in G$.

18. **Existence of Primitive**: Let $f(z)$ be analytic on G, a simply connected region. Then f has a primitive.

19. **Laurent Series**: Let f be analytic on $R_1 < |z - a| < R_2$, then there exists a sequence $\{a_n\}_{n=-\infty}^{\infty}$ and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

with absolute convergence in the open annulus and uniform convergence on every compact subset of the annulus. This series is called a Laurent series, and if γ is a closed curve in the annulus, then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dw.$$

(Note that this is just the same as number 11).

20. **Classification of Singularities**: Suppose f analytic on $B(a;R)\setminus\{a\}$ and f has an isolated singularity at a. Then a is

   (a) *Removable singularity* if there is a function g analytic on $B(a;R)$ such that $f(z)=g(z)$ for all $z\in B(a;R)\setminus\{a\}$.
   The singularity is removable if and only if $\lim_{z\to a}(z-a)f(z)=0$.
   Also, the singularity is removable if and only if the Laurent series of f has no coefficients a_n for $n<0$.

   (b) *Pole* if $\lim_{z\to a}|f(z)|=\infty$.
   If a is a pole, then there is a unique $m\geq 1$ and an analytic function g such that $f(z)=\dfrac{g(z)}{(z-a)^m}$ for all $z\in B(a;R)\setminus\{a\}$ and $g(a)\neq 0$.
   The singularity is a pole if and only if the Laurent series of f has only finitely many coefficients a_n for $n<0$. The partial series for these coefficients is called the singular part of f.

   (c) *Essential singularity* if a is not removable and not a pole.
   The singularity is essential if and only if the Laurent series of f has infinitely many coefficients a_n for $n<0$.

21. **Casorati-Weierstrass**: If f has an essential singularity at a, then for all $\delta>0$, $f(\{z\mid 0<|z-a|<\delta\})$ is dense in $\mathbb{C}$.

22. **Residues**: If f has an isolated singularity at a, then the residue of f at a, $\operatorname{Res}(f;a)=a_{-1}$. We can calculate the residue using the formula for Laurent coefficients:

$$\operatorname{Res}(f;a)=\frac{1}{2\pi i}\int_\gamma f(z)dz.$$

   If a is a pole of order m, then if $g(z)=(z-a)^m f(z)$

$$\operatorname{Res}(f;a)=\frac{g^{(m-1)}(a)}{(m-1)!}.$$

23. **Residue Theorem**: Let f be analytic on a region G except for singularities at $a_1,\ldots,a_m$. Let $\gamma\approx 0$ be a closed curve in G with $a_1,\ldots,a_m\notin\{\gamma\}$. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz=\sum_{k=1}^{m}n(\gamma;a_k)\cdot\operatorname{Res}(f;a_k).$$

24. **Argument Principle**: Let f be meromorphic with roots $z_1,\ldots,z_m$ and poles $p_1,\ldots,p_n$ with $z_1,\ldots,z_m,p_1,\ldots,p_n\notin\{\gamma\}$. Then

$$\frac{1}{2\pi i}\int_\gamma\frac{f'}{f}=\sum_{k=1}^{m}n(\gamma;z_m)-\sum_{j=1}^{n}n(\gamma;p_n).$$

25. **Rouché's Theorem**: Let f, g be meromorphic on G and let γ be a closed curve in G. Then if, for all $z \in \{\gamma\}$,
$$|f(z) + g(z)| < |f(z)| + |g(z)|$$
then $Z_f - P_f = Z_g - P_g$.

