Synopsis of Complex Analysis

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Chapter 1

Complex Numbers

1.1 The Parts of a Complex Number

A complex number, z, is an ordered pair of real numbers similar to the points in the real plane, $\mathbb{R}^2$.

$$z \equiv (x, y) \tag{1.1}$$

The first and second components of z is called the *real* and *imaginary* parts respectively.

$$\text{Re}(z) = x, \quad \text{Im}(z) = y \tag{1.2}$$

The imaginary unit, i, is defined by

$$i \equiv (0, 1) \tag{1.3}$$

Thus, an equivalent notation to the ordered pair, is to write a complex number in terms of its components by

$$z = x + iy \tag{1.4}$$

with the real unit $(1, 0)$ assumed next to the term without an i because that term is real.

1.2 The Complex Product

The plane of complex numbers, $\mathbb{C}$, deviates from $\mathbb{R}^2$ in that the product of two complex numbers is defined by

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) \equiv (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{1.5}$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{1.6}$$

Multiplying two imaginary units, one can see why i is sometimes identified with $\sqrt{-1}$.

$$(0,1)\cdot(0,1) = (-1,0) \tag{1.7}$$
$$i^2 = -1 \tag{1.8}$$

However, one should take caution when writing the imaginary unit as $\sqrt{-1}$, or else one can be lead to contradicting conclusions like

$$-1 = i \cdot i = \sqrt{-1}\cdot\sqrt{-1} \stackrel{?}{=} \sqrt{-1\cdot -1} = \sqrt{1} = 1 \tag{1.9}$$

One should always fall back to the definition of the imaginary unit and the complex product when there is confusion.

1.3 Functions

In this context, a function f takes a complex number z as its argument and returns another complex number. The real and imaginary parts of the returned complex number are often denoted u and v respectively.

$$u \equiv \mathrm{Re}\left(f(z)\right), \quad v \equiv \mathrm{Im}\left(f(z)\right) \tag{1.10}$$

1.4 Euler's Formula

Consider the exponential of an imaginary number, $e^{i\theta}$, where $\theta \in \mathbb{R}$.

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^2}{n!} && (1.11)\\ &= 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 \ldots && (1.12)\\ &= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \ldots\right) && (1.13)\\ &= \cos\theta + i\sin\theta && (1.14) \end{aligned}$$

This gives *Euler's Formula*

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \tag{1.15}$$

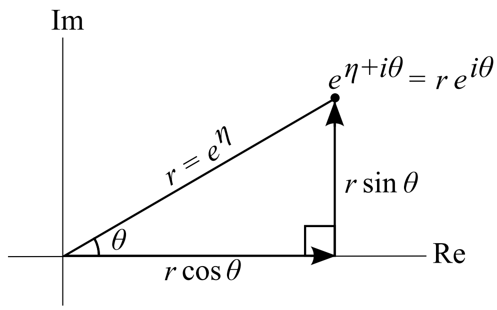


Figure 1.1: Geometric Interpretation of Euler's Formula

Now we see that for a general complex number, $\eta + i\theta$, $\{\eta, \theta\} \in \mathbb{R}$, we have

$$e^{\eta+i\theta} = e^{\eta}\,(\cos\theta + i\sin\theta) \tag{1.16}$$

Defining $r \equiv e^{\eta}$, we see that any nonzero complex number, z, can be written in polar coordinates using

$$z = e^{\eta+i\theta} = r\,e^{i\theta} = r\,(\cos\theta + i\sin\theta) \tag{1.17}$$

Figure 1.1 demonstrates how writing a complex number as $r\,e^{i\theta}$ is a point in the complex plane with polar coordinates (r, θ).

1.5 Argument, Magnitude, and Conjugate

When thinking of a complex number, z, in polar coordinates, it is customary to call the angle "the *argument* of z," denoted

$$\theta = \mathrm{Arg}(z) \tag{1.18}$$

r is known as "the *magnitude* of z," denoted

$$r = |z| \tag{1.19}$$

The *complex conjugate* is an operation which flips the sign of the imaginary part of the complex number on which it acts. The complex conjugate of z is denoted z^*. For $z = x + iy = r\,e^{i\theta}$, we have that

$$z^* = x - iy = r\,e^{-i\theta} \tag{1.20}$$

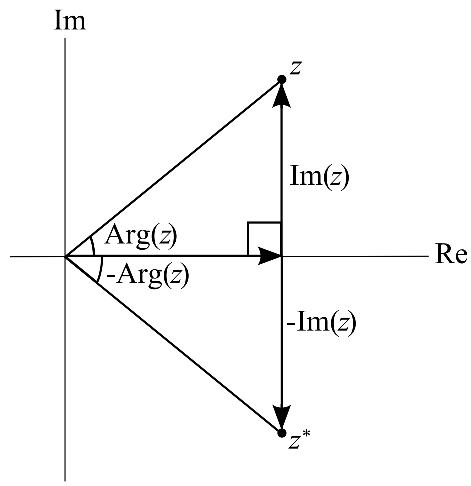


Figure 1.2: Geometric Interpretation of Complex Conjugate

In polar coordinates, we know that flipping the sign of the argument will flip the sign of the imaginary part because

$$\mathrm{Im}(z) = r\sin\theta \tag{1.21}$$

and

$$r\sin(-\theta) = -r\sin\theta = -\mathrm{Im}(z) \tag{1.22}$$

Taking the product of two complex numbers, we have

$$\begin{aligned} z_1\, z_2 &= \left(r_1\, e^{i\theta_1}\right)\left(r_2\, e^{i\theta_2}\right) &(1.23)\\ &= r_1\, r_2\, e^{i(\theta_1+\theta_2)} &(1.24)\end{aligned}$$

From this, we see that the geometric interpretation of the complex product is that it *multiplies the magnitudes and adds the arguments.*

Taking the product of a complex number with its complex conjugate, we have

$$\begin{aligned} z\, z^* &= \left(r\, e^{i\theta}\right)\left(r\, e^{-i\theta}\right) &(1.25)\\ &= r^2 &(1.26)\\ &= |z|^2 &(1.27)\end{aligned}$$

giving us a relation for the magnitude of a complex number.

