Lecture 7:

Complex Analysis

In Lecture 1 we *jumped the gun* and introduced $i = \sqrt{-1}$, which can be regarded as a solution to $x^2 + 1 = 0$. In the early history of mathematics the appearance of i was first regarded as an *embarrassment*, and later as a springboard for philosophical debate. Today the appearance of i is commonplace. In addition to being routinely encountered in solving algebraic equations, it is encountered in many branches of applied mathematics, and often permits simplifications.

As a concrete example

$$x^2 - 2x + 2 = 0 \tag{7.1}$$

has the two solutions: $x = 1 + i$; $x = 1 - i$. Typically, complex numbers are viewed as locations in a 2-dimensional space, termed z-space, where $z = x + iy$, x and y real. The two roots of (7.1) are indicated in the figure as mirror images in the x-axis, and in general pairs of numbers in this relation are said to be (complex) conjugates. If $z = x + iy$ then its conjugate, denoted by z^* or $\overline{z}$ is denoted by $x - iy$. As will be indicated in a moment, a real equation, i.e., one in which i does not explicitly appear such as (7.1), with a complex root must also have the conjugate as a root.

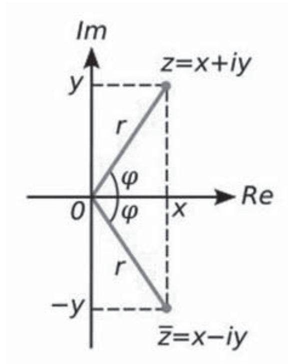

Fig. 7.1: *The complex z-plane, with z and $\overline{z}$ indicated in the cartesian, and polar coordinates*

The addition of the complex numbers,

$$a = a_1 + ia_2 \tag{7.2}$$

and

$$b = b_1 + ib_2 \tag{7.3}$$

is the complex number

$$a + b = (a_1 + b_1) + i(a_2 + b_2). \tag{7.4}$$

and therefore follows vector addition in 2-space. Hence the distance of a point $z = x + iy$ from the origin is still given by the Pythagorean theorem,

$$r^2 = x^2 + y^2 \tag{7.5}$$

Multiplication of complex numbers is given by

$$a \cdot b = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1), \tag{7.6}$$

and therefore (7.5) can be written as

$$r^2 = zz^*. \tag{7.7}$$

Division is given by

$$\frac{a}{b} = \frac{a \cdot b^*}{b \cdot b^*} = \frac{a_1 b_1 + a_2 b_2}{b_1^2 + b_2^2} + i\frac{a_2 b_1 - a_1 b_2}{b_1^2 + b_2^2}. \tag{7.8}$$

(The reals are a subset of the complex numbers, viz., the x-axis of the z-plane.)

These operations, as well as others, are subsumed in Matlab, where e.g. $z = x + iy$ defines the complex number z.[1] It is not necessary to have an $*$ between i and y although you can do so. If a and b are real you write $A =$**comp**(a, b) to get $A = a + ib$. Inversely for any quantity $A = a + ib$

$$\begin{aligned} \mathbf{imag}(A) &= b \\ \mathbf{real}(A) &= a. \end{aligned} \tag{7.9}$$

In polar coordinates

$$z = r(\cos\theta + i\sin\theta) \tag{7.10}$$

where $r\cos\theta = x$ and $r\sin\theta = y$, i.e., r defined by (7.5) is sometimes called the modulus, and

$$\theta = \tan^{-1} y/x. \tag{7.11}$$

Recall that

$$\theta = \mathbf{atan2}(y, x) \tag{7.12}$$

gives $-\pi < \theta \leq \pi$. The command **angle**(z) gives θ directly and can be applied to arrays. It should be clear that **angle**(z) = **atan2**(**imag**(z), **real**(z)). r, sometimes called the modules of z in Matlab is $r =$ **norm**(z) or $r =$ **abs**(z).

Exercise 7.1. Create an m-File to return $0 \leq \theta < 2\pi$.

It is clear from (1.52) that we can write

$$z = re^{i\theta}, \tag{7.13}$$

and hence the product of two complex numbers is

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{7.14}$$

so that angles add. It also follows from (7.13) that the logarithm of z is

[1] In deference to electrical engineers, Matlab also allows j to represent $i = \sqrt{-1}$.

$$\log z = \log r + i\theta. \tag{7.15}$$

But from the multivaluedness of $\theta = tan^{-1}y/x$ we can also write

$$\log z = \log r + i(\theta + 2n\pi) \tag{7.16}$$

for any positive or negative integer n.

