Complex Analysis

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Contents

0 Introduction

These notes come from a semester course on complex analysis taught by Dr. Richard Carmichael at Wake Forest University during the fall of 2010. The main topics covered include

- Complex numbers and their properties
- Complex-valued functions
- Line integrals
- Derivatives and power series
- Cauchy's Integral Formula
- Singularities and the Residue Theorem

The primary reference for the course and throughout these notes is Fisher's *Complex Variables*, 2^{nd} edition.

1 The Complex Plane

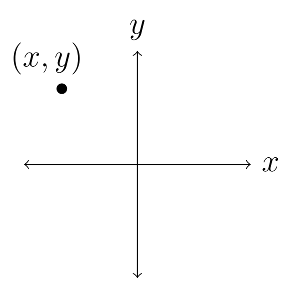
1.1 A Formal View of Complex Numbers

We begin with a description of the complex number system. In the 16th century, mathematicians sought solutions to polynomial equations such as $x^3 + x + 1$, but struggled to find a 'complete' way of describing the solutions. Recall for instance that the roots of a quadratic polynomial $ax^2 + bx + c$ is given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Of course if $b^2 - 4ac < 0$ this has no real solutions. This led Gerolamo Cardano to create the imaginary value $i = \sqrt{-1}$ to compensate for a perceived lack of completeness of solutions.

Formally, complex numbers are numbers of the form $z = x + iy$ where x and y are real numbers. These numbers lie on what is known as the *complex plane*, denoted $\mathbb{C}$.



In this way we can view the **real part** x and the **imaginary part** y of $x + iy$ separately. The set of all complex numbers is denoted $\mathbb{C}$, and they form an algebraic field under the operations

- Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.
- Scaling: $k(x, y) = (kx, ky)$ where k is a real scalar.
- Multiplication: $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. Note that this multiplication differs from the usual multiplication on $\mathbb{R}$, as in Euclidean geometry.

In this class we will freely use both notations for a complex number, that is $x + iy = (x, y)$. For example,

$$\begin{aligned} x &= (x, 0) \\ i &= (0, 1) \\ i^2 &= (0, 1)(0, 1) = (-1, 0). \end{aligned}$$

For $z = x + iy$ we will also denote the real and imaginary parts by $x = \text{Re}(z)$ and $y = \text{Im}(z)$. As a vector space, $\mathbb{C}$ has the following special attributes for each vector (complex number).

Definition. *For a complex number $z = x + iy$, the* **modulus** *or absolute value of z is $|z| = \sqrt{x^2 + y^2}$ and the* **complex conjugate** *of z is $\bar{z} = x - iy$.*

Note that $|z|$ and $|\bar{z}|$ are always equal. Geometrically, the modulus represents the distance in the complex plane from the origin $(0,0)$ to (x,y).

Proposition 1.1.1. *For $z, w \in \mathbb{C}$,*

(i) $|zw| = |z|\,|w|$.

(ii) $\overline{zw} = \bar{z}\bar{w}$.

Since $\mathbb{C}$ is a field, there is also a notion of divisibility for complex numbers. In particular if $x+iy, u+iv \in \mathbb{C}$ and $u+iv \neq 0$, we define

$$\frac{x+iy}{u+iv} = \frac{xu+yv+i(yu-xv)}{u^2+v^2}.$$

One can check that this is the appropriate formula by multiplying and dividing $\frac{x+iy}{u+iv}$ by the conjugate $u-iv$.

As in the xy-plane, there is a polar coordinate system for complex numbers: if $z = x+iy$ then we set $r = |z|$, $x = r\cos\theta$ and $y = r\sin\theta$ where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. This gives us

$$z = |z|(\cos\theta + i\sin\theta).$$

Multiplication is compatible with polar representations, for if $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\psi + i\sin\psi)$ we have

$$\begin{aligned} zw &= |z|\,|w|(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= |z|\,|w|(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi) \\ &= |z|\,|w|(\cos(\theta+\psi) + i\sin(\theta+\psi)). \end{aligned}$$

Likewise, $\frac{z}{w} = \frac{|z|}{|w|}(\cos(\theta-\psi) + i\sin(\theta-\psi))$.

Taking powers of complex numbers, e.g. z^n, is sometimes difficult to compute, since multiplication isn't quite as straightforward in the complex plane. However, there is a result which utilizes the polar representation of a complex number to simplify the expression.

Theorem 1.1.2 (De Moivre's Theorem)**.** *For all integers n, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.*

Proof. We prove this using induction on n. For the base case $n = 1$, we simply have

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta.$$

Now assume De Moivre's Theorem holds for n. Then we have

$$\begin{aligned} (\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\ &= (\cos(n\theta) + i\sin(n\theta))(\cos\theta + i\sin\theta) \\ &= (\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta) + i(\sin\theta\cos(n\theta) + \cos\theta\sin(n\theta)) \\ &= \cos((n+1)\theta) + i\sin((n+1)\theta). \end{aligned}$$

□

Definition. *When we write* $z = |z|(\cos\theta + i\sin\theta)$*, the angle* θ *is called the* **argument** *of* z*, denoted* $\arg z$*.*

We often want to restrict our attention to a single, canonical value of θ for any z. Thus we define the **principal argument** $\theta = \operatorname{Arg} z$, where $-\pi \leq \theta \leq \pi$.

Proposition 1.1.3. $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$*, where these may differ by a multiple of* 2π*.*

Example 1.1.4. Let $z = -1 + i$ and $w = i$. Then $zw = -1 - i$, $\operatorname{Arg}(zw) = -\frac{3\pi}{4}$ and

$$\operatorname{Arg} z + \operatorname{Arg} w = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4} \equiv -\frac{3\pi}{4} \mod 2\pi.$$

1.2 Properties of Complex Numbers

Continuing with the geometric parallels between Euclidean space and the complex plane, we have the important **triangle inequality** for complex numbers:

$$|z + w| \leq |z| + |w|.$$

There is also a related inequality, sometimes called the reverse triangle inequality:

$$\big||z| - |w|\big| \leq |z - w|.$$

The original purpose of complex numbers was to compute roots of all polynomials, so it will be desirable to be able to compute roots of complex numbers. In other words, if $w = |w|(\cos\psi + i\sin\psi)$, what is $w^{1/n}$? Let $z = w^{1/n}$, so that $z^n = w$. Then using De Moivre's Theorem (1.1.2) we have

$$|w|(\cos\psi + i\sin\psi) = (|z|(\cos\theta + i\sin\theta))^n = |z|^n(\cos(n\theta) + i\sin(n\theta)).$$

Solving for θ, we see that

$$\cos\psi = \cos(n\theta) \implies n\theta = \psi + 2\pi k \implies \theta = \frac{\psi + 2\pi k}{n}$$

for some integer k. Hence our expression for $w^{1/n}$ is

$$z = w^{1/n} = |w|^{1/n}\left(\cos\left(\frac{\psi + 2\pi k}{n}\right) + i\sin\left(\frac{\psi + 2\pi k}{n}\right)\right).$$

For the nth root of w, that is $w^{1/n}$, this formula gives all possible roots. In fact there are n distinct roots; all others are repeated values.

Recall that the equation of a circle in $\mathbb{R}^2$ is $\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$ for $r > 0$. In the complex plane, this is expressed by $|z - z_0| = r$.

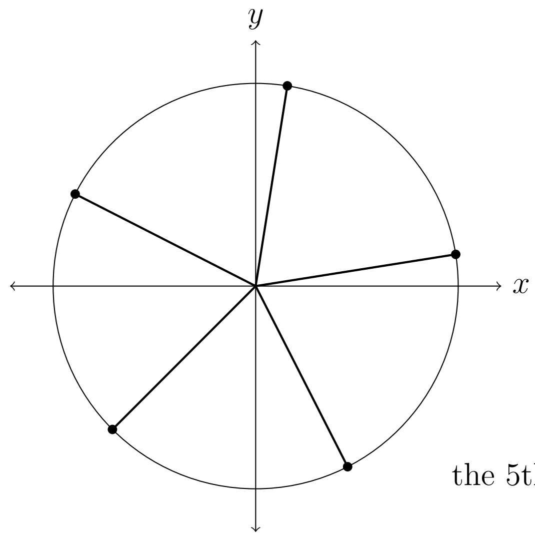
Example 1.2.1. Let's find the 5th roots of $z = 1 + i$. The polar representation of $1 + i$ is

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

The modulus of all the 5th roots of unity is $2^{1/10} \approx 1.07171$. Our work above gives all of these roots as

$$(1+i)^{1/5} = 2^{1/10}\left(\cos\left(\frac{\pi}{20} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{20} + \frac{2\pi k}{5}\right)\right).$$

These are shown on the circle of radius $2^{1/10}$ below.



the 5th roots of $1 + i$

Example 1.2.2. Consider the equation $z^4 - 4z^2 + 4 - 2i = 0$. This may be rewritten as $(z^2 - 2)^2 = 2i = (1+i)^2$ which has solutions

$$z^2 - 2 = \pm(1+i) \implies z^2 = \begin{cases} 3 + i \\ 1 - i. \end{cases}$$

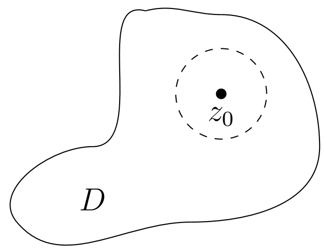
Using the expression for roots above, this yields the following solutions to the original equation:

$$z = \pm\sqrt[4]{10}\left(\cos\left(\frac{1}{2}\arctan\frac{1}{3}\right) + i\sin\left(\frac{1}{2}\arctan\frac{1}{3}\right)\right)$$
$$\text{and } z = \pm\sqrt[4]{2}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right).$$

1.3 Subsets of the Complex Plane

In Chapter 2 we will define functions on the complex plane, i.e. functions whose domain and range are subsets of the complex plane. The following topological terms will be useful.

Definition. *A subset $D \subseteq \mathbb{C}$ is* **open** *if all its points are interior points, that is, any circle drawn around a point (called a neighborhood of the point) lies entirely within D.*



Circles are actually a specific case of a more general notion of 'neighborhood' or open set in topology. Since the open disks (sometimes called open balls) $B(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ form a basis for $\mathbb{C}$ (see any introductory topology text, e.g. Adams and Franzosa or Munkres) it suffices to consider open sets as those 'composed' of smaller open balls.

Example 1.3.1. The half plane $H = \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$ is an open set. Likewise, for any $a \in \mathbb{R}$, $\{z \in \mathbb{C} \mid \text{Re}(z) > a\}$ and $\{z \in \mathbb{C} \mid \text{Re}(z) < a\}$ are open sets, and the same is true for $\text{Im}(z)$.

Definition. *A point z_0 in a set D is called a* **boundary point** *if every neighborhood of z_0 contains both interior and exterior points. D is said to be* **closed** *if it contains its boundary, or the set of all boundary points of D.*

Definition. *An open set D is* **connected** *if all points in D may be joined by a series of contiguous, direct line segments, each of which is completely contained within D. Furthermore, D is* **convex** *if it is connected and* any *single line segment joining two points in D also lies in D.*

2 Complex-Valued Functions

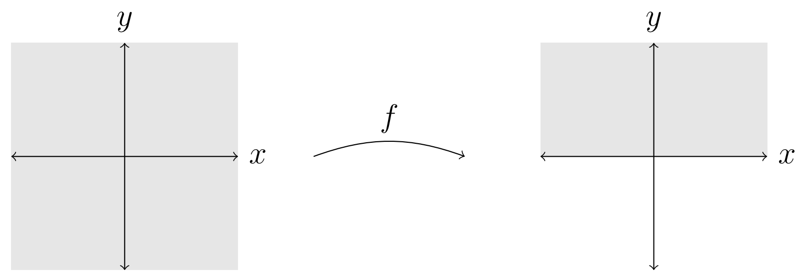
2.1 Functions and Limits

In this section we introduce functions that have values in the complex plane.

Definition. *A* **function** *of a complex variable* z *is a map* $f : D \to \mathbb{C}$ *for some subset* $D \subseteq \mathbb{C}$*, i.e.* f *assigns a complex number to each* $z \in D$*.*

Definition. *The* **domain** *of a complex-valued function* f *is the set of all values* z *for which the function operates; this is usually denoted* D*. The* **range** *is all possible values of the function, denoted* $\operatorname{Im} f$ *or* $f(D)$*.*

Example 2.1.1. Let $f(z) = z^2$. The domain of f is all of $\mathbb{C}$, while the range of f is the closed upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$.



Example 2.1.2. $f(z) = \frac{1}{z-1}$ has domain $D = \{z \in \mathbb{C} \mid z \neq 1\}$ and range $f(D) = \{z \in \mathbb{C} \mid z \neq 0\}$.

Definition. *A* **sequence** *is a complex-valued function whose domain is the set of positive integers, written* $(z_n) = (z_1, z_2, z_3, \ldots)$ *where each* z_i *is a complex number.*

Definition. *A sequence* (z_n) *is said to have a* **limit** L *if, given any* $\varepsilon > 0$ *there is some* $N \in \mathbb{N}$ *such that* $|z_n - L| < \varepsilon$ *for all* $n \geq N$*. In this case we write* $\lim_{n\to\infty} z_n = L$ *and say that* (z_n) **converges** *to* L*. If no such* L *exists, then* (z_n) *is said to* **diverge***.*

The definitions of sequence and limit are nearly identical to their counterparts in real analysis. However, in the complex plane every number has a real and an imaginary part. The following proposition helps us relate the definition of a complex limit to its real and imaginary parts.

Proposition 2.1.3. *Let* $z_n = x_n + iy_n$ *and* $z = x + iy$*. Then* $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*.*

Proof. ($\Longrightarrow$) If $\lim_{n\to\infty} z_n = z$ then the inequalities $|x_n - x| \leq |z_n - z|$ and $|y_n - y| \leq |z_n - z|$ directly imply that (x_n) and (y_n) converge to x and y, respectively.

($\Longleftarrow$) On the other hand, suppose $(x_n) \to x$ and $(y_n) \to y$. If $\varepsilon > 0$ is given, we may choose N_1 and N_2 such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N_1$ and $|y_n - y| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ the triangle inequality gives us

$$|z_n - z| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (z_n) converges to $z = x + iy$. □

As a result, we have

Corollary 2.1.4. *If $z_n \to z$ then $|z_n| \to |z|$.*

The converse to this is generally false. For example, the sequence $|i^n|$ converges to 1 since $|i^n| = |i|^n = 1^n = 1$ for all n; however, $i^n = (i, -1, -i, 1, i, -1, \ldots)$ and this fluctuates infinitely often between these four values, so the sequence diverges.

Proposition 2.1.5. *Suppose $\lim_{n\to\infty} z_n = z$. Then*

(i) For any complex scalar $k \neq 0$, $\lim_{n\to\infty} kz_n = kz$.

(ii) If $z_n \neq 0$ for any n and $z \neq 0$, then $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{z}$.

Proof. (i) Let $\varepsilon > 0$ be given. By convergence of (z_n) there exists a positive integer N such that $|z_n - z| < \frac{\varepsilon}{|k|}$. Then for all $n \geq N$,

$$|kz_n - kz| = |k|\,|z_n - z| < |k|\,\frac{\varepsilon}{|k|} = \varepsilon.$$

Hence $(kz_n) \to kz$.

(ii) First we can choose an N_1 such that $|z_n - z| < \frac{|z|}{2}$ for all $n \geq N_1$. Note that by the reverse triangle inequality,

$$|z_n| \geq |z| - |z_n - z| > |z| - \frac{|z|}{2} = \frac{|z|}{2}.$$

We use this to control the $|z_n|$ term in the calculations below. Next for any $\varepsilon > 0$ there is an N_2 such that for all $n \geq N_2$, $|z_n - z| < \frac{|z|^2\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for any $n \geq N$,

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \left|\frac{z - z_n}{z_n z}\right| = \frac{|z_n - z|}{|z_n|\,|z|} \leq \frac{2}{|z|}\,\frac{1}{|z|}\,|z_n - z| < \frac{2}{|z|^2}\,\frac{|z|^2\varepsilon}{2} = \varepsilon.$$

Hence $\left(\frac{1}{z_n}\right) \to \frac{1}{z}$. □

This shows that limits of complex sequences behave as expected (by which we mean they behave as their counterparts do in the real case). We also have

Theorem 2.1.6. *If (z_n) converges to z and (w_n) converges to w, then the sequence $(z_n w_n)$ converges to zw.*

Definition. *Given a function $f(z)$ with domain D and a point z_0 either in D or in the boundary ∂D of D, we say f has a* **limit** *at z_0 if*

$$\lim_{z\to z_0} f(z) = L$$

for some $L \in \mathbb{C}$. Explicitly, $f(z)$ has limit L at z_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z) - L| < \varepsilon$.