Chapter 2

Differentiation

2.1 Motivation

One might expect that we could define the complex derivative in a way analogous to that of the reals.

$$\frac{df}{dz} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{2.1}$$

But there is an additional complication because $\mathbb{C}$ forms a plane, and therefore we must specify a *direction* to step with the differential element h when taking the derivative, similar to a directional derivative in $\mathbb{R}^2$. In general, the derivative at a single point can take on many values depending on the direction we choose to step. We will consider a subset of the complex functions called *holomorphic* functions that have a single valued derivative at every point in some region. That is, those functions whose derivative at some point is the same in all directions.

2.2 Cauchy-Riemann Equations

Following the expression in equation (2.1), if we pick h to be real, we get

$$\frac{df}{dz} = \lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h} \tag{2.2}$$

$$= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{2.3}$$

$$= \frac{\partial f}{\partial x} \tag{2.4}$$

If we pick h to be imaginary, $h = i\ell$, $\ell \in \mathbb{R}$, we get

$$\begin{aligned} \frac{df}{dz} &= \lim_{\ell \to 0} \frac{f(x + i(y+\ell)) - f(x+iy)}{i\ell} && (2.5) \\ &= \lim_{\ell \to 0} \frac{f(x, y+\ell) - f(x,y)}{i\ell} && (2.6) \\ &= \frac{1}{i}\frac{\partial f}{\partial y} && (2.7) \end{aligned}$$

Requiring that the derivative be the same in all directions, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y} && (2.8) \\ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) && (2.9) \end{aligned}$$

$\Rightarrow$ The *Cauchy-Riemann Equations*:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{2.10}$$

If the partial derivatives of a function are continuous in some region and satisfy the Cauchy-Riemann Equations in that region, then the function is holomorphic in that region.

2.3 Expressions for the Complex Derivative

Using the fact that the derivative is the same in both the x and y directions for holomorphic functions, and also using the Cauchy-Riemann Equations, one can see that all of the following are equivalent expressions for the derivative of a holomorphic function.

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{2.11}$$

2.4 Conjugate Harmonic Functions

If we take the Laplacian of the real part of a holomorphic function, using the Cauchy-Riemann Equations, we get

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} && (2.12) \\ &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} && (2.13) \\ &= 0 && (2.14) \end{aligned}$$

Similarly, for the imaginary part

$$\nabla^2 v = 0 \tag{2.15}$$

A function whose Laplacian is zero is called *harmonic*. Any harmonic function is the real or imaginary part of some holomorphic function. A pair of harmonic functions that are the real and imaginary parts of the same holomorphic function are called *conjugate harmonic functions*.

2.5 Same Rules Apply

TODO

2.6 Examples of Holomorphic Functions

TODO

Chapter 3

Integration

3.1 Cauchy's Integral Theorem

Often one is interested in evaluating integrals along closed contours. *Cauchy's Integral Theorem* shows us that if a function is holomorphic everywhere inside the region bound by some closed contour, then the integral of that function along that contour is *zero*. In the following proof, Γ denotes a closed contour that is the boundary of some region R. Green's theorem is used to get from equation (3.2) to (3.3). In equation (3.3), we recognize that the integrands are zero using the Cauchy-Riemann Equations because we are requiring that the function be holomorphic everywhere in R.

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma (u+iv)(dx+i\,dy) \tag{3.1}$$

$$= \oint_\Gamma (u\,dx - v\,dy) + i\oint_\Gamma (v\,dx + u\,dy) \tag{3.2}$$

$$= -\iint_R \underbrace{\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)}_{0} dx\,dy + i\iint_R \underbrace{\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)}_{0} dx\,dy \tag{3.3}$$

$$= 0 \tag{3.4}$$

∴ *Cauchy's Integral Theorem*:
If $f(z)$ is holomorphic everywhere in the region bounded by Γ, then

$$\boxed{\oint_{\Gamma} f(z)\, dz = 0} \tag{3.5}$$

3.2 The Fundamental Theorem of Calculus

TODO

3.3 The Deformation Theorem

Consider a function $f(z)$ that is holomorphic everywhere in some region except for at a point z_0, where it diverges. Γ, shown in Figure 3.1(a), is some closed contour along which we want to integrate $f(z)$. Now consider a small circular contour centered at z_0, denoted Γ_R and shown in Figure 3.1(b). By connecting these two contours, a new closed contour, Γ', shown in Figure 3.1(c), is made that does not enclose the singular point, z_0. The two opposing segments of contour that connect Γ and Γ_R can be made arbitrarily close together such that their contributions to the integral along Γ' completely cancel.