Matlab contains a convenient framework for visualizing complex functions. You can create a grid in the complex plane with **cplxgrid** and plot a function using **cplxmap**[2]. In Figure 7.2 we plot (7.16) with height $=$ **real**$(\log z)$ and gray level $=$ **imag**$(\log z)$.

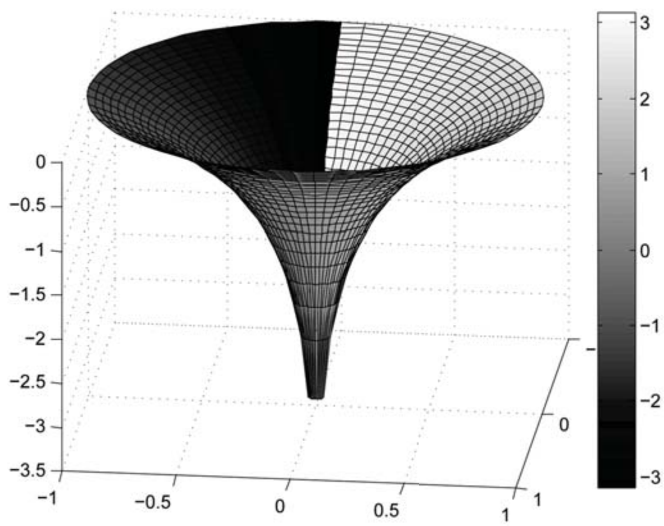

Fig. 7.2: *Visualization of* $\log z$*, (7.15). Height indicates the real part, shading the imaginary parts.*

Note that $.1 < r \leq 1$. The sharp change in shading indicates a phase jump of 2π across the negative x-axis.

A second example is

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2} \tag{7.17}$$

for which the analogous plot is given in Figure 7.3. Along the negative real axis the real $\sqrt{z} = 0$, and also there is a jump of π in phase

[2] Use the Matlab **help** command to view the inputs.

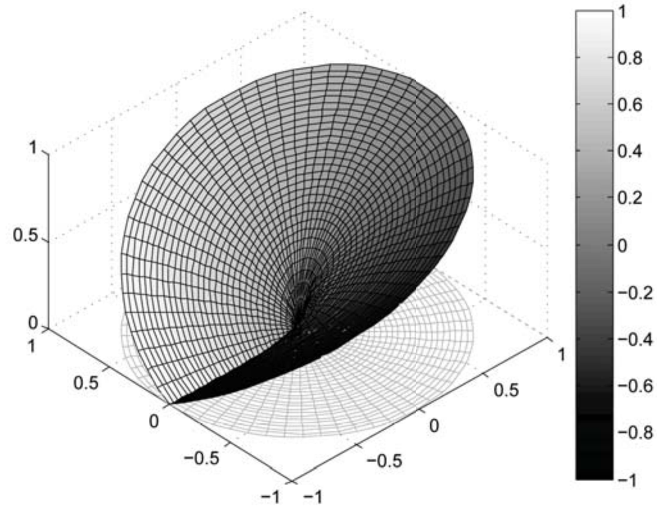

Fig. 7.3: *Visualization of* $\sqrt{z}$, *(7.17). Same as 7.2. Note that* $\mathbf{imag}(\sqrt{2}) = r^{1/2}\sin\theta/2$, *which explains the variation in the color bar.*

Exercise 7.2. (a) Carry out the steps in order to obtain Figures 7.2 & 7.3 and fill in the details of the discontinuity in shading.
(b) For the two cases (7.15) & (7.17) provide separate plots of the real and imaginary parts of functions of the plane. In each case you should get multi-valued surfaces in the phase.

Suppose for complex a and b defined by (7.3) and (7.4) we consider a^b, i.e., a complex number raised to a complex number. To deal with this recall that

$$a^b = e^{b\log a}, \tag{7.18}$$

and therefore for

$$a = |a|e^{i\theta_a}$$
$$a^b = e^{(b_1+ib_2)(ln|a|+i\theta_a)} \tag{7.19}$$

which is clearly computable. For example for $i = e^{i\pi/2}$ and then

$$-2\ln i^i = -2\ln e^{i(i\pi/2)} = \pi, \tag{7.20}$$

is one evaluation and is a purely real quantity! Obviously an eccentric way to compute π.