26. **Jordan's Lemma**: Suppose that:

    (i) $f(z)$ is analytic at all points z in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$;

    (ii) C_R denotes a semicircle $z = Re^{i\theta}$ for $0 \leq \theta \leq 2\pi$ with $R > R_0$;

    (iii) for all points z on C_R there is a positive constant M_R such that $|f(z)| \leq M_R$, with $\lim_{R\to\infty} M_R = 0$

    Then for every positive constant a:
$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0.$$

27. **Fractional Residue**: If z_0 is a simple pole of $f(z)$, and C_R is an arc of the circle $\{|z-z_0| = R\}$ of angle θ, then
$$\lim_{R\to 0} \int_{C_R} f(z)dz = \theta i Res(f(z), z_0).$$

4 Theorems, part 2

1. **Maximum Modulus Theorem**:

    (a) (First Version). If $f : G \to \mathbb{C}$ is analytic and there exists $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

    (b) (Second Version). If G is open and bounded, and f analytic on G and continuous on $\overline{G}$, then
$$\max\{|f(z)| \mid z \in \overline{G}\} = \max |f(z)| \mid z \in \partial G\}.$$
    (Or f attains its maximum on the boundary).

    (c) (Third Version). If $f : G \to \mathbb{C}$ is analytic, and there is a constant M such that $\limsup_{z\to a} |f(z)| \leq M$ for all $a \in \partial_\infty G$, then $|f(z)| \leq M$ for all $z \in G$.
    (Where $\limsup_{z\to a} f(z) = \lim_{r\to 0^+} \sup\{f(z) \mid z \in G \cap B(a; r)\}$.)

2. **Schwarz's Lemma**: Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(0) = 0$. Then

    (i) $|f'(0)| \leq 1$,

    (ii) $|f(z)| \leq z$, and

    (iii) if $|f'(0)| = 1$ or $|f(z)| = z$ for any $z \in \mathbb{D}$, then $f(z) = cz$ for some $|c| = 1$.

3. **Generalized Schwarz's Lemma**: Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic. Then

(i) $|f'(a)| \leq \dfrac{1-|f(a)|^2}{1-|a|^2}$,

(ii) if equality, then $f(z) = \varphi_{-a}(c\varphi_a(z))$.

4. **Logarithmic Convexity**: Let $a < b$, $G = \{z \in \mathbb{C} \mid a < \text{Re } z < b\}$, and $f : \overline{G} \to \mathbb{C}$. If f is continuous on $\overline{G}$, analytic on G and bounded, then $M(x) = \sup\limits_{y \in \mathbb{R}} |f(x+iy)|$ is logarithmically convex.

5. **Phragmèn-Lindelöf**: Let G be simply connected, $f : G \to \mathbb{C}$ analytic, and suppose there exists $\varphi : G \to \mathbb{C}$ analytic, bounded, and nonzero on G. Suppose further that $\partial_\infty G = A \cup B$ and

   (i) for all $a \in A$, $\limsup\limits_{z \to a} |f(z)| \leq M$

   (ii) for all $b \in B$, for all $\eta > 0$, $\limsup\limits_{z \to b} |f(z)||\varphi_(a)|^\eta \leq M$,

   then $|f(z)| \leq M$ on G.

6. **Logic of the ρ metric**: For all $\epsilon > 0$, there exist $\delta > 0$ and $K \subseteq G$ compact such that

$$\rho_K(f, g) < \delta \Longrightarrow \rho(f, g) < \epsilon$$

   and for all $\delta > 0$, K compact, there exists an ϵ such that

$$\rho(f, g) < \epsilon \Longrightarrow \rho_K(f, g) < \delta$$

7. **Spaces of Continuous Functions**: If Ω is complete, then $C(G, \Omega)$ is complete.

8. **Normal Families**: $\mathscr{F} \subseteq C(G, \Omega)$. $\mathscr{F}$ is normal if all sequences have a convergent subsequence.

   $\mathscr{F}$ is normal iff $\overline{\mathscr{F}}$ is compact iff $\mathscr{F}$ is totally bounded (i.e. for all K, $\delta > 0$, there exist $f_1, \ldots, f_n \in \mathscr{F}$ such that $\mathscr{F} \subseteq \bigcup_{i=1}^n \{g \in C(G, \Omega) \mid \rho_K(f; g) < \delta\}$.