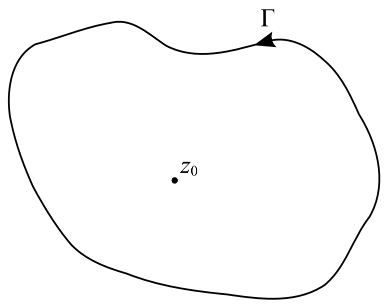
$$\oint_{\Gamma'} f(z)\, dz = \oint_{\Gamma} f(z)\, dz - \oint_{\Gamma_R} f(z)\, dz \tag{3.6}$$

Because Γ' does not include any singularities, its integral is zero by Cauchy's Integral Theorem. Therefore,

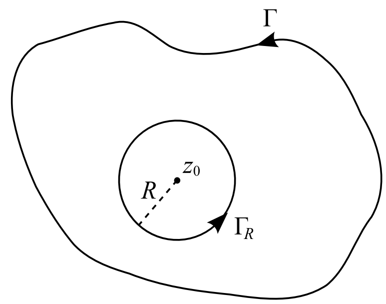
$$\oint_{\Gamma} f(z)\, dz = \oint_{\Gamma_R} f(z)\, dz \tag{3.7}$$

Because the size and shape of Γ_R was arbitrary (though it is often convenient to choose small circles), this demonstrates that a contour can be continuously deformed to any other path enclosing the same singularities without changing the value of its integral. This is the meaning of *Deformation Theorem*.

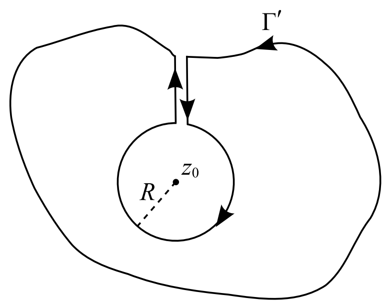
Two contours that can continuously deformed into each other without crossing any singularities are known to be *homotopic* to each other.



(a) Contour of the integral we want to evaluate



(b) Consider a small circular contour



(c) Closed contour with integral zero

Figure 3.1: Demonstrating the Deformation Theorem

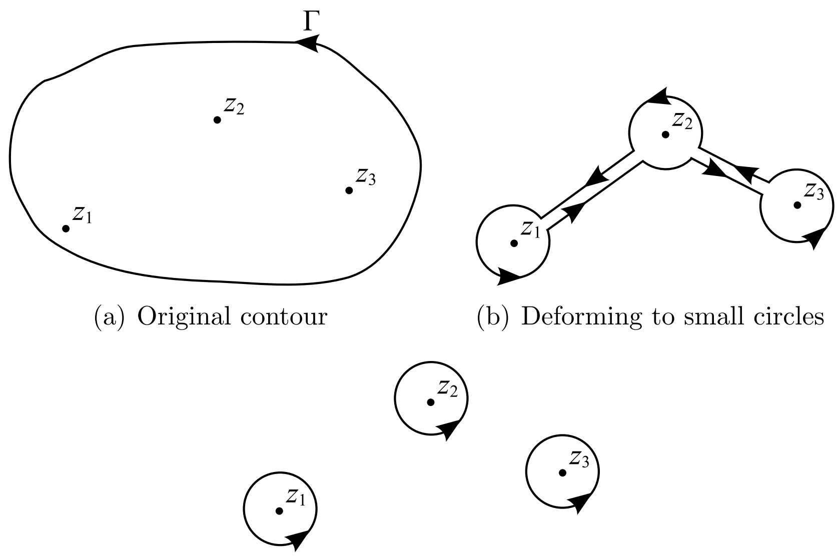
Figure 3.2 shows how a general contour, enclosing some multitude of singularities, can be broken up into several contours, each enclosing one of the singularities.

3.4 Cauchy's Integral Formula

Consider the following integral, where $f(z)$ is holomorophic everywhere in the region integrated.

$$I = \oint \frac{f(z)}{(z - z_0)^n} \, dz \tag{3.8}$$

We deform the contour to an arbitrarily small circle around the singularity, z_0. Therefore, along the entire path of integration, $f(z)$ is arbitrarily close



(a) Original contour

(b) Deforming to small circles

(c) Contours equivalent to the original

Figure 3.2: Deformation Example

to the value of $f(z_0)$ and may be pulled out of the integral.

$$I = f(z_0) \oint \frac{1}{(z - z_0)^n} \, dz \tag{3.9}$$

Then, we make a change of variables: $z - z_0 = r\, e^{i\theta}$, $dz = i\, r\, e^{i\theta}\, d\theta$.

$$\begin{aligned} I &= f(z_0) \int_0^{2\pi} \frac{i\, r\, e^{i\theta}}{r^n\, e^{i\theta}} \, d\theta && (3.10) \\ &= i\, r^{n-1}\, f(z_0) \int_0^{2\pi} e^{i(n-1)\theta} \, d\theta && (3.11) \end{aligned}$$

$$\oint \frac{f(z)}{(z - z_0)^n} \, dz = \begin{cases} 2\pi i\, f(z_0), & n = 1 \\ 0, & n \neq 1 \end{cases} \tag{3.12}$$

Solving for $f(z_0)$ we get *Cauchy's Integral Formula*:

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} \, dz} \tag{3.13}$$

3.5 Holomorphic = Analytic

Taking the n-th derivative of Cauchy's Integral Formula formula, we have *Cauchy's Differentiation Formula*:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\, dz} \tag{3.14}$$

Because the integral on the right is well defined for any n, we conclude that *holomorphic functions are infinitely differentiable.*

Consider a function $f(z)$, for which we want to derive an expansion that is good for values of z near some point z_0. Imagine a circular contour, centered at z_0, and small enough such that $f(z)$ is holomorphic everywhere inside that contour, including at the point z_0. Let z be some point inside the contour, and z', being the integration variable, is some point on the contour.