The fundamental theorem of algebra is a statement of what most people might regard as obvious, viz., that a polynomial of degree N has N *roots*. In particular consider the algebraic equation

$$z^N = 1. \tag{7.21}$$

Obviously any root of (7.21) has unit magnitude and we can write that any root has the form

$$z = e^{i\theta} \tag{7.22}$$

it then follows that with

$$N\theta = 2\pi n \tag{7.23}$$

(7.22) is satisfied. Therefore the solution to (7.21) is

$$z = e^{i\theta_n} \tag{7.24}$$

with

$$\theta_n = \frac{2\pi i n}{N}, \tag{7.25}$$

$n = 0, 1, \cdots N-1$, after which the roots repeat.

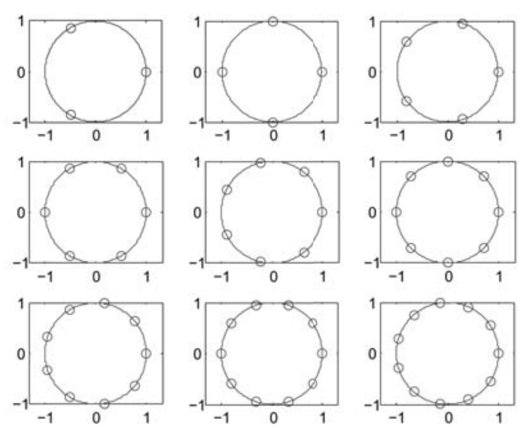

Fig. 7.4: *The roots of unity, (7.21), for* $N = 3, 4, ..., 11$.

As illustrated in Figure 7.2 these roots might be said to uniformly decorate the unit circle with points at the vertices of a regular N-gon. The root

$$z(N) = e^{2\pi i/N} \tag{7.26}$$

is called a primitive N^{th} root of unity since all N roots of (7.21) can be generated from it by

$$z_k = z(N)^k;\; k = 1, \cdots, N. \tag{7.27}$$

As a little study of Figure 7.2 shows there are other possibilities, as well as interesting symmetries. Another illustration is given in Figure 7.5 which shows **real**$(z^3 - 1)$, the monkey saddle mentioned earlier in Lecture 3.

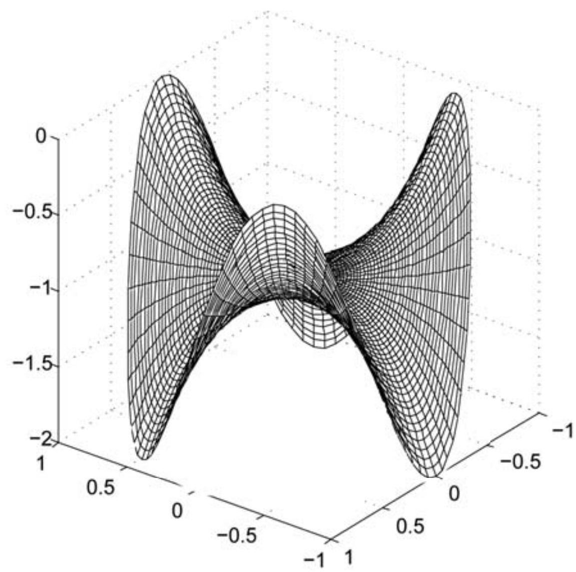

Fig. 7.5: *Real part of $z^3 - 1$. Observe this vanishes at $z = 1, e^{i2\pi/3}, e^{-i2\pi/3}$.*

Taylor Series

The geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{7.28}$$

understandably fails to make sense as $x \uparrow 1$ since the left hand side of (7.28) diverges at $x = 1$. There are problems as $x \downarrow -1$, even though the terms alternate in sign, since they don't get smaller in magnitude, and the series fails to make sense. On the other hand if we consider

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots, \tag{7.29}$$

we see that for $|x| \geq 1$ the series fails to have meaning, since the individual terms increase in magnitude, but there is no apparent reason for the implied convergence failure, since (7.29) tends to 1/2 for $x \to \pm 1$.

However, if we extend our perspective to include the complex plane, it is immediate that the left hand side of (7.29) is *singular* at $x = \pm i$.

When a function tends to ∞ as $O(1/z)$ it is said to possess a pole of order one at $z = 0$, and if it diverges as $O(1/z^n)$, for integer n, it is said to possess a pole of order n at $z = 0$. The function $e^{1/z}$ diverges more rapidly than any pole, as $z \to 0$, and is said to have an essential singularity at the origin.