9. **Arzela-Ascoli**: $\mathscr{F}$ is normal iff

   (i) for all $z \in G$, $\{f(z) \mid f \in \mathscr{F}\}$ has compact closure in Ω, and

   (ii) for all $z \in G$, $\mathscr{F}$ is equicontinuous at z (for all $\epsilon > 0$, there exists $\delta > 0$ such that $|z - w| < \delta \Rightarrow d(f(z), f(w)) < \epsilon$ for all $f \in \mathscr{F}$).

10. **The Space of Holomorphic Functions**: Some useful facts:

   (a) $f_n \to f \Longleftrightarrow$ for all compact $K \subseteq G$, $f_n \to f$ uniformly on K.

   (b) $\{f_n\}$ in $H(G)$, $f \in C(G, \mathbb{C})$, then $f_n \to f \Longrightarrow f \in H(G)$ (If f_n converges, it will converge to an analytic function).

   (c) $f_n \to f$ in $H(G) \Longrightarrow f_n^{(k)} \to f^{(k)}$ (If f converges, its derivatives converge).

   (d) $H(G)$ is complete (Since $H(G)$ is closed and $C(G, \mathbb{C})$ is complete).

11. **Hurwitz's Theorem**: Let $\{f_n\} \in H(G)$, $f_n \to f$, $f \not\equiv 0$. Let $\overline{B(a;r)} \subseteq G$ such that $f \neq 0$ on $|z-a| = r$. Then there exists an N such that $n \geq N \Longrightarrow f_n$ and f have the same number of zeros in $B(a;r)$.

    Corollary: If $f_n \to f$ and $f_n \neq 0$, then either $f(z) \equiv 0$ or $f(z) \neq 0$.

12. **Local Boundedness**: A set $\mathscr{F}$ in $H(G)$ is locally bounded iff for each compact set $K \subset G$ there is a constant M such that $|f(z)| \leq M$ for all $f \in \mathscr{F}$ and $z \in K$. (Also, $\mathscr{F}$ is locally bounded if for each point in G, there is a disk on which $\mathscr{F}$ is uniformly bounded.)

13. **Montel's Theorem**: $\mathscr{F} \subseteq H(G)$, then $\mathscr{F}$ is normal $\Longleftrightarrow \mathscr{F}$ is locally bounded (for all K compact, there exists M such that $f \in \mathscr{F} \Rightarrow |f(z)| \leq M$ for all $z \in K$).

    Corollary: $\mathscr{F}$ is compact iff $\mathscr{F}$ is closed and locally bounded.

14. **Meromorphic/Holomorphic Functions**: If $\{f_n\}$ in $M(G)$ (or $H(G)$) and $f_n \to f$ in $C(G, \mathbb{C}_\infty)$, then either $f \in M(G)$ (or $H(G)$) or $f \equiv \infty$.

15. **Riemann Mapping Theorem**: G simply connected region which is not $\mathbb{C}$. Let $a \in G$, then there is a unique analytic function such that:

    (a) $f(a) = 0$ and $f'(a) > 0$;
    (b) f is one-to-one;
    (c) $f(G) = \mathbb{D}$.

16. **Infinite Products**: Some propositions for convergence of infinite products:

    (a) Re $z_n > 0$. Then $\prod z_n$ converges to a nonzero number iff $\sum \log z_n$ converges.
    (b) Re $z_n > -1$. Then $\sum \log(1+z_n)$ converges absolutely iff $\sum z_n$ converges absolutely.
    (c) Re $z_n > 0$. Then $\prod z_n$ converges absolutely iff $\sum (z_n - 1)$ converges absolutely.

17. **Products Defining Analytic Functions**: G a region and $\{f_n\}$ in $H(G)$ such that $f_n \not\equiv 0$. If $\sum [f_n(z) - 1]$ converges absolutely uniformly on compact subsets of G then $\prod f_n$ converges in $H(G)$ to an analytic function $f(z)$. The zeros of $f(z)$ correspond to the zeros of the f_n's.