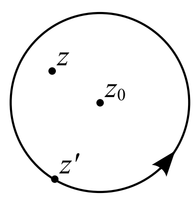


Figure 3.3: Region for Taylor Expansion

We can write $f(z)$ using Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z'-z}\, dz' \tag{3.15}$$

We need to expand the denominator of the integrand in terms of $(z-z_0)$.

$$\frac{1}{z'-z} = \frac{1}{z'-z_0+z_0-z} \tag{3.16}$$

$$= \frac{1}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)} \tag{3.17}$$

Noting that

$$|z-z_0| < |z'-z_0| \tag{3.18}$$

we recognize a geometric series

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n \tag{3.19}$$

Therefore

$$\frac{1}{z'-z} = \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n \tag{3.20}$$

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz' \tag{3.21}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint \frac{f(z')}{(z'-z_0)^{n+1}}\, dz' \tag{3.22}$$

Now we recognize that the integral can be written in terms of derivatives of $f(z_0)$ using Cauchy's Differentiation Formula.

$$f(z) = \frac{1}{\cancel{2\pi i}}\sum_{n=0}^{\infty}(z-z_0)^n \frac{\cancel{2\pi i}\, f^{(n)}(z_0)}{n!} \tag{3.23}$$

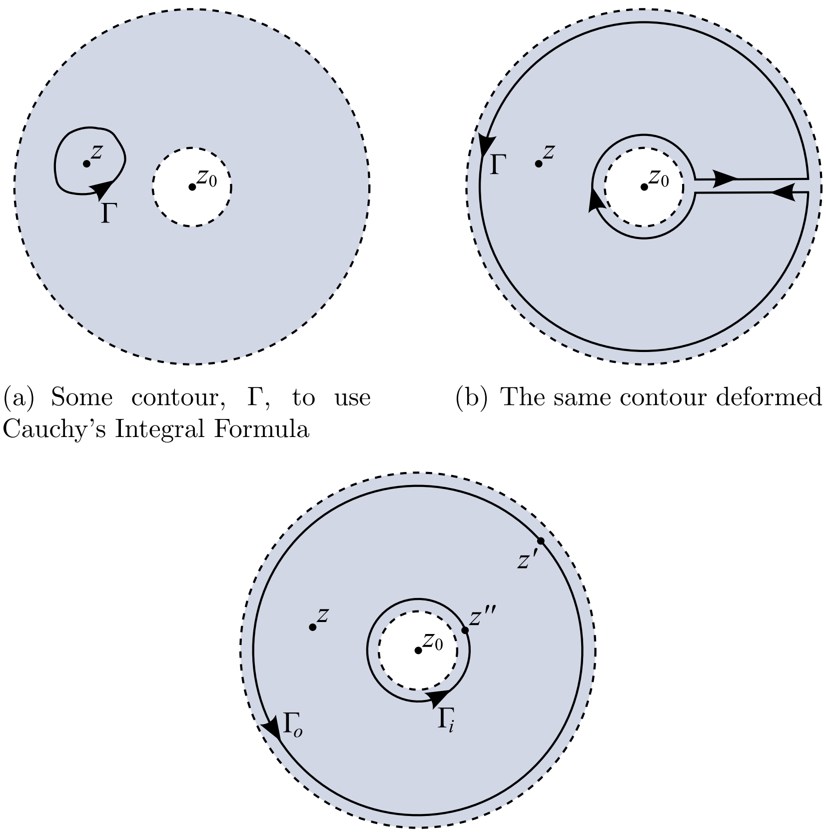
$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \tag{3.24}$$

which is the formula for the Taylor series of $f(z)$. Therefore, every holomorphic function has a convergent Taylor series. That is, *every holomorphic function is analytic.*

3.6 Laurent Series

We will see when we get to the calculus of residues that it is often beneficial to expand a function around a singular point. This can not be done with the Taylor series as we have derived it because the Cauchy Integral Formula we used requires that the function be analytic inside the entire region bounded by the contour one would make to use the formula. This leads us to the development of the Laurent Series as follows.

Consider a function that is analytic in some annulus centered at z_0, bound by the dashed circles shown in Figure 3.4(a). The inner boundary could



(a) Some contour, Γ, to use Cauchy's Integral Formula

(b) The same contour deformed

(c) The integral over Γ can be written in terms of integrals over Γ_i and Γ_o

Figure 3.4: Deriving Laurent Series

enclose a single isolated singularity at z_0 or a collection of singularities. We draw some closed contour Γ within the annulus, around a point z, with hopes of finding a way of using it for Cauchy's Integral Formula. The contour can be deformed within the annulus such that integration along it is equivalent to integration along the inner and outer circular contours, Γ_i and Γ_o, shown in Figure 3.4(c).