More generally consider the *power series*,

$$P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{7.30}$$

of which a Taylor series is an example inside the circle. Then the triangle inequality

$$|a + b| < |a| + |b| \tag{7.31}$$

which simply follows from looking at a triangle of sides a, b and $a + b$, applied to (7.30) gives

$$|P(z)| \leq \sum_{n=0}^{\infty} |a_n||z - z_o|^n. \tag{7.32}$$

If this summation converges for $|z - z_o| < R$, i.e., it is a finite number then $P(z)$ clearly converges. If as $n \uparrow \infty$,

$$|a_n||z - z_o|^n < r^n < 1 \tag{7.33}$$

then we have a geometric series for n large enough, which we know converges. It then follows that

$$R = 1/|a_n|^{1/n}. \tag{7.34}$$

Based on the examples above we should believe that R is the radius of the circle around z_0 containing the nearest singularity of $P(z)$.

Example. Consider

$$\frac{1}{2 - z} = \frac{1}{1 + (1 - z)} = \begin{cases} 1 + (z - 1) + (z - 1)^2 \cdots ; \; |z - 1| < 1 \\ \frac{1}{2}\left(1 + \frac{z}{2} + (\frac{z}{2})^2 + ...\right); \; |z| < 2 \end{cases} \tag{7.35}$$

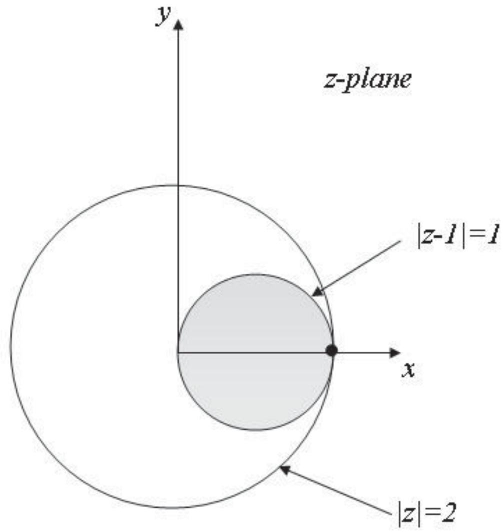

Fig. 7.6: *Regions of convergence for (7.35). The first series converges in the gray circle, $|z-1| < 1$, and the second in $|z| < 2$. In both cases cases convergence is stopped at the singularity at $z = 2$, heavy dot.*

The idea of the Taylor expansion in the complex plane is conceptually very appealing. It says if we know the function on a tiny element, a *micro-dot*, big enough to compute all derivatives then

$$F(z) = \sum_{n=0}^{\infty} F^{(n)}(z_0)\frac{(z-z_0)^n}{n!}; F^n(z) = \frac{d^n F(z)}{dz^n}, \tag{7.36}$$

then the series converges and defines $F(z)$ in the circle $R < |z - z_0|$ which is only limited by the presence of a singularity. Thus knowing e^z in a micro-dot at the origin gives us

$$e^z = 1 + z + \frac{z^2}{2!}\cdots \tag{7.37}$$

which converges for all $z < \infty$. (For any $z, |z|/k! \to 0$ for k large enough.)

The large class of familiar functions such as sine, cosine, etc. discussed in Lectures 1 & 2 immediately carry over to the complex z-plane.

As always functions such as $f(z)$, can be conceptualized as a look up table. Therefore for any $z = z_0$, $f(z_0)$ is a complex number having a real and an imaginary part. If this process is carried out at each possible point of the z-plane

$$f(z) = f(x + iy) = \varphi(x, y) + i\psi(x, y) \tag{7.38}$$

where $\varphi = (f + f^*)/2$ is the real, and $\psi(x, y) = (f - f^*)/2i$, the imaginary part of $f(z)$ at $z = x + iy$. For example if we write

$$e^z = \varphi + i\psi \tag{7.39}$$

then $\varphi = e^x \cos y$ and $\psi = e^x \sin y$.

There is something very special about the complex functions being discussed. To see this consider the transformation

$$\begin{aligned} x &= \frac{1}{2}(z + \overline{z}) \\ y &= \frac{1}{2i}(z - \overline{z}) \end{aligned} \tag{7.40}$$

which in matrix notation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \tag{7.41}$$

so that

$$\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{7.42}$$

which is where we started.