18. **Entire Functions with Prescribed Zeros**: Let $\{a_n\}$ be a sequence with $\lim |a_n| = \infty$ and $a_n \neq 0$. If $\{p_n\}$ is a sequence of integers such that for all $r > 0$

    $$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty,$$

    then $f(z) = \prod E_{p_n}(z/a_n)$ converges in $H(\mathbb{C})$ and f is an entire function with the correct zeros. (Note that you can choose $p_n = n - 1$ and it will always converge).

19. **The (Boss) Weierstrass Factorization Theorem**: Let f be an entire function with non-zero zeros $\{a_n\}$ with a zero of order m at $z = 0$. Then there is an entire function g and a sequence of integers $\{p_n\}$ such that

    $$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n} \right).$$

20. **Existence of Analytic Functions with Given Zeros**: Let G be a region and $\{a_j\}$ a sequence of distinct points with no limit point in G, $\{m_j\}$ a sequence of integers. Then there is an analytic function f defined on G whose only zeros are the a_j's with multiplicity m_j.

21. **Meromorphic Functions as a Quotient of Analytic**: If f is a meromorphic function on the open set G, then there are analytic functions g and h on G such that $f = g/h$.

22. **Runge's Theorem**: Let K be compact and E meet each component of $\mathbb{C}_\infty \setminus K$. If f is analytic in an open set containing K, then for any $\epsilon > 0$, there is a rational function $R(z)$ with poles in E such that $|f(z) - R(z)| < \epsilon$ for all $z \in K$.

    Corollary: Let G be an open subset of the plane and E a subset of $\mathbb{C}_\infty \setminus G$ meeting each component. Let $R(G, E)$ be the set of rational functions with poles in E. If $f \in H(G)$ then there is a sequence $\{R_n\}$ in $R(G, E)$ such that $f = \lim R_n$. (That is, $R(G, E)$ is dense in $H(G)$).

    Corollary: If $\mathbb{C}_\infty \setminus G$ is connected, then polynomials are dense in G.

23. **Polynomially Convex Hull**: Let K be compact. The polynomially convex hull of K ($\hat{K}$) is the set of all points w such that for every polynomial p, $|p(w)| \le \max\{|p(z)| \mid z \in K\}$.

    If K is an annulus, then $\hat{K}$ is the disk obtained by filling in the interior hole.

24. **A Few Words on Simple Connectedness (Ha)**: The following are equivalent for $G \subseteq \mathbb{C}$ open, connected:

    (i) G is simply connected;
    (ii) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every point $a \in \mathbb{C} \setminus G$;
    (iii) $\mathbb{C}_\infty \setminus G$ is connected;
    (iv) For any $f \in H(G)$, there is a sequence of polynomials that converges to f in $H(G)$;
    (v) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;
    (vi) Every function $f \in H(G)$ has a primitive;
    (vii) For any $f \in H(G)$ such that $f(z) \neq 0$, there is a function $g \in H(G)$ such that $f(z) = \exp g(z)$;
    (viii) For any $f \in H(G)$ such that $f(z) \neq 0$, there is a function $g \in H(G)$ such that $f(z) = [g(z)]^2$;
    (ix) G is homeomorphic to $\mathbb{D}$;
    (x) If $u : G \to \mathbb{R}$ is harmonic then there exists a harmonic conjugate.

25. **Mittag-Leffler's Theorem**: Let G be open, $\{a_k\}$ distinct points in G without a limit point in G, and $\{S_k(z)\}$ be a sequence of singular parts at the a_k's. Then there is a meromorphic function f on G whose poles are exactly the $\{a_k\}$ such that the singular part of f at a_k is $S_k(z)$.