Definition. *$f(z)$ is* **continuous** *at a point z_0 in its domain if $\lim_{z\to z_0} f(z)$ exists and it equals $f(z_0)$. In particular, $f(z)$ is continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.*

Example 2.1.7. The function $f(z) = |z|^2$ is continuous on its domain $\mathbb{C}$. For example, $f(z)$ has limit 4 at $z_0 = 2i$. To see this, let $\varepsilon > 0$ and define $\delta_1 = 1$, $\delta_2 = \frac{\varepsilon}{5}$ and $\delta = \min\{\delta_1, \delta_2\}$. Note that by the reverse triangle inequality, $|z| \leq |z - 2i| + |2i| < 1 + 2 = 3$; we will use this below. Then if $0 < |z - 2i| < \delta$ we have

$$\begin{aligned} |f(z) - f(2i)| &= ||z|^2 - 4| \\ &= ||z| + 2| \cdot ||z| - 2| \\ &= (|z| + 2)|z - 2i| \\ &< (3 + 2)\frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

Hence $\lim_{z\to 2i} f(z) = 4$ as claimed.

Example 2.1.8. Consider the function $f(z) = \dfrac{z}{\bar{z}}$ where $z = x + iy \neq 0$ and $\bar{z} = x - iy$, its complex conjugate. Does $\lim_{z\to 0} f(z)$ exist? Well consider this limit along two different paths in the complex plane:

$$\begin{aligned} \lim_{(x,y)\to(0,y)} f(z) &= \frac{0 + iy}{0 - iy} = -1 \\ \lim_{(x,y)\to(x,0)} f(z) &= \frac{x + i0}{x - i0} = 1. \end{aligned}$$

Since these limits are different, the limit of the function must not exist. Hence $\dfrac{z}{\bar{z}}$ is not continuous at $z_0 = 0$.

Definition. *A function $f(z)$ has a* **limit at infinity**, *denoted $\lim_{z\to\infty} f(z) = L$, if for any $\varepsilon > 0$ there is a (large) number M such that $|f(z) - L| < \varepsilon$ whenever $|z| \geq M$. Note that there is no restriction on $\arg z$; only $|z|$ is required to be large.*

Example 2.1.9. The family of functions $f(z) = \frac{1}{z^m}$ has a limit $L = 0$ as $z \to \infty$ for all $m = 1, 2, 3, \ldots$. To see this, let $\varepsilon > 0$ and choose $M = \frac{1}{\varepsilon^{1/m}}$. Then if $|z| \geq M$,

$$\left|\frac{1}{z^m}\right| = \left(\frac{1}{|z|}\right)^m \geq \left(\frac{1}{M}\right)^m = (\varepsilon^{1/m})^m = \varepsilon.$$

By properties of limits, we have

Proposition 2.1.10.

1) *Every polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is continuous on the complex plane.*

2) *If $p(z)$ and $q(z)$ are polynomials, then their quotient $\frac{p(z)}{q(z)}$ is continuous at all points such that $q(z) \neq 0$.*

Every complex-valued function $f(z)$ can be written as $f(z) = u(z) + iv(z)$, where u and v are each real-valued functions. This allows us to view every complex function by its real and imaginary parts. It is easy to see that all of the results on continuity for functions of the real numbers now apply for complex-valued functions. In particular,

Proposition 2.1.11. *Let $f = u + iv$ be a complex-valued function. Then f is continuous at z_0 if and only if u and v are both continuous at z_0.*

2.2 Infinite Series

In this section we briefly review infinite series, since they carry over to the complex case nearly identically.

Definition. *For complex numbers $z_1, z_2, \ldots$ their nth* **partial sum** *is $\sum_{j=1}^{n} z_j = z_1 + \ldots + z_n$.*

Definition. *An* **infinite series** *of complex numbers is a limit of partial sums*

$$\sum_{j=1}^{\infty} z_j = \lim_{n\to\infty} \sum_{j=1}^{n} z_j.$$

Definition. *We say an infinite series of partial sums $s_n = \sum_{j=1}^{n} z_j$* **converges** *if $s = \lim_{n\to\infty} s_n$ exists. Otherwise, the series* **diverges***.*

In the complex case, we can write each $z_j = x_j + iy_j$ so every infinite series may be written as the sum of a real and imaginary series:

$$\sum_{j=1}^{\infty} z_j = \sum_{j=1}^{\infty} x_j + i \sum_{j=1}^{\infty} y_j.$$

As with functions, the series $\sum z_j$ converges if and only if $\sum x_j$ and $\sum y_j$ converge. In other words, $\lim_{n\to\infty} s_n$ only converges when $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ both exist.

Definition. *A series $\sum_{j=1}^{\infty} z_j$ has* **absolute convergence** *if $\sum_{j=1}^{\infty} |z_j|$ converges. If $\sum_{j=1}^{\infty} z_j$ converges but the absolute series does not converge, we say the series* **converges conditionally***.*

Notice that if $\sum_{j=1}^{\infty} z_j$ converges (absolutely) then both $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ converge (absolutely) as well. The triangle inequality for series looks like

$$\left| \sum_{j=1}^{\infty} z_j \right| \leq \sum_{j=1}^{\infty} |z_j|.$$

Example 2.2.1. As in the real case, a geometric series $\sum_{j=1}^{\infty} \alpha^j$ converges to $\frac{1}{1-\alpha}$ if $|\alpha| < 1$ and diverges otherwise. The value α is sometimes called the ratio of the series.

Example 2.2.2. Consider the series $\sum_{j=1}^{\infty} j\left(\frac{1+2i}{3}\right)^j$. Absolute convergence is useful in complex analysis since we can reduce complex numbers to purely real-valued expressions. In this case, we see that

$$\sum_{j=1}^{\infty}\left|j\left(\frac{1+2i}{3}\right)^j\right| = \sum_{j=1}^{\infty} j\left|\frac{1+2i}{3}\right|^j = \sum_{j=1}^{\infty} j\left(\frac{\sqrt{5}}{3}\right)^j$$

which converges by the ratio test, for example. Hence the original series converges absolutely.

Example 2.2.3. The series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converges even though the similar-looking harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To see this, notice that we can write

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} + i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

and both parts converge by the alternating series test.

2.3 Exponential and Logarithmic Functions

Recall from single-variable calculus the exponential function e^x. This function has many definitions, with the two most important being

$$e^x = \lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t$$

$$\text{and} \quad e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

In complex analysis, we define

Definition. *For $z = x + iy$, the* **complex exponential function** e^z *is defined by*

$$e^z = e^x(\cos y + i\sin y).$$

The special case $e^{it} = \cos t + i\sin t$ is called **Euler's formula**. Euler was the first to realize the connection between the exponential function and sine and cosine. This amazing identity, called "the most remarkable formula in mathematics" by Feynman, has been around since 1748 and has far-reaching implications in many branches of mathematics and physics.

The following proposition shows that this definition captures all of the nice properties of e^x from the real case. We will see in a moment that in the complex plane, the exponential function has even deeper properties and an essential connection to the geometry of $\mathbb{C}$.

Proposition 2.3.1. *For complex numbers z and w,*

(a) $e^{z+w} = e^z e^w$.

(b) $\frac{1}{e^z} = e^{-z}$.

(c) $e^{z+2\pi i} = e^z$*, that is, the complex exponential function is periodic with period* $2\pi i$.

(d) If $z = x + iy$, $|e^z| = e^x$ *and therefore* $|e^{iy}| = 1$.

(e) $e^z \neq 0$ *for any* $z \in \mathbb{C}$.

Proof. (a) Let $z = x + iy$ and $w = x' + iy'$. Then

$$\begin{aligned} e^{z+w} &= e^{(x+x')+i(y+y')} = e^{x+x'}(\cos(y+y') + i\sin(y+y')) \\ &= e^x e^{x'}(\cos y + i\sin y)(\cos y' + i\sin y') = e^z e^w \end{aligned}$$

(the last part uses a trick similar to the one used in the proof of De Moivre's Theorem (1.1.2)).

(b) follows from (a) and trig properties.

(c) follows directly from the definition of e^z.

(d) follows from the fact that for any θ, $|\cos\theta + i\sin\theta| = 1$.

(e) By part (d), $|e^{x+iy}| = e^x$, and x is real so e^x is always nonzero. Therefore $|e^z| \neq 0$ which implies $e^z \neq 0$. □

We will see in Chapter 3 that e^z also satisfies one of the nicest properties of the exponential function in the real case: $\frac{d}{dz}e^z = e^z$

Note that part (c) of Proposition 2.3.1 implies that $f(z) = e^z$ is *not* a one-to-one function on the complex plane. This is unfortunate, since that was one of the nice attributes of e^x in the real case, as it allowed us to define an inverse, the logarithm $\log x$. We next show how to construct a partial solution to this problem.

Let $w = e^{x+iy}$. We seek a function F such that $F(w) = x + iy$ and $e^{F(x+iy)} = x + iy$. Note that since $|w| = e^x$ and these are real numbers, we have $x = \ln|w|$. This allows us to define

Definition. *The* **formal logarithm** *is written* $\log z = \ln|z| + i\arg z$.

This is *not* a function (meaning it is not well-defined), since $\arg z$ represents a set of values which differ by $2k\pi$ for integers k.

We remedy this by making *branch cuts* of the complex plane. This is done by taking a ray from the origin, say with angle θ and defining the branch $(\theta, \theta + 2\pi]$ so that $\log z$ is well-defined on this domain. The most important branch is

Definition. *Let* $\operatorname{Arg} z$ *denote the argument of* z *in the branch* $(-\pi, \pi]$*; this is called the* **principal branch**. *Then we define the* **principal logarithm** *by*

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z.$$

Proposition 2.3.2. *On the principal branch,* $\operatorname{Log} e^z = e^{\operatorname{Log} z} = z$.

Proof. Let $z = x + iy$ with $\operatorname{Arg} z = \theta \in (-\pi, \pi]$. Then on one hand,

$$\operatorname{Log} e^z = \ln |e^z| + i \operatorname{Arg} e^z = \ln e^x + iy = x + iy = z$$

and on the other hand,

$$e^{\operatorname{Log} z} = e^{\ln |z| + i \operatorname{Arg} z} = e^{\ln |z|}(\cos\theta + i \sin\theta) = |z|(\cos\theta + i\sin\theta) = z.$$

Note that these require that we restrict our attention to a single branch (it may not even be the principal branch) for the expressions to be well-defined. □

Recall that $f(z) = u(z) + iv(z)$ is continuous if and only if u and v are continuous. Well $\operatorname{Arg} z$ has no limit at values along the negative real axis. Therefore $\operatorname{Log} z$ is not continuous at any point $\operatorname{Re}(z) \leq 0$. However, making a different branch cut allows us to define a function with different continuity.

As in the real case, exponentials for bases other than e are permitted. They relate to the logarithm by

$$a^z = e^{z \log a}$$

where $\log a$ is defined on a fixed branch of the logarithm.

Example 2.3.3. Let's use the complex logarithm to evaluate $(-1)^i$. Note that $(-1)^i = e^{i \log(-1)}$ where log is defined appropriately. We also have

$$\log(-1) = \ln|-1| + i(\arg(-1) + 2k\pi) = 0 + i(-\pi + 2k\pi).$$

Then $e^{i\log(-1)} = e^{-(-\pi+2k\pi)} = e^{\pi - 2k\pi}$ for any integer k. The principal value of $(-1)^i$ is e^π, which is found by

$$(-1)^i = e^{i \operatorname{Log}(-1)} = e^{i(-\pi i)} = e^\pi.$$

Example 2.3.4. We can use logarithms to solve an equation such as $z^{1+i} = 4$. First consider $(1+i)\log z = \log 4 = \ln|4| + 2k\pi i$. This gives us

$$\begin{aligned}\log z &= \frac{\ln|4| + 2k\pi i}{1+i}\left(\frac{1-i}{1-i}\right)\\ &= \frac{(\ln|4| + 2k\pi) - i\ln|4| + 2k\pi i}{2}\\ &= (\ln|2| + k\pi) + i(-\ln|2| + k\pi).\end{aligned}$$

Taking the exponential of both sides yields

$$\begin{aligned}z = e^{\log z} &= e^{(\ln 2 + k\pi) + i(-\ln 2 + k\pi)}\\ &= 2e^{k\pi}((-1)^k \cos(\ln 2) + i(-1)^{k+1}\sin(\ln 2))\\ &= (-1)^k 2e^{k\pi}(\cos(\ln 2) - i\sin(\ln 2)).\end{aligned}$$

Example 2.3.5. To simplify an expression such as $(1+i)^i$, use the logarithm to write $(1+i)^i = e^{i\log(1+i)}$. Then

$$\begin{aligned}\log(1+i) = \ln|1+i| + i(\arg(1+i) + 2\pi k) = \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2\pi k\right)\\ \implies e^{i\log(1+i)} = e^{-\left(\frac{\pi}{4}+2\pi k\right) + i\frac{\ln 2}{2}} = e^{-\frac{\pi}{4}}\left(\cos\left(\frac{\ln 2}{2}\right) + i\sin\left(\frac{\ln 2}{2}\right)\right).\end{aligned}$$

2.4 Trigonometric Functions

The complex trigonometric functions are defined in terms of e^z. This should come as no surprise, given the relation we have seen between exponential and trig functions. By the end of the section we will see that this connection runs even deeper.

Definition. *The* **complex cosine** *and* **complex sine** *functions are defined by*

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad \textit{and} \qquad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}).$$

Note that the complex trig functions coincide with their real counterparts, for if $x \in \mathbb{R}$ we have

$$\begin{aligned} \tfrac{1}{2}(e^{ix} + e^{-ix}) &= \tfrac{1}{2}(\cos x + i\sin x + \cos(-x) + i\sin(-x)) \\ &= \tfrac{1}{2}(\cos x + i\sin x + \cos x - i\sin x) = \cos x \\ \text{and} \quad \tfrac{1}{2i}(e^{ix} - e^{-ix}) &= \tfrac{1}{2i}(\cos x + i\sin x - (\cos(-x) + i\sin(-x))) \\ &= \tfrac{1}{2i}(\cos x + i\sin x - \cos x + i\sin x) = \sin x. \end{aligned}$$

The complex cosine and sine functions are also periodic, with period 2π like the real-valued cosine and sine. Using the fact that e^z is periodic, we can write

$$\begin{aligned} \cos(z + 2\pi) &= \tfrac{1}{2}(e^{i(z+2\pi)} + e^{-i(z+2\pi)}) \\ &= \tfrac{1}{2}(e^{iz}e^{2\pi i} + e^{-iz}e^{-2\pi i}) \\ &= \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z \\ \text{and} \quad \sin(z + 2\pi) &= \tfrac{1}{2i}(e^{i(z+2\pi)} - e^{-i(z+2\pi)}) \\ &= \tfrac{1}{2i})(e^{iz}e^{2\pi i} - e^{-iz}e^{-2\pi i}) \\ &= \tfrac{1}{2i}(e^{iz} - e^{-iz}) = \sin z. \end{aligned}$$

Many other properties of the real trig functions carry over the complex case. Just to name a few,

(a) $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$

(b) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$ and $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$

(c) $\sin(z + w) = \sin z \cos w + \cos z \sin w$

(d) $\cos(z + w) = \cos z \cos w - \sin z \sin w$

(e) $\cos^2 z + \sin^2 z = 1$

(f) $\cos^2 z - \sin^2 z = \cos(2z)$

(g) When we define the derivative of a complex-valued function in Section 3.2, we will see that the derivatives of $\cos z$ and $\sin z$ are similar to the real case.