This suggests that any function $F(x, y)$, defined the (x, y)-plane can be written as,

$$F(x, y) = F\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) = \mathcal{F}(z, \overline{z}) \tag{7.43}$$

But the functions we have been considering *do not depend on* $\overline{z}$. Another way to say this is that $\frac{\partial}{\partial \overline{z}} = 0$, for the cases we have considered. Observe (7.41)

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}} \frac{\partial}{\partial y} = \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}, \tag{7.44}$$

and therefore for $F = \varphi(x, y) + i\psi(x, y)$ (7.44) yields,

$$\frac{\partial}{\partial \overline{z}} F = \left(\frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}\right) \Big(\phi(x, y) + i\psi(x, y)\Big) = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\right), \tag{7.45}$$

For $\frac{\partial}{\partial \overline{z}} F = 0$ this implies that the real and imaginary parts are zero

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{7.46}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7.47}$$

known as the Cauchy-Riemann equations. Functions satisfying (7.46) and (7.47) are said to be analytic functions. Analytic function theory is a stunningly beautiful piece of mathematics, but we will touch on it only very lightly.

Exercise 7.3 (a) Another way of defining analytic functions is by demanding that

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{7.48}$$

be independent of how $\Delta z \to 0$. Show that Cauchy-Riemann equations are obtained if: (1) $\Delta z = \Delta x$; and (2) $\Delta z = i\Delta y$.

(b) Show that $\overline{z}$, which is a continuous function of z, has a derivative that depends on Δz.

An immediate consequence of (7.46) is that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \nabla^2 \psi, \tag{7.49}$$

i.e. both the real and imaginary parts of an analytic function satisfy Laplace's equation. Such functions, also called harmonic, play a central role in heat flow, diffusion, fluid mechanics, electrostatics and so forth. Recall that the Laplacian appears in the model of diffusion discussed in Lecture 3.

If we consider the dynamical system

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial \phi}{\partial x} \\ \frac{dy}{dt} &= \frac{\partial \phi}{\partial y} \end{aligned} \tag{7.50}$$

we obtain a flow in two-dimensions, and according to (3.69) this flow is *volume* (area) preserving if $\varphi(x, y)$ is harmonic, recall (3.70).

Exercise 7.4. Show that

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial \psi}{\partial x} \\ \frac{dy}{dt} &= \frac{\partial \psi}{\partial y} \end{aligned} \tag{7.51}$$

generates a set of lines orthogonal to (7.50).

A few of the beautiful results of analytic function theory bear mention, at least in fine print. To begin, if Γ_{ab} is some curve (contour), from z_a to z_b, in the z-plane the integral of $f(z)$ along it is defined in the usual way by a limit process of the form

$$\int_{\Gamma_{ab}} f(z)dz \approx \sum_j f(z_j)\Delta z_j \tag{7.52}$$

and can be numerically calculated as indicated in (7.52). Next from (7.45)

$$\int_R \frac{\partial f}{\partial \bar{z}} dxdy = \frac{1}{2}\int_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dxdy + \frac{i}{2}\int_R \left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) dxdy \tag{7.53}$$

where R is a region enclosed by a simple closed loop C.

If we refer back to the Green's Theorem (3.65) and for the 1st term on the right take $(V_1, V_2) = (\phi, -\psi)$ and for the 2nd term $(V_1, V_2) = (\psi, \phi)$, (7.53) becomes

$$\begin{aligned} \int_R \frac{\partial f}{\partial \bar{z}} dxdy &= \frac{i}{2}\oint (\psi dx + \phi dy) + \frac{1}{2}\oint_C (-\phi dx + \psi dy) \\ &= -\frac{1}{2}\oint_C f dz. \end{aligned} \tag{7.54}$$

Therefore if f is analytic, $\frac{\partial f}{\partial \bar{z}} = 0$, then

$$\oint_C f dz = 0 \tag{7.55}$$

for Γ any closed loop inside which f is analytic. This is known as Cauchy's theorem. At the risk of trivializing Cauchy's theorem observe that

$$\int z^k dz = \frac{z^k}{k+1} \tag{7.56}$$

except for $k = -1$ in which case

$$\int \frac{dz}{z} = \log z. \tag{7.57}$$

Thus for $k \neq 1$

$$\oint z^k dz = 0. \tag{7.58}$$

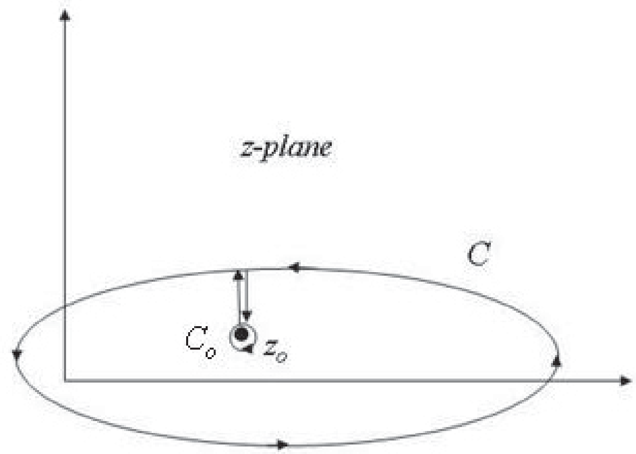

Fig. 7.7: *Construction corresponding to (7.50) demonstrating the equivalence of the contour integrations around C and C_0.*