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z}\, d\xi && (3.25)\\
&= \frac{1}{2\pi i} \oint_{\Gamma_o} \frac{f(z')}{z' - z}\, dz' - \frac{1}{2\pi i} \oint_{\Gamma_i} \frac{f(z'')}{z'' - z}\, dz'' && (3.26)\\
&\equiv \frac{1}{2\pi i}\,\mathcal{O} - \frac{1}{2\pi i}\,\mathcal{I} && (3.27)
\end{aligned}
$$

Following the same reasoning used in section 3.5, the outer integral, $\mathcal{O}$, is given by

$$\mathcal{O} = \sum_{n=0}^{\infty} (z - z_0)^n \oint_{\Gamma_o} \frac{f(z')}{(z' - z_0)^{n+1}}\, dz' \tag{3.28}$$

But for the inner integral, $\mathcal{I}$, the inequality analogous to the one we used to make the geometric series for $\mathcal{O}$ is flipped the other way (compare to equation (3.18)):

$$|z - z_0| > |z'' - z_0| \tag{3.29}$$

Therefore, instead of factoring out $(z'' - z_0)$, like we did in equation (3.17), we will factor out the point $(z - z_0)$, leading us to a suitable geometric series.

$$
\begin{aligned}
\frac{1}{z'' - z} &= \frac{1}{z'' - z_0 + z_0 - z} && (3.30)\\
&= \frac{1}{(z - z_0)\left(\frac{z'' - z_0}{z - z_0} - 1\right)} && (3.31)\\
&= \frac{-1}{z - z_0}\,\frac{1}{1 - \frac{z'' - z_0}{z - z_0}} && (3.32)\\
&= \frac{-1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z'' - z_0}{z - z_0}\right)^n && (3.33)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I} &= -\oint\limits_{\Gamma_i} \frac{f(z'')}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{z''-z_0}{z-z_0}\right)^n dz'' && (3.34)\\
&= -\sum_{n=0}^{\infty} \frac{1}{(z-z_0)^{n+1}} \oint\limits_{\Gamma_i} f(z'')\,(z''-z_0)^n \; dz'' && (3.35)\\
&= -\sum_{n=0}^{\infty} (z-z_0)^{-n-1} \oint\limits_{\Gamma_i} \frac{f(z'')}{(z''-z_0)^{-n}}\, dz'' && (3.36)\\
& \quad n \rightarrow -n-1 && (3.37)\\
&= -\sum_{n=-1}^{-\infty} (z-z_0)^n \oint\limits_{\Gamma_i} \frac{f(z'')}{(z''-z_0)^{n+1}}\, dz'' && (3.38)
\end{aligned}
$$

Putting these results together, we have

$$
\begin{aligned}
f(z) = \quad & \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint\limits_{\Gamma_o} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz' \\
+ \quad & \frac{1}{2\pi i} \sum_{n=-1}^{-\infty} (z-z_0)^n \oint\limits_{\Gamma_i} \frac{f(z'')}{(z''-z_0)^{n+1}}\, dz'' \qquad (3.39)
\end{aligned}
$$

Lastly, we notice that the contours Γ_o and Γ_i are homotopic, and therefore we may write the coefficients of these series in a single compact form:

$$
a_n = \frac{1}{2\pi i} \oint \frac{f(z')}{(z'-z_0)^{n+1}}\, dz' \qquad (3.40)
$$

$$
\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n \; (z-z_0)^n} \qquad (3.41)
$$

Equation (3.40) is more of a formal expression for the Laurent series coeficients than a practical one. In chapters 4 and 5, we will see examples of expanding a function in a Laurent series and why it is useful.

Note that if $f(z)$ analytic in the region in the center of the annulus, then the inner boundary of the annulus can be completely collapsed, changing the annulus to a disk. In this case, the Laurent series of $f(z)$ does not contain any negative powers because the integrand in equation (3.40) is analytic for all negative n, giving $a_n = 0$ by Cauchy's Integral Theorem. Therefore we recover the Taylor series as a special case of the Laurent series.

3.7 Classifying Singularities

Given a function, $f(z)$ has the following Laurent series around z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{3.42}$$

$f(z)$ has the following at z_0

- **simple pole**: if $a_{-1} \neq 0$ and $a_n = 0$ for all $n < -1$. This is also known as a pole of order 1.
- **pole of order** m: if there exists $m < 0$ such that $a_n = 0$ for all $n < m$ and $a_m \neq 0$.
- **essential singularity**: if there does *not* exist m such that $a_n = 0$ for all $n < m$.
- **branch point**: TODO
- **removable singularity**: TODO

3.8 The Residue Theorem

Consider the general problem of evaluating the integral of some function, $f(z)$ along a closed contour, Γ. We have shown by Cauchy's Integral Theorem that if $f(z)$ is analytic everywhere in the region bound by Γ, the integral is zero. We are now equipped with the tools required to answer this problem in the case where $f(z)$ is singular at one or more isolated points in the region bound by Γ. To summarize what we needed to get here: after proving Cauchy's Integral Theorem, we could prove the Deformation Theorem, which allowed us to prove Cauchy's Integral Formula, which we used to demonstrate that a function could be written as a Laurent series.