Example 2.4.1. It is easy to see from the definition of cosine that $\cos z = 0$ if and only if $z = \frac{\pi}{2} + \pi k$ for any integer k.

Example 2.4.2. Complex conjugation commutes with trig and exponential functions:

$$e^{\overline{z}} = \overline{e^z} \qquad \cos\overline{z} = \overline{\cos z} \qquad \sin\overline{z} = \overline{\sin z}.$$

Using the definitions of $\cos z$ and $\sin z$, we can define the other four main trig functions.

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} = -i\frac{e^{2iz}-1}{e^{2iz}+1} \\
\sec z &= \frac{1}{\cos z} \\
\csc z &= \frac{1}{\sin z} \\
\cot z &= \frac{\cos z}{\sin z} = i\frac{e^{2iz}+1}{e^{2iz}-1}.
\end{aligned}$$

3 Calculus in the Complex Plane

3.1 Line Integrals

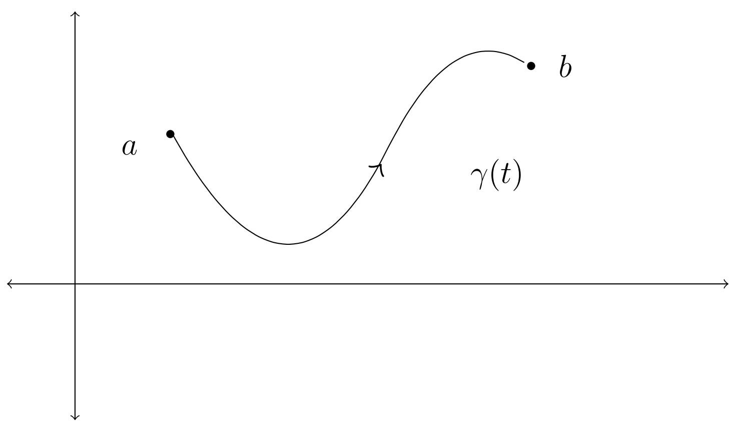
If $f : [a, b] \to \mathbb{C}$ is a complex-valued function which is continuous on some interval $[a, b]$ where $a, b \in \mathbb{R}$, then the integral of f over $[a, b]$ is simply

$$\int_a^b f(t)\,dt = \int_a^b \mathrm{Re}(f(t))\,dt + i \int_a^b \mathrm{Im}(f(t))\,dt.$$

For functions that take on values over some region in the complex plane, we integrate over curves.

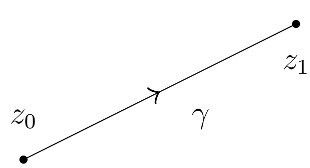
Definition. *Let $f(z)$ be a complex-valued function which is continuous on some region $D \subseteq \mathbb{C}$ and let γ be a smooth curve contained in D that is parametrized by $\gamma(t), a \leq t \leq b$. Then the* **line integral** *of f over γ is*

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$



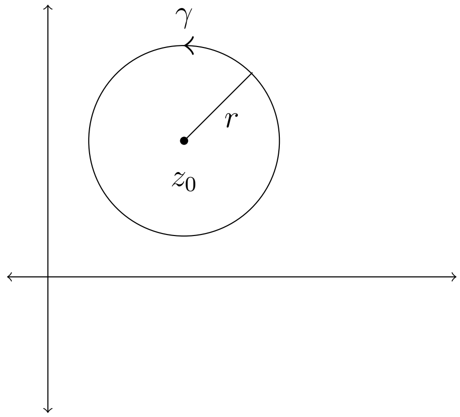
Remember that a curve is **smooth** if its first derivative $\gamma'(t)$ exists and is continuous on $[a, b]$. Since the curves are all functions on a real interval $[a, b]$, we need not worry about complex derivatives yet; $\gamma'(t)$ is just the first derivative in the normal sense. Some important examples of parametrizations in the complex plane are

Example 3.1.1. A curve γ is **simple** if $\gamma(t_1) \neq \gamma(t_2)$ whenever $a < t_1 < t_2 < b$. In plain language, a simple curve does not intersect itself; it is an embedding of the interval $[a, b]$ into $\mathbb{C}$. The easiest simple curve to parametrize is a line:



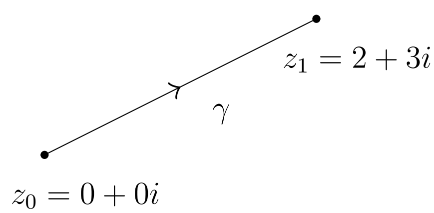
If γ is the line between z_0 and z_1, then we parametrize it by $\gamma(t) = z_0 + t(z_1 - z_0)$ for $0 \leq t \leq 1$.

Example 3.1.2. A curve γ is **closed** if $\gamma(a) = \gamma(b)$, i.e. it starts and ends in the same location. The canonical example of a simple closed curve is a circle:



This is parametrized by $\gamma(t) = z_0 + re^{it}$ for $0 \le t \le 2\pi$.

Example 3.1.3. Let's compute the line integral $\int_\gamma z^2\, dz$ over the line from $(0,0)$ to $(2,3)$ in the complex plane.



We parametrize the curve by $\gamma(t) = 2t + 3it, 0 \le t \le 1$. Then using the formula above, we compute

$$\begin{aligned}\int_\gamma z^2\, dz &= \int_0^1 \gamma(t)^2\gamma'(t)\, dt = \int_0^1 (2t+3it)^2(2+3i)\, dt \\ &= \int_0^1 (4t^2 - 9t^2 + 12it^2)(2+3i)\, dt = \int_0^1 (-5t^2 + 12it^2)(2+3i)\, dt \\ &= \int_0^1 (-46t^2 + 9it^2)\, dt = -\frac{46}{3}t^3\Big|_0^1 + 3it^3\big|_0^1 = -\frac{46}{3} + 3i.\end{aligned}$$

Example 3.1.4. Just as reversing the order of a and b in a real integral changes the integral by -1, one can reverse the orientation of a smooth curve γ to switch the sign of the line integral along γ. Let $-\gamma$ denote the curve γ with orientation reversed. Then

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

Example 3.1.5. Next let's change the path of integration to be the semicircle $\gamma(t) = e^{it}, 0 \le t \le \pi$. We will write $\gamma(t) = \cos t + i \sin t$ so that the derivative may be written

$\gamma'(t) = -\sin t + i\cos t$. Then we compute

$$\begin{aligned}\int_\gamma z^2\,dz &= \int_0^\pi (\cos t + i\sin t)^2(-\sin t + i\cos t)\,dt \\ &= \int_0^\pi (\cos^2 t - \sin^2 t + 2i\cos t\sin t)(-\sin t + i\cos t)\,dt \\ &= \int_0^\pi (\sin^3 t - \cos^2 t\sin t - 2\cos^2 t\sin t)\,dt + i\int_0^\pi (\cos^3 t - \sin^2 t\cos t - 2\sin^2 t\cos t)\,dt \\ &= \int_0^\pi (\sin t - \cos^2 t\sin t - 3\cos^2 t\sin t)\,dt + i\int_0^\pi (\cos t - \sin^2 t\cos t - 3\sin^2 t\cos t)\,dt \\ &= \int_0^\pi (\sin t - 4\cos^2 t\sin t)\,dt + i\int_0^\pi ((\cos t - 4\sin^2 t\cos t)\,dt \\ &= \left[-\cos t + \frac{4}{3}\cos^3 t\right]_0^\pi + i\left[\sin t - \frac{4}{3}\sin^3 t\right]_0^\pi = -\frac{2}{3}.\end{aligned}$$

Example 3.1.6. Compute the line integral $\int_\gamma (z^2 - 3|z| + \operatorname{Im} z)\,dz$ where γ is parametrized by $\gamma(t) = 2e^{it}, 0 \le t \le \frac{\pi}{2}$. First note that $\gamma'(t) = 2ie^{it}$. Then

$$\begin{aligned}\int_\gamma (z^2 - 3|z| + \operatorname{Im} z)\,dz &= \int_0^{\frac{\pi}{2}} (4e^{2it} - 3|2e^{it}| + \operatorname{Im}(2e^{it}))\cdot 2ie^{it}\,dt \\ &= \int_0^{\frac{\pi}{2}} (8ie^{3it} - 12ie^{it} + 4ie^{it}\sin t)\,dt \\ &= \int_0^{\frac{\pi}{2}} \left(8ie^{3it} - 12ie^{it} + 4ie^{it}\left(\frac{1}{2i}(e^{it} - e^{-it})\right)\right)\,dt \\ &= \left[\frac{8}{3}e^{3it} - 12ie^{it} + \frac{1}{2}\sin(2t) - \frac{i}{2}\cos(2t) - 2t\right]_0^{\frac{\pi}{2}} \\ &= \frac{28}{3} - \frac{\pi}{2} - \frac{44}{3}i.\end{aligned}$$

The definition of line integrals can be extended to *piecewise* smooth curves by

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \ldots + \int_{\gamma_k} f(z)\,dz$$

where each γ_i is a smooth curve on an interval $[a_i, b_i] \subset [a, b]$, $\gamma_1(a) = \gamma(a)$, $\gamma_k(b) = \gamma(b)$ and $\gamma_i(b_i) = \gamma_{i+1}(b_i)$ for all i.

Definition. *The* **length of a curve** γ *is given by the integral*

$$\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

where $\gamma(t) = x(t) + iy(t), a \le t \le b$ *is a parametrization of* γ.

Example 3.1.7. Let γ be the unit circle, which has the parametrization $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$. Let's verify the circumference of the circle with the formula for the length of γ:

$$\int_0^{2\pi} |\gamma'(t)|\, dt = \int_0^{2\pi} |ie^{it}|\, dt = \int_0^{2\pi} dt = 2\pi.$$

The next proposition contains some useful properties of the line integral.

Proposition 3.1.8. *Suppose γ is a smooth curve and f and g are continuous, complex-valued functions on a domain containing γ.*

(a) $\displaystyle\int_\gamma (f(z) + g(z))\, dz = \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz.$

(b) For any $c \in \mathbb{C}$, $\displaystyle\int_\gamma cf(z)\, dz = c\int_\gamma f(z)\, dz.$

(c) If τ is a curve whose initial point is the terminal point of γ, then $\gamma\tau$ is defined to be the curve obtained by following γ and then τ. The integral over $\gamma\tau$ is given by

$$\int_{\gamma\tau} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\tau f(z)\, dz.$$

(d) $\displaystyle\left|\int_\gamma f(z)\, dz\right| \leq \max_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$

3.2 Differentiability

Recall that the function $f(z) = \dfrac{z}{\bar{z}}$ is not continuous at $z_0 = 0$. This points to the fact that complex functions are somehow different than their real brethren, and in particular the convergence of a function in $\mathbb{C}$ is much stronger than convergence in $\mathbb{R}$.

Definition. *The* **derivative** *of a complex function $f(z)$ at a point $z_0 \in \mathbb{C}$ is defined by*

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If these limits exist, we say $f(z)$ is **differentiable** *at z_0.*

This definition is the same as in the real case, although as discussed above the notion of a limit is much stronger in $\mathbb{C}$. In the complex world, we have a further notion of differentiability:

Definition. *A complex function $f(z)$ is* **holomorphic** *at $z_0 \in \mathbb{C}$ if $f(z)$ is differentiable on some open disk centered at z_0. Functions which are holomorphic on the whole complex plane $\mathbb{C}$ are called* **entire***.*

Example 3.2.1. Many familiar functions from real analysis have the same derivative in the complex plane. For example, $f(z) = z^2$ has derivative $2z$ which may be confirmed by computing either of the above limits. In fact this holds for all $z \in \mathbb{C}$ so z^2 is an entire function.

Example 3.2.2. $f(z) = \bar{z}^2$ is differentiable at 0 and nowhere else, which means $f(z)$ is *not* holomorphic at 0. To see this, write $z = z_0 + re^{i\theta}$. Then the difference quotient can be written

$$\begin{aligned}\frac{\bar{z}^2 - \bar{z}_0^2}{z - z_0} &= \frac{(\bar{z}_0 + re^{-i\theta})^2 - \bar{z}_0^2}{re^{i\theta}} \\ &= \frac{\bar{z}_0^2 + 2\bar{z}_0 re^{-i\theta} + r^2e^{-2i\theta} - \bar{z}_0^2}{re^{i\theta}} \\ &= \frac{2\bar{z}_0 re^{-i\theta} + r^2e^{-2i\theta}}{re^{i\theta}} = 2\bar{z}_0e^{-2i\theta} + re^{-3i\theta}.\end{aligned}$$

If $r \neq 0$ then we get different answers for the limit $z \to z_0$ (e.g. take $\theta = 0$ and $\theta = \frac{\pi}{2}$) which shows that $f(z)$ is not differentiable at any point other than the origin. At $z_0 = 0$, we see that

$$\lim_{z\to z_0} \frac{\bar{z}^2 - \bar{z}_0^2}{z - z_0} = \lim_{z\to z_0} \frac{\bar{z}^2}{z} = 0.$$

Example 3.2.3. Complex conjugation is not differentiable at any $z_0 \in \mathbb{C}$ since

$$\lim_{z\to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{z\to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{z\to 0} \frac{\bar{z}}{z}$$

does not exist as we have seen.

Most of the nice properties of real derivatives carry over to the complex place.

Proposition 3.2.4. *Let f and g be differentiable at $z \in \mathbb{C}$.*

(a) $(f(z) + g(z))' = f'(z) + g'(z)$.

(b) For any $c \in \mathbb{C}$, $(cf)'(z) = cf'(z)$.

(c) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(d) If $g(z) \neq 0$ then $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$.

(e) $(z^n)' = nz^{n-1}$. *In particular this means that polynomials are entire.*

(f) If g is differentiable at $f(z)$ then $(g(f(z)))' = g'(f(z))f'(z)$.

The fundamental property in this section is a pair of equations called the Cauchy-Riemann Equations, which relate the derivative $f'(z)$ to the partial derivatives with respect to the real and imaginary parts of z.

Theorem 3.2.5 (Cauchy-Riemann Equations)**.** *Let $f(z) = u(x,y) + iv(x,y)$ be a complex function which is continuous at $z_0 = x_0 + iy_0$. Then $f(z)$ is differentiable at z_0 if and only if the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist, are continuous and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \textit{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on some neighborhood of z_0.

Proof. ($\Longrightarrow$) If $f(z)$ is differentiable at $z_0 = x_0 + iy_0$ then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

First consider approaching z along the line $(x_0 + h) + iy_0$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f((x_0+h)+iy_0) - f(x_0+iy_0)}{h} &= \lim_{h\to 0} \frac{u(x_0+h,y_0) + iv(x_0+h,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i\frac{v(x_0+h,y_0) - v(x_0,y_0)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z_0).
\end{aligned}$$

Next, approach along $x_0 + i(y_0 + h)$:

$$\begin{aligned}
\lim_{ih\to 0} \frac{f(x_0+i(y_0+h)) - f(x_0+iy_0)}{ih} &= \lim_{ih\to 0} \frac{u(x_0,y_0+h) + iv(x_0,y_0+h) - u(x_0,y_0) - iv(x_0,y_0)}{ih} \\
&= \lim_{h\to 0} \frac{u(x_0,y_0+h) - u(x_0,y_0)}{ih} + i\frac{v(x_0,y_0+h) - v(x_0,y_0)}{ih} \\
&= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f'(z_0).
\end{aligned}$$

Setting these two expressions for $f'(z_0)$ equal gives the result, since the real and imaginary parts of the resulting expression must be equal.