For a second result again take $f(z)$ analytic in R and consider the ratio $f(z)/(z-z_0)$, where z_0 lies in R. Clearly this function has a pole at z_0. However, if we consider the closed loop of the figure we enclose a region in which the ratio is analytic and if we apply (7.55) to this region we obtain

$$\oint_{C_0} \frac{f(z)}{z-z_0} dz = \oint_C \frac{f(z)}{z-z_0} dz \tag{7.59}$$

since the integrals along the two straight portions just cancel. To evaluate the left hand side of (7.59), which is a circle, $(z-z_0) = Re^{i\theta}$, we can expand in a Taylor series

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots \tag{7.60}$$

and observe under the variable change $(z-z_0) = Re^{i\theta}$

$$\oint_{c_0} \frac{f^{(k)}(z-z_0)^k dz}{(z-z_0)} = i\int_0^{2\pi} f^{(k)} R^k e^{ik\theta} d\theta = \begin{cases} 2\pi i, \; k=0 \\ 0, \quad k \neq 0 \end{cases} \tag{7.61}$$

This yields

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)} dz, \tag{7.62}$$

known as Cauchy's integral formula.

From Cauchy's formula, (7.62), we can make the interesting observation that an analytic function, at any interior point, z_0, is determined by its values on any perimeter closed loop, C.

The real and imaginary parts of $f(z)$, satisfy Laplace's equation, and as mentioned earlier equilibrium heat flow (diffusion, etc) is governed by Laplace's equation. Thus the observation above should make perfectly good sense (physically) to you since the temperature on the boundary fully determines the interior equilibrium temperature.

Julia Sets

As an exercise in exploring the complex plane consider the iterative procedure

$$z_{n+1} = f(z_n) \tag{7.63}$$

where f is a function to be defined and in which z_o will be referred to as the seed. The set of seed points z_o that stay in the finite z-plane is called the Julia set. For example if

$$f = z^2 \tag{7.64}$$

the Julia set is obviously the

$$r = |z| \leq 1, \tag{7.65}$$

although a point on $r = 1$ cycles around the unit circle. A more interesting outcome results if

$$f = z^2 + c \tag{7.66}$$

In fact if $c = -.8 + 0.156i$ the Julia set is indicated by the purple color shown in Figure 7.8. The remaining colors indicate the speed at which seed points march off to infinity, with red indicating the most rapid tendency to become unbounded.

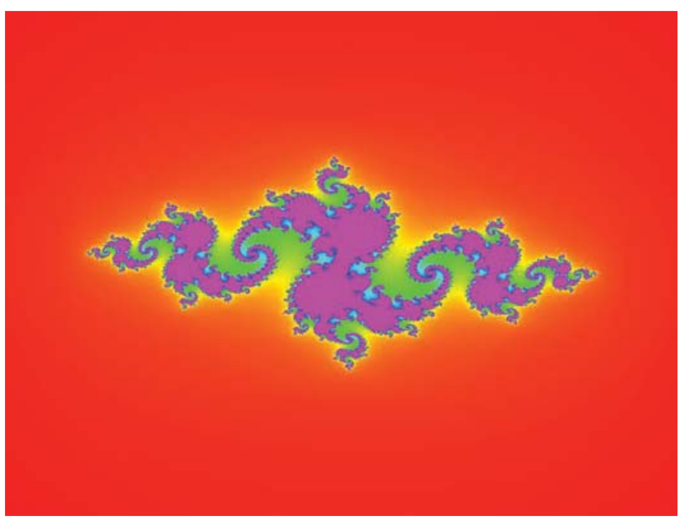

Fig. 7.8: *The Julia set, purple, for $c = -.8 + 0.156i$.*

Exercise 7.5 (Special). Choose a c of your own with which to generate a Julia set within Matlab. Some of the most interesting Julia sets seem to be associated with real $c < 0$ and with a small imaginary part. You can find many examples on the internet. A download of one of these without a Matlab M-file which generates it will not be regarded as acceptable.

There are two main reasons for introducing complex variables. The first, as we will see in a moment, is that our discussion of linear algebra has been incomplete without the topic. The second has to do with Fourier analysis.