Let the singularities of $f(z)$ inside Γ be denoted $\{z_0, z_1, z_2 \ldots z_N\}$. The Deformation Theorem tells us that the contour can be deformed to small circles, $\{\Gamma_0, \Gamma_1, \Gamma_2 \ldots \Gamma_N\}$, surrounding each of the singularities like in Figure 3.2.

$$\oint_{\Gamma} f(z)\, dz = \sum_{k=0}^{N} \oint_{\Gamma_k} f(z)\, dz \tag{3.43}$$

Now consider the integral along one these circular contours, Γ_0. Expanding $f(z)$ in a Laurent series around the singular point z_0, we have

$$\oint_{\Gamma_0} f(z)\, dz = \oint_{\Gamma_0} \sum_{n=-\infty}^{\infty} a_n \, (z - z_0)^n \, dz \tag{3.44}$$

Now we need to restate equation (3.12), an important lemma we arrived at in deriving Cauchy's Integral Formula:

$$\oint (z - z_0)^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases} \tag{3.45}$$

Using this we have

$$\oint_{\Gamma_0} f(z)\, dz = 2\pi i \, a_{-1} \tag{3.46}$$

Seeing that the $n = -1$ coefficient in the Laurent series is especially important, it is customary to give a_{-1} as special name: the "*residue* of $f(z)$ at z_0." It is denoted $\text{Res}(f(z), z_0)$.

Performing the integrals along each Γ_k this way, we have the *Residue Theorem*:

$$\boxed{\oint_{\Gamma} f(z)\, dz = 2\pi i \sum_{k=0}^{N} \text{Res}(f(z), z_k)} \tag{3.47}$$

Therefore, we have shown that the task of evaluating the integral of a function along a closed contour can be reduced to finding the residues of that function at that function's singular points enclosed by the contour. The next section will deal with how one calculates these residues.

3.9 Residue Calculus

3.9.1 Simple Pole

A function, $f(z)$, with a simple pole at z_0 can be written

$$f(z) = \frac{g(z)}{z - z_0} = \frac{1}{z - z_0}\Big(g_0 + g_1(z - z_0) + g_2(z - z_0)^2 + \dots\Big) \tag{3.48}$$

where $g(z)$ is analytic at z_0 and therefore has been expanded in a Taylor series around z_0. This expansion, including the factor of $\frac{1}{z-z_0}$, is the Laurent series expansion of $f(z)$ around z_0. Evidently, g_0 is the residue of $f(z)$ at z_0. A direct way of calculating the residue is given by

$$\boxed{\mathrm{Res}(f(z), z_0) = \Big[f(z) \cdot (z - z_0)\Big]_{z=z_0}} \tag{3.49}$$

3.9.2 Pole of Order n

The more general case is where $f(z)$ has a pole of order n at z_0, and can be written

$$f(z) = \frac{g(z)}{(z - z_0)^n} \tag{3.50}$$

$$= \frac{1}{(z - z_0)^n}\Big(g_0 + g_1(z - z_0) + \ldots + g_{n-1}(z - z_0)^{n-1} + g_n(z - z_0)^n + \ldots\Big) \tag{3.51}$$

where $g(z)$ is analytic at z_0. Evidently, g_{n-1} is the residue of $f(z)$ at z_0. A direct way of calculating this residue is given by

$$\boxed{\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!}\left[\left(\frac{d}{dz}\right)^{n-1} f(z) \cdot (z - z_0)^n\right]_{z=z_0}} \tag{3.52}$$

because

$$= \frac{1}{(n-1)!}\left[\left(\frac{d}{dz}\right)^{n-1}\Big(g_0 + g_1(z - z_0) + \ldots + g_{n-1}(z - z_0)^{n-1} + g_n(z - z_0)^n + \ldots\Big)\right]_{z=z_0} \tag{3.53}$$

$$= \Big[g_{n-1} + g_n(z - z_0) + \ldots\Big]_{z=z_0} \tag{3.54}$$

$$= g_{n-1} \tag{3.55}$$

Chapter 4

Evaluating Real Integrals

The Residue Theorem and Cauchy's Theorem sometimes can be used to calculate *real* integrals that would otherwise be difficult or impossible to calculate using methods from real analysis. The general strategy is to find a way to equate the real integral to one with a closed contour in the complex plane, allowing one to use Residue Theorem or Cauchy's Theorem (in the case that the contour encloses no singularities) to evaluate the integral.

4.1 Example 1

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx \tag{4.1}$$

To evaluate this integral along the real axis we consider a corresponding contour integral

$$\oint_{\Gamma} \frac{1}{z^2+1}\, dz = \int_{-R}^{R} \frac{1}{x^2+1}\, dx + \int_{0}^{\pi} \frac{1}{R^2 e^{i2\theta}+1} iRe^{i\theta}\, d\theta \tag{4.2}$$

By the following argument, we see that the curved part of the path contributes nothing to the integral in the limit $R \to \infty$

$$\begin{aligned} \lim_{R\to\infty} \oint_{\Gamma} \frac{1}{z^2+1}\, dz &= \lim_{R\to\infty} \left(\cancelto{I}{\int_{-R}^{R} \frac{1}{x^2+1}\, dx} + \cancelto{0}{\int_{0}^{\pi} \frac{1}{R^2 e^{i2\theta}+1} iRe^{i\theta}\, d\theta} \right) \\ &= I \end{aligned} \tag{4.3}$$

Now we can use the Residue Theorem to evaluate the integral. Note that the integrand is singular at $\pm i$. Because we have chosen to close the contour in the upper half of the complex plane, the residue at $+i$ is the only one enclosed by the contour. We could have chosen to close the contour in the lower half plane, and then we would use the residue at $-i$. In this case it does not matter which way one closes the contour, but in general, one may have to be careful about chosing which way to close the contour such that the integral along the curved part of the contour goes to zero.