26. **Mean Value Property**: If $u : G \to \mathbb{R}$ is a harmonic function and $\overline{B(a; r)}$ is a closed disk contained in G, then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

In fact, for $z \in B(0;r)$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) u(re^{i\theta})\, d\theta.$$

27. **Jensen's Formula**: Let f be analytic on $\overline{B(0;r)}$ and suppose $a_1, \ldots, a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity. If $f(0) \neq 0$, then

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

28. **Poisson-Jensen Formula**: Let f be analytic on $\overline{B(0;r)}$ and suppose $a_1, \ldots, a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity. If $f(z) \neq 0$, then

$$\log|f(z)| = -\sum_{k=1}^{n} \log\left(\frac{r^2 - \overline{a_k}z}{r(z-a_k)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log|f(re^{i\theta})| d\theta.$$

29. **Genus, Order, and Rank of Entire Functions**:

- *Rank*: Let f be an entire function with zeros $\{a_k\}$ repeated according to multiplicity such that $|a_1| \leq |a_2| \leq \ldots$. Then f is of finite rank if there is a $p \in \mathbb{Z}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty.$$

  If p is the smallest integer such that this occurs, then f is of rank p. A function with only a finite number of zeros has rank 0.

- *Standard Form*: Let f be an entire function of rank p with zeros $\{a_k\}$. Then the canonical product

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

  is the standard form for f.

- *Genus*: An entire function f has finite genus if f has finite rank and $g(z)$ is a polynomial. If the rank is p and the degree of g is q, then the genus $\mu = \max(p,q)$. If f has genus μ, then for each $\alpha > 0$, there exists an r_0 such that $|z| > r_0$ implies

$$|f(z)| < e^{\alpha|z|^{\mu+1}}.$$

- *Order*: An entire function f is of finite order if there exists $a > 0$ and $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. The number

$$\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$$

  is called the order of f.
  If f has order λ and $\epsilon > 0$, then $|f(z)| < \exp(|z|^{\lambda+\epsilon})$ for all $|z|$ sufficiently large, and a z can be found, with $|z|$ as large as desired, such that $|f(z)| \geq \exp(|z|^{\lambda-\epsilon})$.
  If f is of genus μ, then f is of finite order $\lambda \leq \mu + 1$.

30. **Hadamard's Factorization Theorem**: If f is entire with finite order λ, then f has finite genus $\leq \lambda$. Combined with above, we have that f has finite order if and only if f has finite genus. **Corollary**: If f is entire with finite order, then for all $c \in \mathbb{C}$ with one possible exception, we can always solve $f(z) = c$.

    Corollary: If f is entire with finite order $\lambda \notin \mathbb{Z}$, then f has an infinite number of zeros.

5 Special Functions

1. **The Riemann Zeta Function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \quad \text{and} \quad \zeta(s) = \zeta(1-s)$$

   This function has a pole at $s = 1$, zeros at the negative even integers, and its remaining zeros are in the critical strip $\{z \mid 0 < \text{Re } z < 1\}$.

   Riemann's functional equation is

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right).$$

2. **The Gamma Function**: The gamma function is the meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$ defined by:

$$\begin{aligned}
\Gamma(z) &= \int_0^{\infty} e^{-t} t^{z-1} dt \\
&= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \\
&= \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\cdots(z+n)} = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.
\end{aligned}$$

   The residues at each of the poles is given by

$$\text{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}.$$

   The functional equation holds for $z \neq 0, 1, \ldots$

$$\Gamma(z+1) = z\Gamma(z).$$

   Note further that

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad \text{and} \quad \overline{\Gamma(z)} = \Gamma(\overline{z}) \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi}.$$

6 Theorems, part 3

1. **Schwarz Reflection Principle**: Let G be a region such that $G = G^*$ (symmetric with respect to real axis). If $f : G_+ \cup G_0 \to \mathbb{C}$ is continuous and analytic on G_+, and $f(G_0) \subseteq \mathbb{R}$, then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = g(z)$ for $z \in G_+ \cup G_0$.