Linear Algebra

In our discussion of Linear Algebra we avoided complex eigenvalues and complex eigenvectors. We are now in a position to remedy this. For this purpose consider the matrix.

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.67}$$

which since

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \tag{7.68}$$

is easily seen to be just a (counterclockwise) rotation by $\pi/2$. If we ask about the eigentheory of $\mathbf{A}$

$$\mathbf{A}\mathbf{u} = \mathbf{A}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda\mathbf{u}, \tag{7.69}$$

it is immediate that this is fool's quest, since there is no way for a pure rotation to have a *self-reproducing* vector.

Nevertheless, if we pursue eigentheory the characteristic equation is

$$\lambda^2 + 1 = 0, \tag{7.70}$$

so that the eigenvalues are $\pm i$, a reflection of the fact that we cannot have (real) self reproducing vectors of $\mathbf{A}$. Further, the full analysis yields

$$\lambda_1 = i;\ \mathbf{u}^1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{7.71}$$

and

$$\lambda_2 = -i;\ \mathbf{u}^2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{7.72}$$

with the embarrassing result

$$(\mathbf{u}^1)^\dagger \mathbf{u}^1 = 0 \tag{7.73}$$

which seems to say that $\mathbf{u}^1$ is *perpendicular* to itself!

With the inclusion of complex numbers we are in new territory, and to get a desired feature, viz., that $(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|^2$ give the distance from the origin we have to extend the definition of an inner product. What to do is clear from (7.5), and in general for vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \;\&\; \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \tag{7.74}$$

with possible complex components the inner product is defined as

$$(\mathbf{v}, \mathbf{w}) = \sum_{j=1}^{N} \overline{v}_j w_j = \mathbf{v}^\dagger \mathbf{w} \tag{7.75}$$

where the adjoint now is

$$\mathbf{v}^\dagger = [\overline{v}_1, \overline{v}_2 \cdots, \overline{v}_n]. \tag{7.76}$$

The real vector inner product remains the same, but this clearly takes care of the embarrassment of (7.73) since now

$$\|\mathbf{u}^1\|^2 = (\mathbf{u}^1, \mathbf{u}^1) = 2, \tag{7.77}$$

which is the correct squared distance to the origin. Note that this means

$$(c\mathbf{x}, \mathbf{y}) = \overline{c}(\mathbf{x}, \mathbf{y}), \tag{7.78}$$

and

$$(\mathbf{x}, c\mathbf{y}) = c(\mathbf{x}, \mathbf{y}). \tag{7.79}$$

This also changes the definition of what we mean by an adjoint matrix, and therefore the definition of a symmetric matrix. Recall the discussion of adjoint, (4.19) and (4.20) which in the present context is

$$
\begin{aligned}
(\mathbf{u}, \mathbf{A}\mathbf{v}) &= \sum_{m,n} \overline{u}_n A_{nm} v_m \\
&= \sum_{m,n} \overline{(\overline{A}_{nm} u_n)} v_m \\
&= (\mathbf{A}^\dagger \mathbf{u}, \mathbf{v}).
\end{aligned} \tag{7.80}
$$

Therefore the adjoint, $\mathbf{A}^\dagger$, is the complex conjugate transpose of $\mathbf{A}$. A matrix is symmetric (or also said to be Hermitian), if it is equal to its conjugate transpose. E.g.,

$$
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{7.81}
$$

is Hermitian and observe

$$
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \tag{7.82}
$$

implies the real eigenvalues

$$
\lambda^2 = 1; \lambda = \pm 1. \tag{7.83}
$$

As you can easily verify the corresponding eigenvectors are

$$
\lambda_1 = +1;\ \mathbf{u}^1 = \begin{bmatrix} 1 \\ i \end{bmatrix} / \sqrt{2} : \lambda_2 = -1;\ \mathbf{u}^2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} / \sqrt{2}, \tag{7.84}
$$

from which it is easily seen that

$$
\|\mathbf{u}^1\|^2 = 1 = \|\mathbf{u}^2\|^2 \ \ and \ \ (\mathbf{u}^1, \mathbf{u}^2) = 0 \tag{7.85}
$$

This is representative of any hermitian matrix, $\mathbf{A}$, viz, that the eigenvalues are real and the eigenvectors can be taken to be orthonormal.