($\Longleftarrow$) The converse requires a little more care. We will show that $f(z)$ is differentiable at z_0 with derivative $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$. We first break up the difference quotient, using $h = h_x + ih_y$:

$$\begin{aligned}
\frac{f(z_0+h) - f(z_0)}{h} &= \frac{f(z_0+h) - f(z_0+h_x) + f(z_0+h_x) - f(z_0)}{h} \\
&= \frac{f(z_0+h_x+ih_y) - f(z_0+h_x)}{h} + \frac{f(z_0+h_x) - f(z_0)}{h} \\
&= \frac{h_y}{h} \cdot \frac{f(z_0+h_x+ih_y) - f(z_0+h_x)}{h_y} + \frac{h_x}{h} \cdot \frac{f(z_0+h_x) - f(z_0)}{h_x}.
\end{aligned}$$

Elsewhere, we have

$$\frac{\partial f}{\partial x}(z_0) = \frac{h_y}{h} \cdot \frac{\partial f}{\partial y}(z_0) + \frac{h_x}{h} \cdot \frac{\partial f}{\partial x}(z_0).$$

Now we subtract these two expressions and take a limit, which gives

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}-\frac{\partial f}{\partial x}(z_0)=\lim_{h\to 0}\left[\frac{h_y}{h}\left(\frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h_y}-\frac{\partial f}{\partial y}(z_0)\right)\right]$$
$$+\lim_{h\to 0}\left[\frac{h_x}{h}\left(\frac{f(z_0+h_x)-f(z_0)}{h_x}-\frac{\partial f}{\partial x}(z_0)\right)\right].$$

If we can show that the limits on the right are both 0, then we're done. The ratios $\frac{h_x}{h}$ and $\frac{h_y}{h}$ are both bounded by the triangle inequality, so it suffices to prove the the expressions in parentheses tend to 0. The second term goes to 0 since by definition,

$$\frac{\partial f}{\partial x}(z_0)=\lim_{h_x\to 0}\frac{f(z_0+h_x)-f(z_0)}{h_x}.$$

The other expression is more problematic, since it involves both h_x and h_y. However, the Mean Value Theorem from real analysis gives us real numbers $0<a,b<1$ such that

$$\frac{u(x_0+h_x,y_0+h_y)-u(x_0+h_x,y_0)}{h_y}=u_y(x_0+h_x,y_0+ah_y)$$
$$\text{and}\quad \frac{v(x_0+h_x,y_0+h_y)-v(x_0+h_x,y_0)}{h_y}=v_y(x_0+h_x,y_0+bh_y).$$

Substituting these expressions into the first term above gives us

$$\begin{aligned}\frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h_y}-\frac{\partial f}{\partial y}(z_0)&=u_y(x_0+h_x,y_0+ah_y)+iv_y(x_0+h_x,y_0+bh_y)\\&\quad-u_y(x_0,y_0)-iv_y(x_0,y_0)\\&=(u_y(x_0+h_x,y_0+ah_y)-u_y(x_0,y_0))\\&\quad+i(v_y(x_0+h_x,y_0+bh_y)-v_y(x_0,y_0)).\end{aligned}$$

Finally, these two pieces each tend to 0 since u_y and v_y are assumed to be continuous at $z_0=x_0+iy_0$. This finishes the proof. □

Example 3.2.6. Consider the function

$$f(z)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}+i\dfrac{x^3+y^3}{x^2+y^2} & z\neq 0\\ 0 & z=0.\end{cases}$$

It is easy to see that the Cauchy-Riemann equations hold for $f(z)$ at $z_0=0$, but the complex derivative $f'(0)$ does not exist. This is not a failure of the theorem, however, since the partial derivatives u_x,u_y,v_x and v_y are not continuous at any point but 0.

Example 3.2.7. Consider $f(z)=\operatorname{Log} z$ using the principal branch D as its domain. We may write this as

$$f(z)=\ln|z|+i\operatorname{Arg} z=\tfrac{1}{2}\ln(x^2+y^2)+i\arctan\left(\tfrac{y}{x}\right).$$

So one sees that $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$ and $v(x,y) = \arctan\left(\frac{y}{x}\right)$. We calculate the partials:

$$u_x = \frac{x}{x^2+y^2} \qquad v_x = -\frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$
$$u_y = \frac{y}{x^2+y^2} \qquad v_y = \frac{1}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}.$$

Hence $u_x = v_y$ and $u_y = -v_x$ so $f(z)$ satisfies the Cauchy-Riemann equations on D, meaning it is differentiable. Moreover, we can write its derivative as

$$f'(z) = u_x + iv_x = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{|z|}.$$

This is a striking, yet perhaps predictable result that reassures us that our definition of the complex logarithm captures the real case.

3.3 Power Series

Definition. *A* **power series** *is an infinite series of the form*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Such a series is said to be centered about z_0.

Example 3.3.1. Power series are really a generalization of a geometric series

$$\sum_{n=0}^{\infty} z^n$$

centered about $z_0 = 0$, where all the coefficients are 1. We know from Section 2.2 that this series converges to $\dfrac{1}{1-r}$ exactly when $|z| < 1$. We will see that power series behave in similar ways, and when they converge, they converge to complex functions that we may be interested in.

For a power series $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ we have three cases for convergence:

(1) The series only converges at $z = z_0$. In this case, the radius of convergence of the series is 0.

(2) The series converges for all z in a disc of finite radius R centered at z_0.

(3) The series converges for all $z \in \mathbb{C}$, in which case we say the series has an infinite radius of convergence.

Examples.

① Consider the series $\sum_{n=0}^{\infty} n!\, z^n$. By the ratio test,

$$\lim_{n\to\infty}\left|\frac{(n+1)!\, z^{n+1}}{n!\, z^n}\right| = \lim_{n\to\infty}|z|(n+1) = \infty$$

so the series diverges for all positive radii. This is an example of case 1, i.e. the series has no radius of convergence.

② For $\sum_{n=0}^{\infty} 5^n(z-i)^n$, the ratios test gives us

$$\lim_{n\to\infty}\left|\frac{5^{n+1}(z-i)^{n+1}}{5^n(z-i)^n}\right| = \lim_{n\to\infty} 5|z-i|.$$

So the series converges (absolutely) whenever $5|z-i| < 1 \implies |z-i| < \frac{1}{5}$. This is an example of case 2, where the series has positive radius of convergence $R = \frac{1}{5}$.

③ The power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an example of case 3, since it converges (absolutely) for all z as shown again by the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0 < 1.$$

A power series with positive or infinite radius of convergence represents a function that is holomorphic within the disc of convergence of the series. This is one of the most important facts in complex analysis, so we take a moment to formalize it here.

Theorem 3.3.2. *Suppose* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a positive or infinite radius of convergence* R. *Then it represents a function* $f(z)$ *which is holomorphic on* $D = \{z \in \mathbb{C} : |z-z_0| < R\}$.

Proof. This will be proven in Section 3.6. □

Now that we know that power series are holomorphic (differentiable) on their discs of convergence, we can take derivatives.

Theorem 3.3.3. *Suppose* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a positive or infinite radius of convergence* R. *Then its derivative is also a power series:*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

which has radius of convergence R.

This can be applied repeatedly to obtain the **Taylor series expansion** of $f(z)$ about z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Example 3.3.4. The Taylor series for the exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Using the formulas for $\cos z$ and $\sin z$ from Section 2.4, we can derive their Taylor series as well:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z - z_0)^{2n}$$
$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z - z_0)^{2n+1}.$$

3.4 Cauchy's Theorem

We now arrive at a theorem of central importance in complex analysis. The statement of the theorem is simple, but as we will see, this result has far-reaching implications in the complex world.

Theorem 3.4.1 (Cauchy's Theorem)**.** *Let $f(z)$ be a complex function that is holomorphic on domain D, and suppose γ is any piecewise smooth, simple, closed curve in D. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof. By assumption $f'(z)$ is continuous on D and γ has interior Ω within D. We compute

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma (u+iv)(dx + i\,dy) = \int_\gamma (u\,dx - v\,dy + i(v\,dx + u\,dy)) \\
&= \int_\gamma (u\,dx - v\,dx) + i\int_\gamma (v\,dx + u\,dy) \\
&= \iint_\Omega (-v_x - u_y)\,dxdy + i\iint_\Omega (u_x - v_y)\,dxdy \quad \text{by Green's Theorem} \\
&= \iint_\Omega (-v_x + v_x)\,dxdy + i\iint_\Omega (u_x - u_x)\,dxdy \quad \text{by Cauchy-Riemann equations} \\
&= 0 + i0 = 0.
\end{aligned}$$

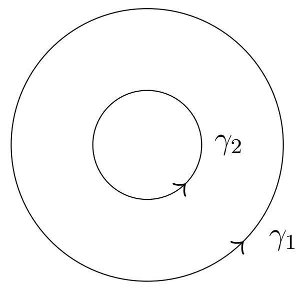
□

Some immediate consequences of Cauchy's Theorem are

Corollary 3.4.2 (Independence of Path)**.** *If γ_1 and γ_2 are curves with the same initial and terminal points lying in a domain on which $f(z)$ is holomorphic, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Corollary 3.4.3 (Deformation of Path)**.** *Suppose γ_1 and γ_2 are two simple, closed curves with the same orientation, with γ_2 lying on the interior of γ_1.*



If $f(z)$ is holomorphic on the region between γ_1 and γ_2 then

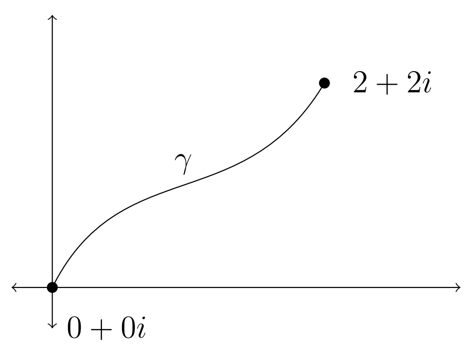
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Corollary 3.4.4 (Fundamental Theorem of Calculus)**.** *If $f(z)$ is holomorphic on a simply-connected domain D, then there is a holomorphic function F satisfying*

$$F(z) = \int_{\gamma} f(z)\,dz$$

for any γ lying in D. Equivalently, F satisfies $F'(z) = f(z)$ on all of D.

Example 3.4.5. Now it's easy to solve an integral such as $\displaystyle\int_\gamma e^z\,dz$ where γ is some path from 0 to $2+2i$:



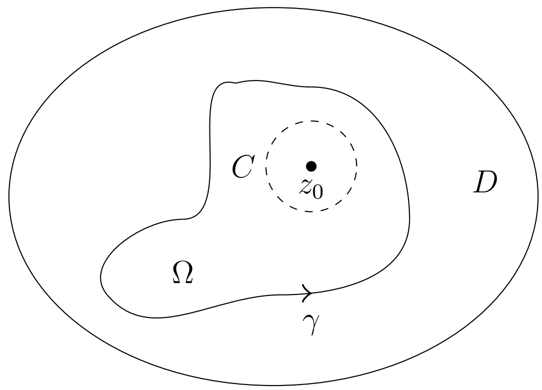
$$\int_\gamma e^z\,dz = e^z|_{2+2i} - e^z|_{0+0i} = e^2(\cos 2 + i\sin 2) - 1.$$

The most important application of Cauchy's Theorem is Cauchy's Integral Formula, which is described in the next section.

3.5 Cauchy's Integral Formula

Theorem 3.5.1 (Cauchy's Integral Formula)**.** *Suppose f is holomorphic on a domain D and γ is a simple closed curve on D, with positive orientation and interior Ω. Then for all $z \in \Omega$,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$



Proof. Fix $z \in \Omega$ and let C be a circle with center z contained in Ω. Note that for any $z \in D$, $\dfrac{f(\zeta)}{\zeta - z}$ is holomorphic on $D \setminus \{z\}$. By deformation of path,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

We parametrize C by $z + re^{it}$ for $0 \le t \le 2\pi$ and write

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{it})}{re^{it}} \, ire^{it} \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt. \end{aligned}$$

Now take the limit as $r \to 0$. Since $f(z)$ is continuous, we can bring the limit inside the integral:

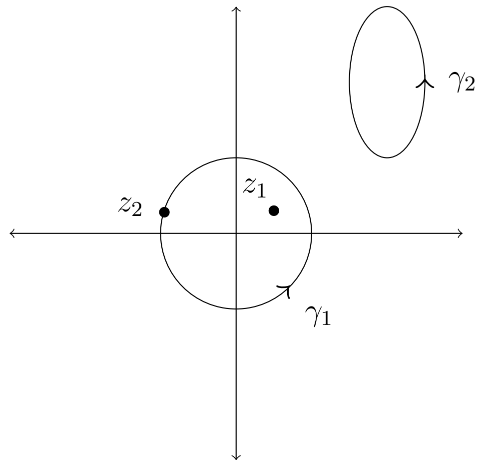
$$\lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z) \, dt.$$

Notice that $f(z)$ doesn't depend on t, so we can integrate this easily and see that it equals $f(z)$. This proves the theorem. □

Example 3.5.2. Cauchy's integral formula allows us to solve path integrals that were previously inaccessible. For example, if γ is a circle about the origin of radius 1, then $z = \frac{1}{2}$ is on its interior and $\dfrac{e^z}{z - \frac{1}{2}}$ is not holomorphic on the interior of γ. However, Cauchy's integral formula lets us compute

$$\int_\gamma \frac{e^z}{z - \frac{1}{2}} \, dz = 2\pi i e^{1/2}.$$

Example 3.5.3. Consider the following contours



First, Cauchy's Theorem (3.4.1) makes it easy to evaluate integrals around γ_2, since z_1 and z_2 are not on the interior of this curve. For example,

$$\int_{\gamma_2} \frac{e^z}{z - z_1}\, dz = 0 \qquad \text{and} \qquad \int_{\gamma_2} \frac{e^z}{z - z_2}\, dz = 0.$$

When a point is on the interior of a curve, we use Cauchy's integral formula (3.5.1):

$$\int_{\gamma_1} \frac{e^z}{z - z_1}\, dz = 2\pi i e^{z_1}.$$

Unfortunately, since z_2 lies directly on γ_1, the integral

$$\int_{\gamma_1} \frac{e^z}{z - z_2}\, dz$$

must be evaluated by hand, e.g. by parametrization.

Example 3.5.4. Using our integration formulas so far, we can break complicated contours down into simple pieces. For example, consider

$$\int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\, dz.$$

The contour of integration is the circle of radius 2 centered at $z_0 = -1$, which contains $z_1 = -2$ on its interior but not $z_2 = 2$. By partial fraction decomposition, we can write

$$\begin{aligned} \int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\, dz &= \int_{|z+1|=2} \left(\frac{-1}{z-2} + \frac{1}{z+2} \right) dz \\ &= \int_{|z+1|=2} \frac{1}{z+2}\, dz - \int_{|z+1|=2} \frac{1}{z-2}\, dz. \end{aligned}$$

The second of these integrals is 0 by Cauchy's Theorem (3.4.1). The first evaluates to $2\pi i$ by Cauchy's integral formula (3.5.1), so we see that the original integral is equal to $2\pi i$.

We can see this another way, by setting $f(z) = \dfrac{-z^2}{z-2}$ and noticing that f is holomorphic on $|z+1| = 2$. Then Cauchy's integral formula (3.5.1) tells us that

$$\int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\,dz = 2\pi i f(-2) = 2\pi i \frac{-4}{-4} = 2\pi i.$$

The next theorem shows that Cauchy's Integral Formula is intimately related to complex power series.

Theorem 3.5.5. *Let f be holomorphic on a domain D and suppose z_0 is a point in D such that the circle $|z - z_0| < R$ for some real R lies in D. Let γ be a simple closed curve lying within this circle and containing z_0 on its interior. Then*

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad \textit{where} \qquad a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta$$

Proof. Let $\Delta = \{z : |z - z_0| < R\}$. By deformation of path, it suffices to consider when γ is a circle. For a fixed $r < R$, we take γ to be the positively-oriented circle $\gamma : |z - z_0| = r$. By Cauchy's Integral Formula (3.5.1),

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for any z on the interior of γ. For any one of these z's, let $s = |z - z_0|$ so that $s < r$. Consider

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\cdot\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}.$$

Note that $\dfrac{|z - z_0|}{|\zeta - z_0|} = \dfrac{s}{r} < 1$. This allows us to introduce the series as a convergent geometric series:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^k.$$

Using this and the expression given by Cauchy's integral formula above, we are able to write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^k\,d\zeta \\ &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}(z - z_0)^k \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta. \end{aligned}$$

□

Corollary 3.5.6. *If $f(z)$ is holomorphic on D, f has derivatives of all orders on D and each derivative is holomorphic on D.*

Proof. By Theorem 3.5.5, $f(z)$ can be written as a power series with positive radius of convergence,

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta,$$

for some γ about z_0. We will see in Section 3.6 that we can differentiate (and antidifferentiate) power series, so $f(z)$ is infinitely differentiable on the region of convergence of the power series. □

3.6 Analytic Functions

Theorem 3.5.5 suggests a powerful connection between power series and holomorphic functions in the complex plane. In this section we prove that every power series represents a holomorphic function on its region of convergence and every holomorphic function has a power series representation on its domain. First, we need a converse to Cauchy's Theorem (3.4.1).