$$I = 2\pi i \text{ Res}\left(\frac{1}{z^2+1}, i\right) \tag{4.4}$$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} \tag{4.5}$$

$$\text{Res}\left(\frac{1}{z^2+1}, i\right) = \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i} \tag{4.6}$$

$$\therefore \quad I = \pi \tag{4.7}$$

4.2 Example 2

$$I = \int_0^\infty \frac{1}{1+x^4} \tag{4.8}$$

$$f(z) \equiv \frac{1}{1+z^4} \tag{4.9}$$

Finding the singularites of $f(z)$:

$$-1 = z^4, \quad z = e^{i\theta} \tag{4.10}$$

$$e^{i(\pi+2\pi N)} = e^{i4\theta} \tag{4.11}$$

$$\theta = \frac{\pi}{4} + \frac{\pi}{2}N \tag{4.12}$$

The closed contour integral is given by the following three contour integrals.

$$\oint f(z)\,dz = \cancelto{I}{\int_{C_1} f(z)\,dz} + \cancelto{0}{\int_{C_2} f(z)\,dz} + \int_{C_3} f(z)\,dz \tag{4.13}$$

We have not yet chosen which angle, θ, should be used to evaluate the integral along C_3.

$$\int_{C_3} f(z)\, dz = \int_\infty^0 dr\; e^{i\theta} \frac{1}{1 + r^4 e^{i4\theta}}, \quad z = r\, e^{i\theta} \tag{4.14}$$

We can conveniently choose $4\theta = 2\pi \Rightarrow \theta = \frac{\pi}{2}$ such that $e^{i4\theta} = 1$.

$$\int_\infty^0 dr\; e^{i\pi/2} \frac{1}{1+r^4} = -i \int_0^\infty dr\; e^{i\pi/2} \frac{1}{1+r^4} = -iI \tag{4.15}$$

$$\therefore \quad 2\pi i \,\mathrm{Res}\left(f(z), e^{i\pi/4}\right) = I - iI = I(1-i) \tag{4.16}$$

$$\begin{aligned}
\mathrm{Res}\left(f(z), e^{i\pi/4}\right) &= \left.\frac{z - e^{i\pi/4}}{1+z^4}\right|_{e^{i\pi/4}} && (4.17)\\
&= \left.\frac{1}{4z^3}\right|_{e^{i\pi/4}} && (4.18)\\
&= \frac{1}{4e^{i3\pi/4}} && (4.19)\\
&= \frac{1}{4\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)} && (4.20)\\
&= \frac{\sqrt{2}}{4\,(i-1)} && (4.21)
\end{aligned}$$

$$\begin{aligned}
I(1-i) &= 2\pi i\, \mathrm{Res}\left(f(z), e^{i\pi/4}\right) && (4.22)\\
&= 2\pi i\, \frac{\sqrt{2}}{4\,(i-1)} && (4.23)
\end{aligned}$$

$$\begin{aligned}
I &= \frac{\pi i}{\sqrt{2}}\, \frac{1}{(i-1)\,(i-1)} && (4.24)\\
&= \frac{\pi i}{\sqrt{2}}\, \frac{1}{i - \not{1} + \not{1} + i} && (4.25)\\
&= \frac{\pi}{2\sqrt{2}} && (4.26)
\end{aligned}$$

Chapter 5

Summing Series

5.1 Non-alternating Series

Consider the residues of $\cot(\pi z)$ when z is an integer, n. Take note that $\cot(\pi z + \pi n) = \cot(\pi z)$, and therefore it sufficent to expand $\cot(\pi z)$ near $z = 0$ to have an expansion for $\cot(\pi z)$ when z is near any integer.

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} \tag{5.1}$$

$$= \frac{1 - \frac{1}{2}(\pi z)^2 + O[(\pi z)^4]}{\pi z - \frac{1}{6}(\pi z)^3 + O[(\pi z)^5]} \tag{5.2}$$

$$= \frac{1 - \frac{1}{2}(\pi z)^2 + O[(\pi z)^4]}{\pi z \left(1 - \frac{1}{6}(\pi z)^2 + O[(\pi z)^4]\right)} \tag{5.3}$$

Let $x = \frac{1}{6}(\pi z)^2 + O[(\pi z)^4]$, and use the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots \tag{5.4}$$

then

$$\cot(\pi z) = \left(\frac{1}{\pi z} - \frac{1}{2}\pi z + O[(\pi z)^3]\right)\left(1 + \frac{1}{6}(\pi z)^2 + O[(\pi z)^4]\right) \tag{5.5}$$

$$= \frac{1}{\pi z} - \frac{1}{2}\pi z + \frac{1}{6}\pi z + O[(\pi z)^3] \tag{5.6}$$

$$= \frac{1}{\pi z} - \frac{1}{3}\pi z + O[(\pi z)^3] \tag{5.7}$$

$$\Rightarrow \quad \text{Res}\left(\cot(\pi z), n\right) = \frac{1}{\pi} \tag{5.8}$$

$$\text{Res}\left(\pi \ \cot(\pi z), n\right) = 1 \tag{5.9}$$

If we have a function $g(z)$ that is not singular at $z = n$, then

$$\boxed{\text{Res}\left(g(z)\, \pi \ \cot(\pi z), n\right) = g(n)} \tag{5.10}$$

Thus, a summation of terms $g(n)$ can be related to a contour integral of the function $g(z)\, \pi \ \cot(\pi z)$ using the Residue Theorem. This can be helpful in summing difficult series, as will be shown in the following example.