Suppose $\mathbf{A}$ is hermitian and

$$
\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{7.86}
$$

then

$$
(\mathbf{v}, \mathbf{A}\mathbf{v}) = (\mathbf{A}\mathbf{v}, \mathbf{v}) = (\lambda \mathbf{v}, \mathbf{v}) = (\mathbf{v}, \lambda \mathbf{v}) \tag{7.87}
$$

therefore

$$
\lambda^*(\mathbf{v}, \mathbf{v}) = \lambda(\mathbf{v}, \mathbf{v}) \tag{7.88}
$$

so λ is real.

Next if $\lambda \neq \mu$ and

$$\mathbf{A}\mathbf{w} = \mu\mathbf{w} \tag{7.89}$$

then

$$(\mathbf{v}, \mathbf{A}\mathbf{w}) = (\mathbf{A}\mathbf{v}, \mathbf{w}) = \lambda(\mathbf{v}, \mathbf{w}) = \mu(\mathbf{v}, \mathbf{w}) \tag{7.90}$$

and hence

$$(\lambda - \mu)(\mathbf{v}, \mathbf{w}) = 0, \tag{7.91}$$

which by hypothesis demonstrates that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. As usual the eigenvectors can be taken to have unit length, $\|\mathbf{v}\| = 1$.

Fourier Analysis

Suppose $f(z) = 2\sum_{n=0}^{\infty} a_n z^n$ is a power series that defines $f(z)$ and for which the series converges in the disk $|z| < R > 1$. In particular on the circle, $r = 1$, we can write, with $z = re^{i\theta}$,

$$f(z)\big|_{r=1} = f(e^{i\theta}) = 2\sum_{n=0}^{\infty} a_n e^{in\theta} = \phi(\theta) + i\psi(\theta), \tag{7.92}$$

where the last form gives f in terms of its real and imaginary parts. Since the real part φ, is $\frac{1}{2}(f + \overline{f})$ we have

$$\phi(\theta) = Re\ f(|z| = 1) = \sum_{n=0}^{\infty} a_n e^{in\theta} + \sum_{n=0}^{\infty} \overline{a}_n e^{-in\theta} = \sum_{-\infty}^{\infty} a_n e^{in\theta};\ a_{-n} = \overline{a}_n, \tag{7.93}$$

and a_0 is twice its value in (7.92). Equation (7.84) says that a function defined on the unit circle $\varphi(\theta)$, which is therefore 2π-period, can be expressed as a sum over the infinite set of 2π-periodic functions $\{e^{in\theta}\}, n = 0, \pm1, \pm2\cdots$. There is a certain reasonableness to this statement, and we have demonstrated this, at least in the framework of analytic functions. However, it is also reasonable to expect that any 2π-periodic function has a *Fourier expansion*, (7.93). So for an arbitrary, but known, 2π-periodic function $g(\theta)$ we expect that exists a Fourier series representation

$$g(\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta}. \tag{7.94}$$

In order to determine a_n, first note that

$$\int_0^{2\pi} e^{im\theta} d\theta = \begin{cases} 0;\ m \neq 0 \\ 2\pi;\ m = 0 \end{cases}. \tag{7.95}$$

No analysis is necessary to see this, just the observation that any, non-constant, sinusoid over its period has equal positive and negative area. Next, based on our earlier discussion of the complex inner product, the inner product of two complex functions, $v(\theta)$ and $u(\theta)$ say on $(0, 2\pi)$, is given by

$$(u, v) = \int_0^{2\pi} \overline{u}(\theta) v(\theta) d\theta. \tag{7.96}$$

It therefore follows from (7.94) and (7.96) that

$$(e^{im\theta}, g(\theta)) = \int_0^{2\pi} e^{-im\theta} g(\theta) d\theta = 2\pi a_m. \tag{7.97}$$

At this point we can imagine $g(\theta)$ to be any function defined on $(0, 2\pi)$, and by expanding it in a Fourier series we extend its definition to be a 2π-periodic function as a result of (7.94). For a continuous function the summation in (7.94) converges pointwise to $g(\theta)$, and if it is discontinuous say at θ_0 then the summation, (7.94), converges to the average value, $\frac{1}{2}\{g(\theta_0^+) + g(\theta_0^-)\}$. [3] An alternate form is obtained if we set $\theta = 2\pi s$ then the interval $(0, 2\pi)$ goes to $(0, 1)$, and

$$f(s) = \sum_n a_n e^{2\pi i n s};\ a_n = \int_0^1 e^{-2\pi i n t} f(s) ds. \tag{7.98}$$

In either case the index n informs us of the number of cycles in the period for the corresponding sinusoid.

[3] This is more or less obvious from the continuous case if we regard the discontinuous case as a limit of the continuous approximations.