Theorem 3.6.1 (Morera's Theorem)**.** *Suppose $f(z)$ is continuous on a domain D and*

$$\int_\gamma f(z)\,dz = 0$$

for all smooth, closed curves γ in D. Then f is holomorphic on D.

Proof. We may assume D is connected; otherwise the proof can be repeated on each connected component of D. Fix $z_0 \in D$ and define $F(z) = \int_\gamma f(\zeta)\,d\zeta$ where γ is any smooth curve connecting z_0 and z. By independence of path, $F(z)$ is well-defined for all $z \in D$. Since all *closed* curves γ give $F = 0$ and $f(z)$ is continuous, it follows that $F'(z) = f(z)$, that is, F is an antiderivative of f. Then $F(z)$ is holomorphic on D, which by Corollary 3.5.6 implies that $f(z)$ is also holomorphic on D. □

We prove the first direction of the power series-holomorphic function connection below.

Theorem 3.6.2. *Suppose $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ has a positive radius of convergence R. Then f is a holomorphic function on the domain $D = \{z \in \mathbb{C} : |z - z_0| < R\}$.*

Proof. Given any closed curve γ in D,

$$\int_\gamma \sum_{k=0}^{\infty} a_k(z - z_0)^k\,dz = 0$$

by continuity of the power series on its region of convergence. Then Morera's Theorem says that $f(z)$ is holomorphic on D. □

Now we know that power series are differentiable on their region of convergence. The next result says that we can differentiate power series term-by-term, just as in the real case.

Theorem 3.6.3. *Suppose* $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ *has positive radius of convergence* R. *Then* $f(z)$ *is differentiable with*

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

which also has radius of convergence R.

Example 3.6.4. In this example we verify the derivatives for $e^z, \cos z$ and $\sin z$. In Example 3.3.4 we saw that the Taylor series expansions for these functions are

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z - z_0)^{2n}$$

$$\text{and} \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z - z_0)^{2n+1}.$$

Differentiating the power series for e^z term-by-term shows that

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

We can use the definitions of $\cos z$ and $\sin z$ in terms of the complex exponential function (Section 2.4) to prove that their derivatives are

$$\frac{d}{dz} \cos z = -\sin z \qquad \text{and} \qquad \frac{d}{dz} \sin z = \cos z.$$

We can repeatedly apply Theorem 3.6.3 to subsequent derivatives of f to obtain a statement of Taylor's Theorem for complex functions:

Theorem 3.6.5. *Suppose* $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ *has a positive radius of convergence. Then*

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

We now turn to the other connection between holomorphic functions and power series. Well actually, we have already proven (Corollary 3.5.6) that holomorphic functions have power series representations, which we recall here.

Theorem 3.6.6. *Let f be holomorphic on a domain D. Then*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{for} \qquad a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta$$

where $z_0 \in D$ and γ is a simple closed curve lying in D and containing z_0 on its interior.

We immediately obtain the following generalization of Cauchy's integral formula (3.5.1).

Corollary 3.6.7. *Suppose f is holomorphic on a domain D and γ is a simple closed curve in D, positively oriented and with interior Ω. Then for all $z \in \Omega$ and $n \in \mathbb{N}$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

We now define what it means for a function to be analytic on a certain region in the complex plane.

Definition. *A function $f(z)$ that is continuous on a region $D \subseteq \mathbb{C}$ is* **analytic** *at $z_0 \in D$ if f equals its Taylor series expansion about z_0 and f is analytic on D if it is analytic at every point in D.*

The following theorem summarizes everything we have learned so far about holomorphic functions in the complex plane.

Theorem 3.6.8. *For a complex function $f(z)$ which is continuous on a domain D, the following are equivalent:*

(1) $f(z)$ is differentiable on some open disk centered at $z_0 \in D$, that is, f is holomorphic at z_0.

(2) The Taylor series expansion of $f(z)$ about z_0 converges to $f(z)$ with positive radius of convergence, i.e. f is analytic.

(3) $f(z)$ satisfies the Cauchy-Riemann equations on some neighborhood of z_0.

(4) $\int_{\gamma} f(z) \, dz = 0$ for every simple closed curve γ inside D with z_0 on its interior (Cauchy's Theorem and Morera's Theorem).

We conclude with a consequence of the generalized Cauchy's integral formula to entire functions that are bounded.

Theorem 3.6.9 (Liouville's Theorem)**.** *If $f(z)$ is entire and there exists a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant function.*

Proof. Let $z_0 \in \mathbb{C}$ and take C_r to be the circle centered at z_0 with radius $r > 0$. By Corollary 3.6.7,

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2} \, d\zeta.$$

Parametrize the circle by $C_r : z_0 + re^{it}, 0 \leq t \leq 2\pi$. Then

$$f'(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{r^2e^{2it}} ire^{it}\,dt$$
$$= \frac{1}{2\pi r}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{e^{it}}\,dt.$$

Taking the modulus of both sides and applying the triangle inequality for integrals, we have

$$|f'(z_0)| \leq \frac{1}{2\pi r}\int_0^{2\pi} \left|\frac{f(z_0 + re^{it})}{e^{it}}\right| dt$$
$$= \frac{1}{2\pi r}\int_0^{2\pi} \frac{|f(z_0 + re^{it})|}{|e^{it}|}\,dt$$
$$\leq \frac{1}{2\pi r}\int_0^{2\pi} M\,dt.$$

As we take $r \to 0$, this expression tends to 0 as well, showing $|f'(z_0)| = 0$. Since z_0 was arbitrary, we have shown that $f(z)$ is constant. □

3.7 Harmonic Functions

There is a certain class of holomorphic functions which are important in physics. We study them here.

Definition. *A complex function $f = u + iv$ is* **harmonic** *on a domain D if it has continuous second partial derivatives on D that satisfy the* **Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The next result says that the real and imaginary parts of a holomorphic function are harmonic.

Proposition 3.7.1. *Suppose $f = u + iv$ is a holomorphic function on a domain D. Then u and v are harmonic on D.*

Proof. Since f is holomorphic, it is infinitely differentiable and so are u and v. In particular, u and v have continuous second partial derivatives. Moreover, f satisfies the Cauchy-Riemann equations:

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

which imply $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ since these are continuous. Hence u is harmonic. The proof is the same for v. □

Given a harmonic function u, one may be interested in finding a **harmonic conjugate** of u, i.e. another harmonic function v such that $f = u + iv$ is holomorphic in some region of the complex plane.

Example 3.7.2. Consider the function $u(x, y) = \dfrac{x}{x^2 + y^2}$. We first show that u is harmonic by computing second partials.

$$\begin{aligned}
u_x &= \frac{-x^2 + y^2}{(x^2 + y^2)^2} \\
u_{xx} &= \frac{-2x(x^2 + y^2)^2 - 4x(x^2 + y^2)(-x^2 + y^2)}{(x^2 + y^2)^4} \\
&= \frac{(x^2 + y^2) \cdot 2x(-x^2 - y^2 + x^2 - y^2)}{(x^2 + y^2)^4} \\
&= \frac{-4xy^2}{(x^2 + y^2)^3} \\
\text{and} \qquad u_y &= \frac{-2xy}{(x^2 + y^2)^2} \\
u_{yy} &= \frac{-2x(x^2 + y^2)^2 - 4y(x^2 + y^2)(-2xy)}{(x^2 + y^2)^4} \\
&= \frac{(x^2 + y^2) \cdot 2x(4y^2 - x^2 - y^2)}{(x^2 + y^2)^4} \\
&= \frac{4xy^2}{(x^2 + y^2)^3}.
\end{aligned}$$

Thus $u_{xx} + u_{yy} = 0$ so $u(x, y)$ is harmonic. Now for $f = u + iv$ to be a holomorphic function, it will need to satisfy the Cauchy-Riemann equations, so $u_x = v_y$ and $u_y = -v_x$. The above shows that we must have $v_x = \dfrac{2xy}{(x^2 + y^2)^2}$. Integrating with respect to x,

$$v = \int 2xy(x^2 + y^2)^{-2}\, dx = \frac{-y}{x^2 + y^2} + y\Psi(y)$$

for some function $\Psi(y)$. Now if we differentiate this with respect to y, we have

$$v_y = \frac{-(x^2 + y^2) - 2y(-y)}{(x^2 + y^2)^2} + y\Psi'(y) + \Psi(y) = \frac{-x^2 + y^2}{(x^2 + y^2)^2} + y\Psi'(y) + \Psi(y).$$

By the expression for u_x determined above, we must have $y\Psi'(y) + \Psi(y) = 0$. A general solution to this differential equation is $\Psi(y) = \frac{c}{|y|}$, which gives us

$$v(x, y) = \frac{-y}{x^2 + y^2} + y\frac{c}{|y|} = \frac{-y}{x^2 + y^2} \pm c$$

and this is holomorphic for all $(x, y) \in \mathbb{C}$ such that $y \neq 0$.

Proposition 3.7.3. *If $u(x,y) = k$ is a constant function, then it has a harmonic conjugate $v(x,y)$ which is also constant.*

Proof. To begin with, u clearly satisfies the Laplace equation so it is harmonic. A harmonic conjugate v must satisfy $v_x = v_y = 0$ by the Cauchy-Riemann equations, so

$$v = \int v_y\,dy = k_1 + \chi(x)$$

$$\text{and} \qquad v = \int v_x\,dx = k_2 + \psi(y).$$

Setting these equal, we have $k_1 + \chi(x) = k_2 + \psi(y)$, so $\chi_x(x) = \psi_y(y) = 0$, showing that each of the functions must be constant, say $\chi(x) = c_1$ and $\psi(y) = c_2$. Therefore $v(x,y) = k_1 + c_1 = k_2 + c_2$, showing v is a constant function. □

In general, the existence of harmonic conjugates is characterized by

Theorem 3.7.4. *Suppose $u(x,y)$ is a harmonic function on the simply connected region $D \subseteq \mathbb{C}$. Then there exists a harmonic conjugate $v(x,y)$ such that $f = u + iv$ is holomorphic on D.*

Proof. Fix $(x_0, y_0) \in D$ and define $v(x,y)$ by

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t,y_0)\,dt.$$

Then $f = u + iv$ is holomorphic in D since it satisfies the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt = \frac{\partial u}{\partial x} \\
\text{and} \quad \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) = \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) \quad \text{using the Laplace equation} \\
&= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,y_0) - \frac{\partial u}{\partial y}(x,y_0) = -\frac{\partial u}{\partial y}(x,y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}(x,y_0)\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(x,y)\right) - \frac{\partial}{\partial y}\int_{x_0}^{x} \frac{\partial^2 u}{\partial y^2}(t,y_0)\,dt \\
&= -\frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) + \frac{\partial^2 u}{\partial y \partial x}(x,y) + \frac{\partial}{\partial y}\int_{x_0}^{x} \frac{\partial^2 u}{\partial x^2}(t,y_0)\,dt \\
&= -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,y_0)\right) - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) + \frac{\partial^2 u}{\partial y \partial x}(x,y) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(x,y_0) - \frac{\partial u}{\partial x}(x_0,y_0)\right) \\
&= 0.
\end{aligned}$$

So $v(x,y)$ is indeed a harmonic conjugate of $u(x,y)$. □

Corollary 3.7.5. *Every harmonic function is infinitely differentiable on its domain.*

3.8 The Maximum Principle

4 Meromorphic Functions and Singularities

4.1 Laurent Series

With Theorem 3.6.6, we saw that an analytic function can be written

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{where} \qquad a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta$$

for all z in its domain D. This is highly useful, but when $f(z)$ is *not* analytic on a domain D we still want a way of representing f as a series. This motivates the introduction and application of Laurent series:

Definition. *A* **Laurent series** *is a series expansion of a function $f(z)$ about a point z_0 not in the domain of f in terms of two infinite power series, a positive and negative one:*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k (z - z_0)^{-k} = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k.$$

Remark. A Laurent series converges if and only if both the positive and negative series converge. Absolute and uniform convergence are defined analagously. Notice that any Taylor series is a Laurent series whose negative part vanishes.

Example 4.1.1. $f(z) = e^{1/z}$ is not analytic at $z_0 = 0$, but we can write its Laurent series expansion

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}.$$

In this case only the $k = 0$ term of the positive series is nonzero.

Example 4.1.2. Consider the function $f(z) = \frac{z^3+z^2}{(z-1)^2}$ about $z_0 = 1$. First we write the regular Taylor series expansion of the numerator about z_0:

$$z^3 + z^2 = \sum_{k=0}^{\infty} a_n (z-1)^k = 2 + 5(z-1) + \frac{8}{2!}(z-1)^2 + \frac{6}{3!}(z-1)^3.$$

Dividing by $(z-1)^2$ yields

$$\frac{z^3 + z^2}{(z-1)^2} = \frac{2}{(z-1)^2} + \frac{5}{z-1} + 4 + (z-1)$$

which is a Laurent series for $f(z)$ about $z_0 = 1$. The coefficients are $b_2 = 2, b_1 = 5, a_0 = 4, a_1 = 1$ and the rest are zero.

Example 4.1.3. Similarly, we use the Taylor series for $\sin z$ to write the Laurent series for $f(z) = \frac{\sin z}{z^3}$ about $z_0 = 0$ as

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots$$

We should take a moment to explicitly describe the region of convergence of a Laurent series. Suppose

$$\sum_{k\in\mathbb{Z}} c_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}.$$

The positive series has some radius convergence R_1, that is, the series converges on the region $\{z \in \mathbb{C} : |z-z_0| < R_1\}$. Similarly, the negative series is just a power series in $\frac{1}{z-z_0}$ so it has radius of convergence $\frac{1}{R_2}$, i.e. it converges when $\frac{1}{|z-z_0|} < \frac{1}{R_2}$. This can be written as the complement of a closed disk, $\{z \in \mathbb{C} : |z-z_0| > R_2\}$. Thus we see that the Laurent series is convergent on an annular region $\{z \in \mathbb{C} : R_2 < |z-z_0| < R_1\}$ (as long as $R_2 < R_1$). By Theorem 3.6.2, the Laurent series represents an analytic function $f(z)$ on the region $D = \{z \in \mathbb{C} : R_2 < |z-z_0| < R_1\}$. This is made explicit in the next theorem.

Theorem 4.1.4. *Suppose f is a holomorphic function on $D = \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$. Then f is equal to its Laurent series expansion about z_0 which can be written*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}$$

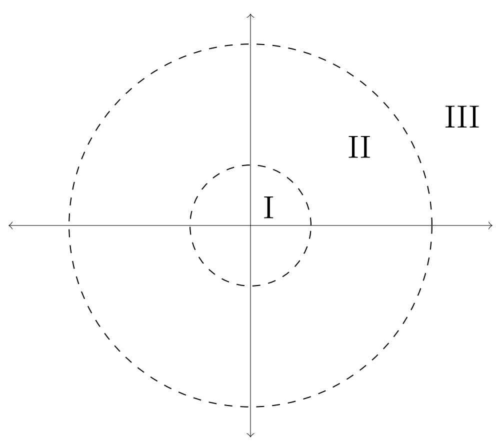
$$\text{where} \qquad a_k = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta \quad \text{and} \quad b_k = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta-z_0)^{-k+1}}\,d\zeta$$

for circles C_1 and C_2 centered at z_0 with radii R_1 and R_2, respectively.

Proof. Apply Cauchy's Theorem (3.4.1) and related results to both series. □

Remark. By the definition of their coefficients in terms of the integrals above, Laurent series expansions are unique.