2. **Analytic Continuations**: Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve and $[f]_a$ be a germ at $a = \gamma(0)$. An analytic continuation of $[f]_a$ along γ is a family $(f_t, G_t), t \in [0, 1]$ such that

   (i) $\gamma(t) \in G_t$

   (ii) $[f_0]_a = [f]_a$

   (iii) $\forall t \in [0, 1]$, $\exists \delta > 0$ such that $|s - t| < \delta \Longrightarrow \gamma(s) \in G_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$

3. **Uniqueness of Analytic Continuations**: Let $\gamma : [0, 1] \to \mathbb{C}$ be a path from a to b and let (f_t, G_t) and (g_t, B_t) be two analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$.

4. **Analytic Continuations along FEP Homotopic Curves**: Let $a \in \mathbb{C}$ and $[f]_a$ a germ at a. If γ_0 and γ_1 are FEP homotopic and $[f]_a$ admits analytic continuation along every $\gamma_s, s \in [0, 1]$, then the analytic continuations of $[f]_a$ along γ_0 and γ_1 are equal.

5. **Monodromy Theorem**: Let G be a region, $a \in G$, $[f]_a$ a germ at a. If G is simply connected and admits unrestricted continuation of $[f]_a$. then there exists $F \in H(G)$ such that $[F]_a = [f]_a$.

6. **Neighborhood Systems**: Let X be a set and for all $x \in X$, $\mathscr{N}_x$ a collection of subsets of X such that

   (i) for each $U \in \mathscr{N}_x$, $x \in U$;

   (ii) if $U, V \in \mathscr{N}_x$, $\exists W \in \mathscr{N}_x$ such that $W \subseteq U \cap V$;

   (iii) if $U \in \mathscr{N}_x$ and $V \in \mathscr{N}_y$ then for $z \in U \cap V$ $\exists W \in \mathscr{N}_z$ such that $W \subseteq U \cap V$.

   Then $\{\mathscr{N}_x \mid x \in X\}$ is a neighborhood system on X.

7. **Sheaf of Germs**: For an open set G in $\mathbb{C}$ let

   $$\mathscr{S}(G) = \{(z, [f]_z) \mid z \in G, f \text{ is analytic at } z\},$$

   and define a map $\rho : \mathscr{S}(G) \to \mathbb{C}$ by $\rho(z, [f]_z) = z$. Then $(\mathscr{S}(G), \rho)$ is the sheaf of germs of analytic functions on G.

   We put a topology on the sheaf of germs by defining a neighborhood system. For $D \subseteq G$, and $f \in H(D)$, define

   $$N(f, D) = \{(z, [f]_z) \mid z \in D\}.$$

   For each point $(a, [f]_a) \in \mathscr{S}(G)$, let

   $$\mathscr{N}_{(a,[f]_a)} = \{N(g, B) \mid a \in B \text{ and } [g]_a = [f]_a\}.$$

   This is a neighborhood system on $\mathscr{S}(G)$ and the induced topology is Hausdorff.

8. **Components of the Sheaf of Germs**:
   - There is a path in $\mathscr{S}(G)$ from $(a, [f]_a)$ to $(b, [g]_b)$ iff there is a path γ in G from a to b such that $[g]_b$ is the analytic continuation of $[f]_a$ along γ.
   - Let $\mathscr{C} \subseteq \mathscr{S}(G)$ and $(a, [f]_a) \in \mathscr{C}$. Then $\mathscr{C}$ is a component of $\mathscr{S}(G)$ iff

$$\mathscr{C} = \{(b, [g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\}.$$

9. **Riemann Surfaces**: Fix a function element (f, D). The complete analytic function $\mathscr{F}$ associated with (f, D) is the collection

$$\mathscr{F} = \{[g]_z \mid [g]_z \text{ is an analytic continuation of } [f]_a \text{ for any } a \in D\}.$$

   Then $\mathscr{R} = \{(z, [g]_z) \mid [g]_z \in \mathscr{F}\}$ is a component of $\mathscr{S}(\mathbb{C})$, and $(\mathscr{R}, \rho)$ is the Riemann Surface of $\mathscr{F}$.