Consider the Riemann zeta-function, $\zeta(s)$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{5.11}$$

Let's say we want to evaluate $\zeta(2)$.

$$g(n) \equiv \frac{1}{n^2}, \quad \zeta(2) = \sum_{n=1}^{\infty} g(n) \tag{5.12}$$

$$f(z) \equiv g(z)\, \pi \ \cot(\pi z) \tag{5.13}$$

$$\begin{aligned}
\frac{1}{2\pi i} \oint f(z)\, dz &= \sum_{n=-1}^{-\infty} \text{Res}\left(f(z), n\right) + \text{Res}\left(f(z), 0\right) \\
&\quad + \sum_{n=1}^{\infty} \text{Res}\left(f(z), n\right) && (5.14) \\
&= \sum_{n=-1}^{-\infty} g(n) + \text{Res}\left(f(z), 0\right) + \sum_{n=1}^{\infty} g(n) && (5.15) \\
&= \sum_{n=-1}^{-\infty} \frac{1}{n^2} + \text{Res}\left(f(z), 0\right) + \sum_{n=1}^{\infty} \frac{1}{n^2} && (5.16) \\
&= \text{Res}\left(f(z), 0\right) + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} && (5.17) \\
&= \text{Res}\left(f(z), 0\right) + 2\, \zeta(2) && (5.18)
\end{aligned}$$

Note that because $g(z)$ is singular at $z = 0$, the residue of $f(z)$ at $z = 0$ cannot be calculated with equation 5.10. It must be calculated independently.

$$
\begin{aligned}
f(z) &= \frac{\pi}{z^2}\cot(\pi z) && (5.19)\\
&= \frac{i\pi}{z^2}\,\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}, \quad z = R\,e^{i\theta} && (5.20)\\
&\text{as } R \to \infty \\
&\to \frac{i\pi}{R^2\,e^{i2\theta}}\,\frac{e^{i\pi R\cos\theta}\,\cancelto{0}{e^{-\pi R\sin\theta}} + e^{-i\pi R\cos\theta}\,e^{\pi R\sin\theta}}{e^{i\pi R\cos\theta}\,\cancelto{0}{e^{-\pi R\sin\theta}} - e^{-i\pi R\cos\theta}\,e^{\pi R\sin\theta}} && (5.21)\\
&\to \frac{i\pi}{R^2\,e^{i2\theta}}\,\cancelto{-1}{\frac{e^{-i\pi R\cos\theta}\,e^{\pi R\sin\theta}}{-e^{-i\pi R\cos\theta}\,e^{\pi R\sin\theta}}} && (5.22)\\
&\to 0 && (5.23)
\end{aligned}
$$

$$\therefore \quad \oint f(z)\,dz = 0, \text{ as } R \to \infty \tag{5.24}$$

$$\therefore \quad \zeta(2) = \frac{-1}{2}\,\mathrm{Res}\,(f(z), 0) \tag{5.25}$$

Now we need to find the residue at $z = 0$.

$$
\begin{aligned}
f(z) &= \frac{\pi}{z^2}\cot(\pi z) && (5.26)\\
&= \frac{\pi}{z^2}\left(\frac{1}{\pi z} - \frac{1}{3}\pi z + O[(\pi z)^3]\right) && (5.27)\\
&= \frac{1}{z^3} - \frac{\pi^2}{3z} + O[\pi^2 z] && (5.28)
\end{aligned}
$$

$$\Rightarrow \quad \mathrm{Res}\,(f(z), 0) = \frac{-\pi^2}{3} \tag{5.29}$$

$$\therefore \quad \zeta(2) = \frac{-1}{2}\,\frac{-\pi^2}{3} = \frac{\pi^2}{6} \tag{5.30}$$

5.2 Alternating Series

For alternating series, instead of considering the residues of $\cot(\pi z)$, as we did with non-alternating series, we will consider the residues of $\csc(\pi z)$ at $z = n$, an integer.

Near $z = n$, $w \equiv z - n$ is near 0.

$$\begin{aligned}
\csc(\pi z) &= \csc(\pi w + \pi n) && (5.31)\\
&= (-1)^n \csc(\pi w) && (5.32)\\
&= \frac{(-1)^n}{\sin(\pi w)} && (5.33)\\
&= \frac{(-1)^n}{\pi w - \frac{1}{6}(\pi w)^3 + O[(\pi w)^5]} && (5.34)\\
&= \frac{(-1)^n}{\pi w\left(1 - \frac{1}{6}(\pi w)^2 + O[(\pi w)^4]\right)} && (5.35)\\
&\text{Using the geometric series,}\\
&= \frac{(-1)^n}{\pi w}\left(1 + \frac{1}{6}(\pi w)^2 + O[(\pi w)^4]\right) && (5.36)\\
&= (-1)^n\left(\frac{1}{\pi w} + \frac{1}{6}\pi w + O[(\pi w)^3]\right) && (5.37)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \mathrm{Res}\,(\csc(\pi z), n) &= \frac{(-1)^n}{\pi} && (5.38)\\
\mathrm{Res}\,(\pi \csc(\pi z), n) &= (-1)^n && (5.39)
\end{aligned}$$

If we have a function $g(z)$ that is not singular at $z = n$, then

$$\boxed{\mathrm{Res}\,(g(z)\,\pi\ \csc(\pi z), n) = (-1)^n\, g(n)} \qquad (5.40)$$

Let's say we want to evaluate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad (5.41)$$