Example 4.1.5. Consider $f(z) = \frac{1}{(z-1)(z-3)}$ on three different regions centered about the origin:



The three regions are given by I : $\{z \in \mathbb{C} : |z| < 1\}$, II : $\{z \in \mathbb{C} : 1 < |z| < 3\}$ and III : $\{z \in \mathbb{C} : |z| > 3\}$. We want to compute Laurent series for $f(z)$ in each of the regions. First we use partial fraction decomposition to write

$$\frac{1}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{1/2}{z-3}.$$

On various regions, we compute the following using geometric series:

$$
\begin{aligned}
-\frac{1}{2}\cdot\frac{1}{z-1} &= \frac{1}{2}\cdot\frac{1}{1-z} = \frac{1}{2}\sum_{n=0}^{\infty} z^n, \quad |z|<1 \\
\frac{1}{2}\cdot\frac{1}{z-3} &= -\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{1-{}^2\!/_3} = -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \quad |z|<3 \\
-\frac{1}{2}\cdot\frac{1}{z-1} &= -\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-{}^1\!/_z} = -\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty} z^{-n}, \quad |z|>1 \\
\frac{1}{2}\cdot\frac{1}{z-3} &= \frac{1}{2}\cdot\frac{1}{z}\frac{1}{1-{}^3\!/_z} = \frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n, \quad |z|>3.
\end{aligned}
$$

Putting these together into Laurent series on each region, we have

$$
\begin{aligned}
\mathrm{I}: f(z) &= \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{6}\cdot\frac{1}{3^n}\right) z^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2}(1-3^{-n-1})z^n \\
\mathrm{II}: f(z) &= -\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \\
&= \sum_{n=0}^{\infty} -\frac{1}{2}3^{-n-1}z^n + \sum_{n=0}^{\infty} -\frac{1}{2}z^{-n-1} \\
\mathrm{III}: f(z) &= -\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty} 3^n z^{-n} \\
&= \sum_{n=0}^{\infty}\frac{1}{2}(3^n-1)z^{-n-1}.
\end{aligned}
$$

In these we see examples of a Laurent series that is a Taylor series (I), corresponding to a disk on which $f(z)$ is holomorphic; a Laurent series with both positive and negative parts (II), which is holomorphic on an annulus; and a Laurent series with only negative part (III), holomorphic on the complement of a disk.

Laurent series give us a way to deal with 'holes' in the domain of a function which is otherwise holomorphic on the region. Such functions have a special name:

Definition. *A complex function $f(z)$ is* **meromorphic** *on a domain D if it is holomorphic on $D \setminus \{z_1, z_2, \ldots, z_r\}$ where r is finite.*

4.2 Isolated Singularities

A singularity is the name we give to a 'hole' in the domain of a complex function. Below we describe the three different types of singularities a function may have.

Definition. *If $f(z)$ is holomorphic on the punctured disk $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ (R may be infinite) but not at z_0 then z_0 is called an* **isolated singularity** *of f. The three types of isolated singularities are*

(a) z_0 is a **removable singularity** *if there is a function g which is holomorphic on the disk $D \cup \{z_0\} = \{z \in \mathbb{C} : |z - z_0| < R\}$ such that $f(z) = g(z)$ for all $z \in D$.*

(b) z_0 is a **pole** *if $\lim_{z \to z_0} |f(z)| = \infty$. In particular, z_0 is a* **pole of order m** *if z_0 is a root of $\frac{1}{f(z)}$ with multiplicity m. Equivalently, m is the smallest integer such that $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$.*

(c) z_0 is an **essential singularity** *if it is neither removable nor a pole.*

The isolated singularities of a function may be characterized in terms of Laurent series expansions of the function.

Proposition 4.2.1. *Let z_0 be an isolated singularity of $f(z)$ and suppose $f(z)$ has a Laurent series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

in the region $0 < |z - z_0| < R$.

(a) z_0 is a removable singularity if and only if $b_n = 0$ for all n and there is a function g,

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ a_0 & z = z_0, \end{cases}$$

which is analytic in $|z - z_0| < R$.

(b) z_0 is a pole of $f(z)$ if and only if all but a finite number of the b_n vanish. Specifically, if $b_n = 0$ for all $n > m$ then z_0 is a pole of order m and f can be written

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \ldots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(c) z_0 is an essential singularity if and only if infinitely many of the b_n are nonzero.

Examples. We examine functions with each type of isolated singularity.

① The function $f(z) = \frac{\sin z}{z}$ has a Laurent series which is a Taylor series:

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$$

on the region $\{z \in \mathbb{C} : |z| > 0\}$. Therefore $z_0 = 0$ is removable and the function g that removes the singularity is

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

Note that $g(z)$ is analytic everywhere; it is an entire function. This shows that $f(z)$ is meromorphic on $\mathbb{C} \smallsetminus \{0\}$.

② Consider $f(z) = \frac{\sin z}{z^4}$ whose Laurent series is given by

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-3} = \frac{1}{z^3} - \frac{1/6}{z} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+5)!}z^{2n+1}.$$

This shows that $z_0 = 0$ is a pole of order 3. Moreover, by Theorem 4.1.4 we can use the coefficients of the Laurent series to integrate $f(z)$ around some contour C containing $z_0 = 0$ on its interior:

$$\int_C \frac{\sin z}{z^4}\,dz = b_1 \cdot 2\pi i = \left(-\frac{1}{6}\right)2\pi i = -\frac{\pi i}{3}.$$

③ $\sin\left(\frac{1}{z}\right), \cos\left(\frac{1}{z}\right)$ and $e^{1/z}$ are all functions with essential singularities at $z_0 = 0$. For example, consider the Laurent series expansion of $f(z) = e^{1/z}$:

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = 1 + z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{6}z^{-3} + \dots .$$

Although there is not a nice extension of $e^{1/z}$ to an analytic function about $z_0 = 0$, we can still use the b_1 coefficient of its Laurent series to compute contour integrals:

$$\int_C e^{1/z}\,dz = b_1 \cdot 2\pi i = (1)2\pi i = 2\pi i.$$

The next result is rather neat. It says that if $f(z)$ has an essential singularity at z_0 then the image $f(D)$ of any disk D centered at z_0 is dense in $\mathbb{C}$ (in the topological sense).

Theorem 4.2.2 (Casorati-Weierstrass)**.** *If z_0 is an essential singularity of $f(z)$ and $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some positive R, then for any $z \in \mathbb{C}$ and $\varepsilon > 0$, there is some $z' \in D$ such that $|z - f(z')| < \varepsilon$.*

Proof. To contradict, suppose there is some $z \in \mathbb{C}$ and an $\varepsilon > 0$ such that for all $z' \in D$, $|z - f(z')| \geq \varepsilon$. Then $g(z) = \frac{1}{f(z')-z}$ is bounded as $z \to z_0$, so

$$\lim_{z\to z_0} (z - z_0)g(z) = \lim_{z\to z_0} \frac{z - z_0}{f(z') - z} = 0.$$

By Proposition 4.2.1, g has a removable singularity at z_0, and therefore

$$\lim_{z\to z_0} \left| \frac{f(z') - z}{z - z_0} \right| = \infty.$$

This implies that $\frac{f(z')-z}{z-z_0}$ has a pole at z_0, say of order m. By definition,

$$\lim_{z\to z_0} (z - z_0)^{m+1} \frac{f(z') - z}{z - z_0} = \lim_{z\to z_0} (z - z_0)^n (f(z') - z) = 0.$$

Finally, this shows that $f(z') - z$ has a pole or removable singularity at z_0 which implies the same of $f(z)$, but this cannot be the case since z_0 was essential. Hence $f(D)$ must be dense. □

4.3 The Residue Theorem

The examples in Section 4.2 illustrate the connection between the coefficients of the negative part of the Laurent series of a function and contour integrals of the function about its singularities. The coefficient b_1 is of particular importance, so much so that it has a special name.

Definition. *Let z_0 be an isolated singularity of $f(z)$. The* **residue** *of f at z_0 is*

$$Res(f; z_0) := \frac{1}{2\pi i} \int_C f(z)\, dz$$

where $C : |z - z_0| = r$ for some $0 < r < R$, the radius of convergence of the Laurent series for f. This is in turn equal to the b_1 coefficient of the Laurent series.

There is a nice formula for the residues of removable singularities and poles.

Proposition 4.3.1. *Suppose z_0 is a nonessential singularity of $f(z)$.*

(a) If z_0 is a removable singularity, $Res(f; z_0) = 0$.

(b) If z_0 is a pole of order m, then

$$Res(f; z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z).$$

Proof. (a) follows from Cauchy's Theorem (3.4.1), and (b) is a simple application of Taylor's Theorem to the series

$$(z - z_0)^m f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^{n+m}.$$

The formula for $\text{Res}(f; z_0)$ follows from the identification of the residue and b_1. □

Example 4.3.2. Let $f(z) = \dfrac{e^z}{z^2(z-i\pi)^4}$. Then f has a pole of order 2 at $z_0 = 0$, so we define $g(z) = z^2 f(z)$ which is analytic on a small enough neighborhood of 0 (so that it avoids $i\pi$). By Proposition 4.3.1,

$$\operatorname{Res}(f;0) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}g(z) = \lim_{z\to 0} g'(z).$$

The first derivative of g is

$$\begin{aligned} g'(z) &= \frac{e^z(z-i\pi)^4 - e^z\cdot 4(z-i\pi)^3}{(z-i\pi)^8} \\ &= \frac{e^z(z-i\pi)^3(z-i\pi-4)}{(z-i\pi)^8} \\ &= \frac{e^z(z-i\pi-4)}{(z-i\pi)^5}. \end{aligned}$$

Then the formula for the residue above allows us to compute

$$\operatorname{Res}(f;0) = \lim_{z\to 0}\frac{e^z(z-i\pi-4)}{(z-i\pi)^5} = \frac{-i\pi-4}{(-i\pi)^5} = \frac{1}{\pi^4} + \frac{4}{i\pi^5}.$$

Proposition 4.3.3. *Suppose f and g are analytic on $|z-z_0| < r$ for some $z_0 \in \mathbb{C}$ and $r > 0$, and suppose $g(z_0) = 0$ but $g'(z_0) \neq 0$. Then*

$$Res\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. Let $g(z)$ have the following power series centered at z_0 (by assumption the series has no c_0 coefficient):

$$g(z) = \sum_{k=1}^{\infty} c_k(z-z_0)^k = (z-z_0)\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

where $a_k = c_{k-1}$; call the analytic function represented by this new series $h(z)$. Note that $h(z_0) = c_1 \neq 0$, so

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)h(z)}$$

and $\frac{f}{h}$ is analytic at z_0. Using the definition of residue in terms of the Laurent series coefficients, the residue of $\frac{f}{g}$ is equal to the constant term of the series for $\frac{f}{h}$ (the $n = -1$ term of the series for $\frac{f}{g}$). This is computed to be $\frac{f(z_0)}{h(z_0)}$, but by the way we defined h, $h(z_0) = g'(z_0)$. Hence

$$\operatorname{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

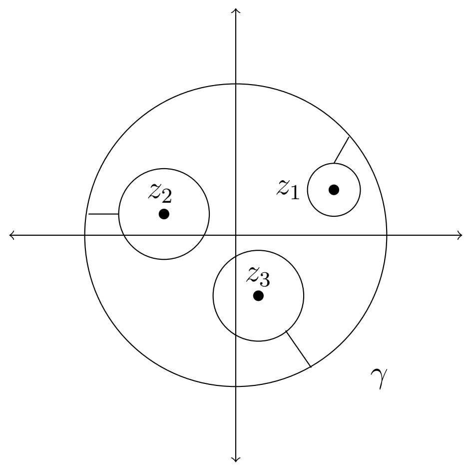
□

We finally arrive at the central theorem in basic complex analysis: the Residue Theorem.

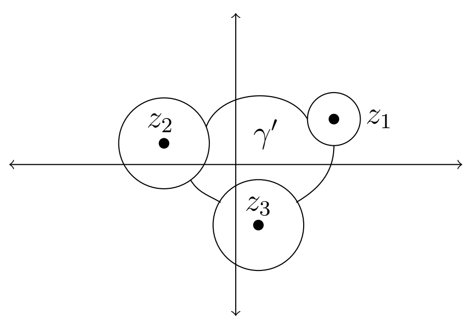
Theorem 4.3.4 (The Residue Theorem)**.** *Suppose $f(z)$ is meromorphic on a region D; let $z_1, \ldots, z_n$ be the isolated singularties of f inside D. If γ is a piecewise smooth, positively oriented, simple closed curve lying in D that does not pass through any of the z_i then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} Res(f; z_i).$$

Proof. Draw a positively-oriented circle C_i around each singularity z_i such that z_i is the only singularity of f on its interior. The case where $n = 3$ is illustrated below.



Then γ is contractible to a curve γ' which connects the C_i together and otherwise contains no singularities on its interior. Such a contraction is shown in the next figure.

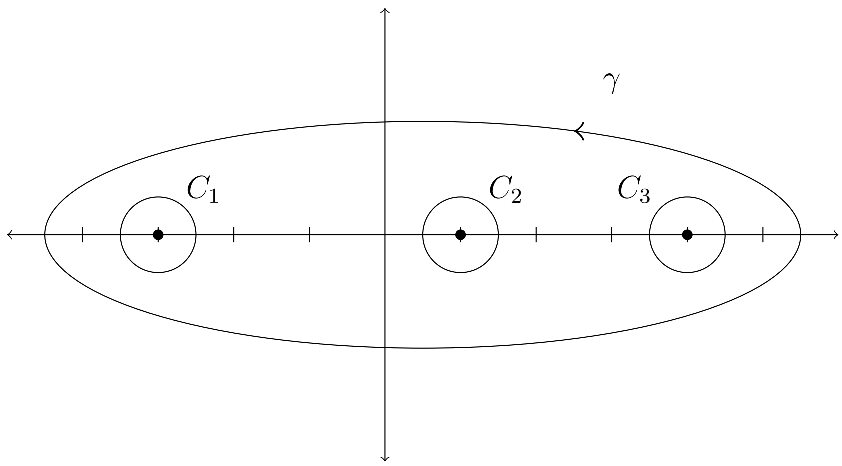


Then $\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$ but by construction, $f(z)$ is holomorphic on the interior of γ', so by Cauchy's Theorem (3.4.1) this part equals 0. Evaluate the remaining terms using the definition of residue to produce the main summation formula:

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = \sum_{i=1}^{n} 2\pi i\,\mathrm{Res}(f; z_i).$$

□

Example 4.3.5. Evaluate $\displaystyle\int_\gamma \frac{z^2-2z+1}{(z-1)(z-4)(z+3)}\,dz$ about the given contour.



Set $f(z) = \dfrac{z^2-2z+1}{(z-1)(z-4)(z+3)}$. By the Residue Theorem we may evalute the integral of f over γ as

$$\int_\gamma \frac{z^2-2z+1}{(z-1)(z-4)(z+3)}\,dz = 2\pi i(\mathrm{Res}(f;-3)+\mathrm{Res}(f;1)+\mathrm{Res}(f;4)).$$

First, note that the function $g(z) = \frac{z-1}{(z-4)(z+3)}$ is holomorphic on C_2 and $f(z) = g(z)$ on the interior of C_1 minus 1. Thus $z_2 = 1$ is a removable singularity, so by Proposition 4.3.1, $\mathrm{Res}(f;1) = 0$. Next, it is easy to see that $z_1 = -3$ and $z_3 = 4$ are both simple poles, so we compute their residues using the pole formula (Proposition 4.3.1):

$$\mathrm{Res}(f;-3) = \lim_{z\to-3}(z+3)f(z) = \lim_{z\to-3}\frac{z^2-2z+1}{(z-1)(z-4)} = \frac{4}{7}$$
$$\mathrm{Res}(f;4) = \lim_{z\to4}(z-4)f(z) = \lim_{z\to4}\frac{z^2-2z+1}{(z-1)(z+3)} = \frac{3}{7}.$$

Putting this together, we have

$$\int_\gamma \frac{z^2-2z+1}{(z-1)(z-4)(z+3)}\,dz = 2\pi i\left(\frac{4}{7}+0+\frac{3}{7}\right) = 2\pi i.$$

4.4 Some Fourier Analysis

Techniques in Fourier analysis are vital in many areas of mathematics and the physical sciences, especially when signal or wave data needs to be broken down into simple components. By studying heat diffusion and wave equations, Joseph Fourier discovered that every continuous function can be approximated with arbitrarily small error by a series of the form

$$\sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

Extending these functions to the complex plane, we can take advantage of Euler's formula $e^{it} = \cos t + i \sin t$.