10. **Complex Manifolds**: Let X be a topological space.
    - A coordinate chart is a pair (U, φ) such that $U \subseteq X$ is open and $\varphi : U \to V \subseteq \mathbb{C}$ is a homeomorphism.
    - A complex manifold is a pair (X, Φ) where X is connected, Hausdorff and Φ is a collection of coordinate patches on X such that
      (i) each point of X is contained in at least one member of Φ and
      (ii) if $(U_a, \varphi_a), (U_b, \varphi_b) \in \Phi$ with $U_a \cap U_b \neq \emptyset$, then $\varphi_a \circ \varphi_b^{-1}$ is analytic.

11. **Analytic Functions**: Let (X, Φ) and (Ω, Ψ) be analytic manifolds, $f : X \to \Omega$ continuous, $a \in X$, and $(a) = \alpha$. Then f is analytic at a if for any patch $(\Lambda, \psi) \in \Psi$ which contains α, there is a patch $(U, \varphi) \in \Phi$ which contains a such that

    (i) $f(U) \subseteq \Lambda$;
    (ii) $\psi \circ f \circ \varphi^{-1}$ is analytic on $\varphi(U) \subseteq \mathbb{C}$.

12. **Some Results on Analytic Functions**:
    - Let $\mathscr{F}$ be a complete analytic function with Riemann surface $(\mathscr{R}, \rho)$. If $\mathscr{F} : \mathscr{R} \to \mathbb{C}$ is defined by $\mathscr{F}(z, [f]_z) = f(z)$ then $\mathscr{F}$ is an analytic function.
    - Compositions of analytic function are analytic
    - (Limit Points) If f and g are analytic functions $X \to \Omega$ and if $\{x \in X : f(x) = g(x)\}$ has a limit point in X, then $f = g$.
    - (Maximum Modulus) If $f : X \to \mathbb{C}$ is analytic and there is a point $a \in X$ and a neighborhood U of a such that $|f(a)| \geq |f(x)|$ for all $x \in U$, then f is constant.
    - (Liouville) If (X, Φ) is a compact analytic manifold, then there is no non-constant analytic function from X into $\mathbb{C}$.
    - (Open Mapping) Let $f : X \to \Omega$ be a non-constant analytic function. If U is an open subset of X, then $f(U)$ is open in Ω.

13. **Mean Value Property**: If $u : G \to \mathbb{R}$ is a harmonic function and $\overline{B(a;r)} \subset G$ then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

14. **Maximum Principles**:

    I. Suppose $u : G \to \mathbb{R}$ has the MVP. If there is a point $a \in G$ such that $u(a) \geq u(z)$ for all z in G, then u is constant. (Analogously, there is a Minimum Principle).

    II. Let $u, v : G \to \mathbb{R}$ be bounded and continuous functions with the MVP. If for each point $a \in \partial_\infty G$,

    $$\limsup_{z \to a} u(z) \leq \liminf_{z \to a} v(z)$$

    then $u(z) < v(z)$ for all z in G or $u = v$.

    Corollary: If a continuous function satisfying the MVP is 0 on the boundary, then it is identically 0.

    III. If $\varphi : G \to \mathbb{R}$ is a subharmonic function and there is a point $a \in G$ with $\varphi(a) \geq \varphi(z)$ for all z in G, then φ is constant.

    IV. If $\varphi, \psi : G \to \mathbb{R}$ are bounded functions such that φ is subharmonic and ψ is superharmonic and for each point $a \in \partial_\infty G$,

    $$\limsup_{z \to a} \varphi(z) \leq \liminf_{z \to a} \psi(z)$$

    then $\varphi(z) < \psi(z)$ for all z in G or $\varphi = \psi$ is harmonic.