Definition. *Let f be an integrable, complex-valued function defined on $\mathbb{R}$ (or defined on $\mathbb{C}$ but restricted to $\mathbb{R}$ for this section). The* **Fourier transform** *of F is*

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{2\pi iwx}\,dx.$$

The reason the Residue Theorem (4.3.4) is important to the study of Fourier series becomes evident in the next theorem.

Theorem 4.4.1. *Let $f(z)$ be analytic on the half-plane $\mathfrak{H} : \mathrm{Im}(z) \geq 0$ except possibly at a finite number of singularities $\{z_1, \ldots, z_n\}$, all of which have positive imaginary part. Suppose $|f(z)|$ gets arbitrarily small for all $z \in \mathfrak{H}$ with sufficiently large modulus, i.e.*

$$\lim_{R\to\infty} \max_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} |f(z)| = 0.$$

Then for all real numbers $w > 0$,

$$\hat{f}(w) = 2\pi i \sum_{j=0}^{n} Res(f(z)e^{2\pi iwz}; z_j).$$

Similarly, if all of the above conditions hold for the negative half-plane $\mathfrak{H}' : \mathrm{Im}(z) \leq 0$, then

$$\hat{f}(w) = -2\pi i \sum_{j=1}^{n} Res(f(z)e^{2\pi iwz}; z_j).$$

Example 4.4.2. Consider the real-valued function $f(x) = \dfrac{1}{1+x^2}$. We can extend this to a complex function $f(z) = \dfrac{1}{1+z^2}$ which clearly satisfies

$$\lim_{R\to\infty} \max_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} |f(z)| = 0.$$

Then Theorem 4.4.1 above tells us we can compute the Fourier transform of $f(x)$ in terms of residues of $f(z)$: $\hat{f}(w) = 2\pi i\,\mathrm{Res}(f(z)e^{2\pi iwz}; i)$. Since $f(z)$ only has a simple pole in the upper half-plane at $z_0 = i$, we use Proposition 4.3.1 to compute this residue:

$$\mathrm{Res}(f(z)e^{2\pi iwz}; i) = \lim_{z\to i}(z-i)\frac{e^{2\pi iwz}}{(z-i)(z+i)} = \lim_{z\to i}\frac{e^{2\pi iwz}}{z+i} = \frac{e^{-2\pi w}}{2i}.$$

Hence the Fourier transform of $f(z)$ is $\hat{f}(w) = \pi e^{-2\pi w}$.

4.5 The Argument Principle

5 Complex Mappings

5.1 Möbius Transformations

5.2 Conformal Mappings

5.3 The Riemann Mapping Theorem

6 Riemann Surfaces

Riemann surfaces are a mix of the topology of covering spaces and the complex analysis of analytic continuation. The main problem one encounters in the latter setting is that a holomorphic function does not always admit a uniquely defined analytic continatuion. The normal strategy then is to employ 'branch cuts', but this tactic seems ad hoc and not suited to generalization. Riemann's idea was to replace the branches of a function with a covering space on which the analytic continuation is an actual function.

6.1 Holomorphic and Meromorphic Maps

Definition. *Let X be a surface, i.e. a two-dimensional manifold. A* **complex atlas** *on X is a choice of open covering $\{U_i\}$ of X together with homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{C}$ such that for each pair of overlapping charts U_i, U_j, the transition map*

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

and its inverse are holomorphic. A **complex structure** *on X is the choice of a complex atlas, up to holomorphic equivalence of charts, defined by a similar condition to the above. A connected surface which admits a complex structure is called a* **Riemann surface***.*

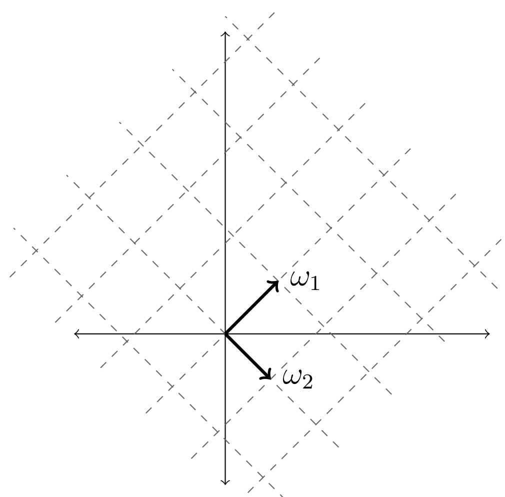
Example 6.1.1. The complex plane $\mathbb{C}$ is a trivial Riemann surface. Any connected open subset U in $\mathbb{C}$ is also a Riemann surface via the given embedding $U \hookrightarrow \mathbb{C}$.

Example 6.1.2. The complex projective line $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ admits a complex structure defined by the open sets $U_0 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ and $U_1 = \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^\times \cup \{\infty\}$, together with charts

$$\varphi_0 : U_0 \to \mathbb{C}, z \mapsto z \quad \text{and} \quad \varphi_1 : U_1 \to \mathbb{C}, z \mapsto \frac{1}{z},$$

where $\frac{1}{\infty} = 0$ by convention. Note that $\varphi_1 \circ \varphi_0^{-1}$ is the function $z \mapsto \frac{1}{z}$ on $\mathbb{C}^\times$ which is holomorphic.

Example 6.1.3. Let $\Lambda \subseteq \mathbb{C}$ be a lattice with basis $[\omega_1, \omega_2]$.



Then the quotient $\mathbb{C}/\Lambda$ admits a complex structure as follows. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map and suppose $\Pi \subseteq \mathbb{C}$ is a *fundamental domain* for Λ, meaning no two points in Π are equivalent mod Λ. Set $U = \pi(\Pi) \subseteq \mathbb{C}/\Lambda$. Then $\pi|_\Pi : \Pi \to U$ is a homeomorphism, so let $\varphi : U \to \Pi$ be its inverse. Letting $\{U_i\}$ be the collection of all images under π of fundamental domains for Λ, we get a complex atlas on $\mathbb{C}/\Lambda$ (one can easily check that the transition functions between the U_i are locally constant, hence holomorphic). Topologically, $\mathbb{C}/\Lambda$ is homeomorphic to a torus.

Definition. *A function $f : U \to \mathbb{C}$ on an open subset U of a Riemann surface X is* **holomorphic** *if for every complex chart $\varphi : V \to \varphi(V) \subseteq \mathbb{C}$, the function $f \circ \varphi^{-1} : \varphi(U \cap V) \to U \cap V \to \mathbb{C}$ is holomorphic.*

Let $\mathcal{O}(U)$ denote the set of all holomorphic functions $U \to \mathbb{C}$.

Lemma 6.1.4. *For any open set U of a Riemann surface X, $\mathcal{O}(U)$ is a commutative $\mathbb{C}$-algebra.*

Proposition 6.1.5 (Holomorphic Continuation)**.** *For any open set $U \subseteq X$ of a Riemann surface and any $x \in U$, if $f \in \mathcal{O}(U \setminus \{x\})$ is bounded in a neighborhood of x, then f extends uniquely to some $\tilde{f} \in \mathcal{O}(U)$.*

More generally, we can define holomorphic maps between two Riemann surfaces.

Definition. *A continuous map $f : X \to Y$ between Riemann surfaces is called* **holomorphic** *if for every pair of charts $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ on X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ on Y such that $f(U) \subseteq V$, the map*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to U \to V \to \psi(V)$$

is holomorphic. We say f is **biholomorphic** *if it is a bijection and its inverse f^{-1} is also holomorphic. In this case X and Y are said to be* **isomorphic as Riemann surfaces***.*

Lemma 6.1.6. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are holomorphic maps between Riemann surfaces, then $g \circ f : X \to Z$ is also holomorphic.*

Proposition 6.1.7. *Let $f : X \to Y$ be a holomorphic map. Then for all open $U \subseteq X$, there is an induced $\mathbb{C}$-algebra homomorphism*

$$\begin{aligned} f^* : \mathcal{O}(U) &\longrightarrow \mathcal{O}(f^{-1}(U)) \\ \psi &\longmapsto f^*\psi := \psi \circ f. \end{aligned}$$

Proof. The fact that $f^*\psi$ is an element of $\mathcal{O}(f^{-1}(U))$ follows from the above definitions of $\mathcal{O}$ and a holomorphic map between Riemann surfaces. The ring axioms are also easy to verify. □

Theorem 6.1.8. *Suppose $f, g : X \to Y$ are holomorphic maps between Riemann surfaces such that there exist a set $A \subseteq X$ containing a limit point $a \in A$ and $f|_A = g|_A$. Then $f = g$.*

Proof. Let $U \subseteq X$ be the set of all $x \in X$ with an open neighborhood W on which $f|_W = g|_W$. Then U is open and $a \in U$; we will show it is also closed. If $x \in \partial U$, we have $f(x) = g(x)$ since f and g are continuous. Choose a neighborhood $W \subseteq X$ of x and charts $\varphi : W \to \varphi(W) \subseteq \mathbb{C}$ and $\psi : W' \to \psi(W') \subseteq \mathbb{C}$ in Y with $f(W) \subseteq W'$ and $g(W) \subseteq W'$. Consider

$$F = \psi \circ f \circ \varphi^{-1} : \varphi(W) \to \psi(W') \quad \text{and} \quad G = \psi \circ g \circ \varphi^{-1} : \varphi(W) \to \psi(W').$$

Then F and G are holomorphic and $W \cap U \neq \varnothing$, so we must have $F = G$. Therefore $f|_W = g|_W$, so $x \in U$ after all. This implies $U = X$. □

Definition. *A* **meromorphic function** *on an open set $U \subseteq X$ consists of an open subset $V \subseteq U$ and a holomorphic function $f : V \to \mathbb{C}$ such that $U \setminus V$ contains only isolated points, called the* **poles** *of f, and $\lim_{x\to p} |f(x)| = \infty$ for every pole $p \in U \setminus V$.*

Denote the set of meromorphic functions on U by $\mathcal{M}(U)$. Then $\mathcal{M}(U)$ is a $\mathbb{C}$-algebra, where $f + g$ and fg are defined by meromorphic continuation.

Example 6.1.9. Any polynomial $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ is a holomorphic function $\mathbb{C} \to \mathbb{C}$. Viewing $\mathbb{C} \subseteq \mathbb{P}^1$, f is a meromorphic function on $\mathbb{P}^1$ with only a pole at ∞ of order n (assuming $c_n \neq 0$).

Example 6.1.10. Any meromorphic function $f \in \mathcal{M}(X)$ may be represented by a Laurent series expansion about any of its poles p by choosing a complex chart $U \to \mathbb{C}$ containing p, lifting z to a parameter t on U and writing

$$f(t) = \sum_{n=-N}^{\infty} c_n t^n \text{for some } c_n \in \mathbb{C}.$$

Theorem 6.1.11. *Suppose X is a Riemann surface. Then the set of meromorphic functions $\mathcal{M}(X)$ is in bijection with the set of holomorphic maps $X \to \mathbb{P}^1$.*

Proof. If $f \in \mathcal{M}(X)$ is a meromorphic function, then setting $f(p) = \infty$ for every pole p of f defines a holomorphic map $f : X \to \mathbb{P}^1$. Indeed, it is clear that f is continuous. Let P be the set of its poles. If $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ is a chart on X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ is a chart on $\mathbb{P}^1$ with $f(U) \subseteq V$, then since f is holomorphic on $X \setminus P$, $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic on $\varphi(U) \setminus \varphi(P)$. By Proposition 6.1.5, $\psi \circ f \circ \varphi^{-1}$ is actually holomorphic on $\varphi(U)$, so f is a holomorphic map of Riemann surfaces.

Conversely, if $g : X \to \mathbb{P}^1$ is holomorphic, then by Theorem 6.1.8, either $g(X) = \{\infty\}$ or $g^{-1}(\infty)$ is a set of isolated points in X. It is then easy to see that $g : X \setminus g^{-1}(\infty) \to \mathbb{C}$ is meromorphic. □

Corollary 6.1.12 (Meromorphic Continuation)**.** *For any open set $U \subseteq X$ and any $x \in U$, if $f \in \mathcal{M}(U \setminus \{x\})$ is bounded in a neighborhood of x, then f extends uniquely to some $\tilde{f} \in \mathcal{M}(U)$.*

Proof. Apply Proposition 6.1.5 and Theorem 6.1.11. □

Corollary 6.1.13. *Any nonzero function in $\mathcal{M}(X)$ has only isolated zeroes. In particular, $\mathcal{M}(X)$ is a field.*

Theorem 6.1.14. *Let $f : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then for every $x \in X$ with $y = f(x) \in Y$, there exists $k \in \mathbb{N}$ and complex charts $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ of X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ of Y with $f(U) \subseteq V$ such that*

(1) $x \in U$ with $\varphi(x) = 0$ and $y \in V$ with $\psi(y) = 0$.

(2) $F = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is given by $F(z) = z^k$ for all $z \in \varphi(U)$.

Proof. It is easy to arrange (1) by replacing (U, φ) with another chart obtained by composing φ with an automorphism of $\mathbb{C}$ taking $\varphi(x) \mapsto 0$. So without loss of generality assume (1) is satisfied. By Theorem 6.1.8, $F = \psi \circ f \circ \varphi^{-1}$ is nonconstant. Thus since $f(0) = 0$, we may write $F(z) = z^k g(z)$ for some $k \geq 1$ and some $g \in \mathcal{O}(\varphi(U))$ with $g(0) \neq 0$. Then $g(z) = h(z)^k$ for some holomorphic function h on $\varphi(U)$, and $H(z) = zh(z)$ defines a biholomorphic map α of some open neighborhood $W \subseteq \varphi(U)$ of 0 onto another open neighborhood of 0. Finally, replace (U, φ) by $(\varphi^{-1}(W), \alpha \circ \varphi)$. By construction, $F = \psi \circ f\varphi^{-1}$ is now of the form $F(z) = z^k$. □

Definition. *The integer k for which F can be written $F(z) = z^k$ about $x \in X$ is called the* **multiplicity** *of f at x.*

Corollary 6.1.15. *If $f : X \to Y$ is a nonconstant holomorphic map between Riemann surfaces, then f takes open sets to open sets.*

Corollary 6.1.16. *If $f : X \to Y$ is an injective holomorphic map, then f is biholomorphic $X \to f(X)$.*

Proof. If f is injective, then locally $F(z) = z^k$ with $k = 1$. Hence f^{-1} is holomorphic. □

Corollary 6.1.17 (Maximum Principle)**.** *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a nonconstant holomorphic function. Then $|f|$ does not attain its maximum.*

Proof. Suppose $x_0 \in X$ exists such that $|f(x_0)| = \sup\{|f(x)| : x \in X\}$. Set

$$D = \{z \in \mathbb{C} : |z| \leq |f(x_0)|\}$$

so that $f(x_0)$ lies in the boundary of D. Then $f(X) \subseteq D$, but by Corollary 6.1.15, $f(X)$ is open in D, contradicting $f(x_0) \in \partial D$. □

Theorem 6.1.18. *If $f : X \to Y$ is a nonconstant holomorphic map and X is compact, then Y is also compact and f is surjective.*

Proof. By Corollary 6.1.15, $f(X)$ is open but since X is compact, $f(X)$ is also compact and in particular closed. Therefore $f(X) = Y$. □

Corollary 6.1.19 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ with $c_i \in \mathbb{C}$ has a root.*

Proof. Such an f extends to a holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ by setting $f(\infty) = \infty$. Since $\mathbb{P}^1$ is compact, Theorem 6.1.18 says f is surjective, so $f(z) = 0$ for some $z \in \mathbb{C}$. □

Corollary 6.1.20. *Every holomorphic function on a compact Riemann surface is constant.*

Proof. $\mathbb{C}$ is not compact, so Theorem 6.1.18 implies that every holomorphic function from a compact space into $\mathbb{C}$ must be constant. □

Corollary 6.1.21. *Every meromorphic function on $\mathbb{P}^1$ is rational.*

Proof. First, note that the only way for such an $f \in \mathcal{M}(\mathbb{P}^1)$ to have infinitely many poles is if it had a limit point, but then Theorem 6.1.8 would imply $f \equiv \infty$. Thus f has finitely many poles, say $a_1, \ldots, a_n \in \mathbb{P}^1$; we may assume ∞ is not one of the poles, or else consider the function $\frac{1}{f}$ instead. For $1 \leq i \leq n$, expand f as a Laurent series about a_i:

$$f_i(z) = \sum_{j=1}^{m_i} c_{ij}(z - a_i)^{-j} \quad \text{for } c_{ij} \in \mathbb{C}.$$

Then $g = f - (f_1 + \ldots + f_n)$ is holomorphic on $\mathbb{P}^1$ and thus constant by Corollary 6.1.20 since $\mathbb{P}^1$ is compact. This shows f is rational. □

Corollary 6.1.20 gives another proof of Liouville's Theorem (3.6.9):

Corollary 6.1.22 (Liouville's Theorem)**.** *Every bounded holomorphic function on $\mathbb{C}$ is constant.*

Proof. By Proposition 6.1.5, f has a holomorphic continuation to $\tilde{f} : \mathbb{P}^1 \to \mathbb{C}$, but by Corollary 6.1.20, $\tilde{f}$ must be constant. □

6.2 Covering Spaces

The idea in this section is to relate holomorphic maps between Riemann surfaces to covering space theory. Recall the following definition from topology.