15. **The Poisson Kernel**: For $0 \leq r < 1, -\infty < \theta < \infty$, the Poisson kernel is the following:

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

16. **Dirichlet Problem in the Disk**: If $f : \partial\mathbb{D} \to \mathbb{R}$ is a continuous function, then there is a continuous harmonic function $u : \overline{\mathbb{D}} \to \mathbb{R}$ such that $u(z) = f(z)$ for all $z \in \partial\mathbb{D}$. Moreover, u is unique and defined by

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, dt.$$

17. **Harmonicity vs. MVP**: If $u : G \to \mathbb{R}$ is a continuous function which has the MVP, then u is harmonic.

18. **Harnack's Inequality**: If $u : \overline{B(a;R)} \to \mathbb{R}$ is continuous, harmonic in $B(a;R)$, and $u \geq 0$ then for $0 \leq r < R$ and all θ

$$\frac{R - r}{R + r} u(a) \leq u(a + re^{i\theta}) \leq \frac{R + r}{R - r} u(a).$$

19. **Harnack's Theorem**: Let G be a region. The metric space $\text{Har}(G)$ is complete. If $\{u_n\}$ is a sequence in $\text{Har}(G)$ such that $u_1 \leq u_2 \leq \ldots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in $\text{Har}(G)$ to a harmonic function.

20. **Subharmonic Functions**: Let $\varphi : G \to \mathbb{R}$ be continuous. Then φ is subharmonic iff for every $G_1 \subseteq G$ and every harmonic u_1 on G_1, $\varphi - u_1$ satisfies the Maximum Principle on G_1

    Corollary: φ is subharmonic iff for every bounded region G_1 such that $\overline{G_1} \subset G$ and for every continuous function $u_1 : \overline{G_1} \to \mathbb{R}$ that is harmonic on G_1 and satisfies $\varphi(z) \leq u_1(z)$ on ∂G_1, $\varphi(z) \leq u_1(z)$ for $z \in G_1$.

21. **Maxima of Subharmonic Functions**: If φ_1 and φ_2 are subharmonic functions on G then $\varphi(z) = \max\{\varphi_1(z)\varphi_2(z)\}$ is a subharmonic function.

22. **Bumping** Let $\varphi : G \to \mathbb{R}$ be subharmonic and $\overline{B(a;r)} \subset G$. Define $\varphi'(z) = \varphi(z)$ if $z \in G \setminus B(a;r)$ and $\varphi'(z)$ be the solution to the Dirichlet problem for $z \in B(a;r)$. Then φ' is subharmonic.

23. **The Perron Function**: Let $f : \partial_\infty G \to \mathbb{R}$ be continuous. Then $u(z) = \sup\{\varphi(z) \mid \varphi \in \mathscr{P}(f, G)\}$ defines a harmonic function on G.

    ($\mathscr{P}(f, G) = \{\varphi : G \to \mathbb{R} \mid \varphi \text{ subharmonic}, \limsup_{z \to a} \varphi(z) \leq f(a)\ \forall a \in \partial_\infty G\}$)

24. **General Dirichlet Problem**: A region G is a Dirichlet Region iff there is a barrier for G at each point of $\partial_\infty G$.

    (A barrier for G at a is a family $\{\psi_r\}$ such that ψ_r is superharmonic on $G(a;r)$ with $0 \leq \psi_r(z) \leq 1$, $\lim\limits_{z \to a} \psi_r(z) = 0$, and $\lim\limits_{z \to w} \psi_r(z) = 1$ for $w \in G \cap \{w \mid |w - a| = r\}$.)

    Corollary: Let G be a region such that no component of $\mathbb{C}_\infty \setminus G$ reduces to a point, then G is a Dirichlet region.

    Corollary: A simply connected region is a Dirichlet region.