Definition. *A map $p : Y \to X$ between connected, Hausdorff spaces is a* **covering map** *if each point $x \in X$ has a neighborhood U such that $p^{-1}(U) \subseteq Y$ is a nonempty disjoint union $p^{-1}(U) = \coprod U_\alpha$ such that the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for each U_α. Such a neighborhood U is called an* **evenly covered** *neighborhood of x, and the U_α are called the* **sheets** *of the cover over x. The domain space Y is called a* **covering space** *of X.*

Theorem 6.2.1. *If $p : Y \to X$ is a nonconstant holomorphic map between Riemann surfaces then p is open and has discrete fibres.*

Proof. By Corollary 6.1.15, p is open and Theorem 6.1.8 implies each fibre is discrete. □

Let $p : Y \to X$ be a cover of Riemann surfaces. Traditionally, holomorphic functions $f : Y \to \mathbb{C}$ are treated as multi-valued functions on X by setting $f(x) = \{f(y_1), \ldots, f(y_n)\}$ where $p^{-1}(x) = \{y_1, \ldots, y_n\}$.

Example 6.2.2. Let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the exponential map $z \mapsto e^z$ and $f = id : \mathbb{C} \to \mathbb{C}$ the identity map. Then the resulting multi-valued function $\mathbb{C}^\times \to \mathbb{C}$ is the complex logarithm, which is only defined as a function after making a particular choice of *branch* of the function. We can describe this idea more cleanly with Riemann surfaces and branched covers.

Definition. *Suppose $p : Y \to X$ is a nonconstant holomorphic map. A* **ramification point** *of p is a point $y \in Y$ such that for every neighborhood $V \subseteq Y$ of y, $p|_V : V \to p(V)$ is not injective. The image $x = p(y)$ is called a* **branch point** *of p. If p has no ramification points (and hence no branch points), then we call p an* **unramified map**.

Theorem 6.2.3. *A nonconstant holomorphic map $p : Y \to X$ is unramified if and only if it is a local homeomorphism.*

Proof. Suppose p is unramified. Then for any $y \in Y$, there exists a neighborhood $V \subseteq Y$ of y such that $p|_V : V \to p(V)$ is injective and open. Therefore $p|_V$ is a homeomorphism onto $p(V)$. The converse follows from basically the same argument. □

Example 6.2.4. For each $n \geq 2$, the map $p_n : \mathbb{C} \to \mathbb{C}$ defined by $p_n(z) = z^n$ is ramified at $0 \in \mathbb{C}$ and unramified everywhere else. Therefore $p_n : \mathbb{C}^\times \to \mathbb{C}$ is an unramified cover. Moreover, Theorem 6.1.14 says that every ramified cover of Riemann surfaces $Y \to X$ is locally of the form $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$.

Example 6.2.5. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$ is an unramified cover. In fact, as in the topological case, exp gives a universal cover of $\mathbb{C}$ via the inverse system of the covers p_n.

Example 6.2.6. The quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ from Example 6.1.3 is an unramified cover of Riemann surfaces.

Theorem 6.2.7. *Suppose $p : Y \to X$ is a local homeomorphism of Hausdorff topological spaces and X is a Riemann surface. Then Y admits a unique complex structure making p a holomorphic map.*

Proof. Let $\varphi : V \to \mathbb{C}$ be a chart of X. Then there exists an open subset $U \subseteq V$ over which $p|_U : p^{-1}(U) \to U$ is a homeomorphism. Set $\widetilde{U} = p^{-1}(U)$ and $\widetilde{\varphi} = \varphi \circ p|_U : \widetilde{U} \to \mathbb{C}$. Then $\widetilde{\varphi}$ is a complex chart on Y and the collection $\{\widetilde{U}, \widetilde{\varphi}\}$ obtained in this way forms a complex atlas on Y. Since $p : Y \to X$ is locally biholomorphic by construction, it is a holomorphic map between Riemann surfaces. Uniqueness is easy to check. □

Example 6.2.8. Now that we can view nonconstant holomorphic maps as local homeomorphisms, and in most cases covering spaces, we can rephrase the language of branch cuts as a lifting problem. For example, let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the exponential map and suppose $f : X \to \mathbb{C}^\times$ is a holomorphic map of Riemann surfaces, with X simply connected. Then by covering space theory, for each fixed $x_0 \in X$ and $z_0 \in \mathbb{C}$ such that $f(x_0) = e^{z_0}$, there exists a unique lift $F : X \to \mathbb{C}$ making the diagram

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{F}{\nearrow} & \big\downarrow {\scriptstyle \exp} \\ X & \xrightarrow[f]{} & \mathbb{C}^\times \end{array}$$

commute. Theorem 6.2.7 can be used to show that any such F is holomorphic. Moreover, any other lift G of f differs from F by $2\pi i n$ for some $n \in \mathbb{Z}$. For the special case of a simply connected open set $X \subseteq \mathbb{C}^\times$, any lift F is a branch of the complex logarithm on X.

Example 6.2.9. Similarly, one can construct the complex root functions $z \mapsto z^{1/n}, n \geq 2$, as lifts along the cover $p_n : \mathbb{C}^\times \to \mathbb{C}$.

Let $f : Y \to X$ be a nonconstant holomorphic map that is *proper*, i.e. the preimage of any compact set in X is compact in Y. For each $x \in X$, define the *multiplicity* of f at x to be

$$\operatorname{ord}_x(f) = \sum_{y \in f^{-1}(x)} v_y(f)$$

where $v_y(f)$ is the multiplicity of f at y.

Example 6.2.10. If $f : Y \to X$ is unbranched at $x \in X$, then $p^{-1}(x) = \{y_1, \ldots, y_n\}$ for some n and $v_{y_i}(f) = 1$ for each $1 \leq i \leq n$. Thus $\operatorname{ord}_x(f) = n$.

Theorem 6.2.11. *If $f : Y \to X$ is a proper, nonconstant holomorphic map between Riemann surfaces, then there exists a number $n \in \mathbb{N}$ such that for every $x \in X$, $\operatorname{ord}_x(f) = n$.*

Proof. By Theorem 6.2.1, the set B of ramification points of f is a closed, discrete subset of Y. Let $A = f(B) \subseteq X$. Then since f is proper, A is also closed and discrete. The restriction $f|_{Y \smallsetminus B} : Y \smallsetminus B \to X \smallsetminus A$ is unramified, so it is a finite-sheeted covering space; say n is the number of sheets of $f|_{Y \smallsetminus B}$, i.e. the size of any fibre $f^{-1}(x)$ for an unbranched point $x \in X$. By the above example, f has multiplicity n at every $y \in Y \smallsetminus B$. Suppose $a \in A$ and write $f^{-1}(a) = \{b_1, \ldots, b_k\} \subseteq B$ and $m_i = v_{b_i}(f)$. For each $1 \leq i \leq k$, we may choose neighborhoods $V_i \subset Y$ of b_i and $U_i \subset X$ of a such that for all $x \in U_i \smallsetminus \{a\}$, $f^{-1}(x) \cap V_i$ consists of exactly m_i points. Then there is a neighborhood $U \subseteq U_1 \cap \cdots \cap U_k$ of a such that $f^{-1}(U) \subseteq V_1 \cup \cdots \cup U_k$ and for every $x \in U \cap (X \smallsetminus A)$, $f^{-1}(x)$ consists of exactly $m_1 + \ldots + m_k$ points. However we showed that $|f^{-1}(x)| = n$, so $n = m_1 + \ldots + m_k$ as required. □

Corollary 6.2.12. *Let X be a compact Riemann surface and $f : X \to \mathbb{C}$ a nonconstant meromorphic function. Then the number of zeroes of f equals the number of poles of f, counted with multiplicity.*

Proof. View f as a holomorphic function $X \to \mathbb{P}^1$. Since X and $\mathbb{P}^1$ are compact, f is a proper map so $\operatorname{ord}_0(f) = \operatorname{ord}_\infty(f)$. But $\operatorname{ord}_0(f)$ is precisely the number of zeroes of f, while $\operatorname{ord}_\infty(f)$ is the number of poles. □

Corollary 6.2.13. *Any complex polynomial $f(z) \in \mathbb{C}[z]$ of degree n has exactly n zeroes, counted with multiplicity.*

Proof. We may view f as a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$. Then it is easy to see $\operatorname{ord}_\infty(f) = n$, so once again $\operatorname{ord}_0(f) = n$. □

7 Elliptic Functions

In this chapter we review the classical theory of Jacobians for complex curves, starting with the construction and basic properties of elliptic functions, their connection to elliptic curves and their Jacobians, and then describing the construction in arbitrary dimension.

7.1 Elliptic Functions

Let $\Lambda \subseteq \mathbb{C}$ be a lattice, i.e. a free abelian subgroup of rank 2. Then Λ can be written

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \quad \text{for some } \omega_1, \omega_2 \in \mathbb{C} \text{ such that } \frac{\omega_1}{\omega_2} \notin \mathbb{R}.$$

Definition. *A function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *is* **doubly periodic** *with* **lattice of periods** Λ *if* $f(z + \ell) = f(z)$ *for all* $\ell \in \Lambda$ *and* $z \in \mathbb{C}$.

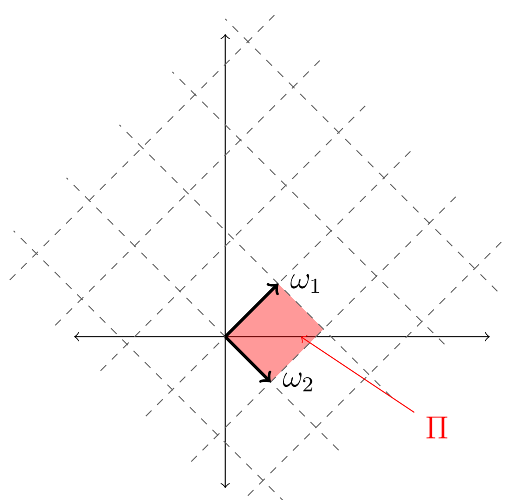
Definition. *An* **elliptic function** *is a function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *that is meromorphic and doubly periodic.*

It is not obvious that doubly periodic functions even exist! We will prove this shortly.

Definition. *Let* $\Lambda \subseteq \mathbb{C}$ *be a lattice. The set*

$$\Pi = \Pi(\omega_1, \omega_2) = \{t_1\omega_1 + t_2\omega_2 \mid 0 \leq t_i < 1\}$$

is called the **fundamental parallelogram***, or* **fundamental domain***, of* Λ*. We say a subset* $\Phi \subseteq \mathbb{C}$ *is* **fundamental** *for* Λ *if the quotient map* $\mathbb{C} \to \mathbb{C}/\Lambda$ *restricts to a bijection on* Φ*.*



Lemma 7.1.1. *For any choice of basis* $[\omega_1, \omega_2]$ *of* Λ*,* $\Pi(\omega_1, \omega_2)$ *is fundamental for* Λ*.*

Lemma 7.1.2. *Let* Λ *be a lattice. Then*

(a) *If* Π *is the fundamental domain of* Λ, *then for any* $\alpha \in \mathbb{C}$, $\Pi_\alpha := \Pi + \alpha$ *is fundamental for* Λ.

(b) *If* Φ *is fundamental for* Λ, *then* $\mathbb{C} = \bigcup_{\ell \in \Lambda} \Phi + \ell$.

Corollary 7.1.3. *Suppose* f *is an elliptic function with lattice of periods* Λ *and* Φ *fundamental for* Λ. *Then* $f(\mathbb{C}) = f(\Phi)$.

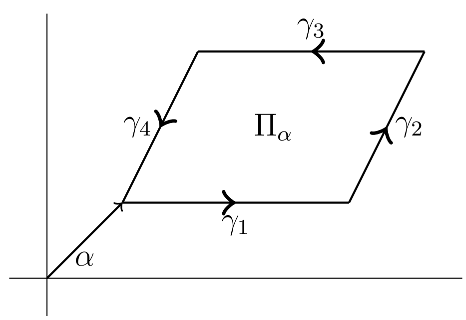
Proposition 7.1.4. *A holomorphic elliptic function is constant.*

Proof. Let f be such an elliptic function and let Φ be the fundamental domain for its lattice of periods. Then $\overline{\Pi}$ is compact and hence $f(\overline{\Pi})$ is as well. In particular, $f(\mathbb{C}) = f(\Pi) \subseteq f(\overline{\Pi})$ is bounded, so by Liouville's theorem, f is constant. □

Proposition 7.1.5. *Let* f *be an elliptic function. If* $\alpha \in \mathbb{C}$ *is a complex number such that* $\partial\Pi_\alpha$ *does not contain any of the poles of* f, *then the sum of the residues of* f *inside* $\partial\Pi_\alpha$ *equals 0.*

Proof. Fix a basis $[\omega_1, \omega_2]$ of Λ and set $\Delta = \partial\Pi_\alpha$. By the residue theorem, it's enough to show $\int_\Delta f(z)\,dz = 0$. We parametrize the boundary of Π as follows:

$$\begin{aligned}
\gamma_1 &= \alpha + t\omega_1 \\
\gamma_2 &= \alpha + \omega_1 + t\omega_2 \\
\gamma_3 &= \alpha + (1-t)\omega_1 + \omega_2 \\
\gamma_4 &= \alpha + (1-t)\omega_2.
\end{aligned}$$



We show that $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0$ and leave the proof that $\int_{\gamma_2} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 0$ for exercise. Consider

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz &= \int_0^1 f(\alpha + t\omega_1)(\omega_1\,dt) + \int_0^1 f(\alpha + (1-t)\omega_1 + \omega_2)(-\omega_1\,dt) \\
&= \omega_1 \int_0^1 f(\alpha + t\omega_1)\,dt + \omega_1 \int_1^0 f(\alpha + s\omega_1)\,ds \quad \text{since } f \text{ is elliptic} \\
&= \omega_1 \left(\int_0^1 f(\alpha + t\omega_1)\,dt - \int_0^1 f(\alpha + s\omega_1)\,ds \right) = 0.
\end{aligned}$$

Hence the sum of the residues equals 0. □

Corollary 7.1.6. *Any elliptic function has either a pole of order at least 2 or two poles on the fundamental domain of its lattice of periods.*

Proposition 7.1.7. *Suppose f is an elliptic function with fundamental domain Π and $\alpha \in \mathbb{C}$ such that $\Delta = \partial\Pi_\alpha$ does not contain any zeroes or poles of f. Let $\{a_j\}_{j=1}^n$ be a finite set of zeroes and poles in Π_α, with m_j the order of the pole a_j. Then $\sum_{j=1}^n m_j = 0$.*

Proof. For a pole z_0, we can write $f(z) = (z - z_0)^m g(z)$ for some holomorphic function $g(z)$, with $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = (z - z_0)^{-1}\left(m + (z - z_0)\frac{g'(z)}{g(z)}\right).$$

Hence $\mathrm{Res}\left(\frac{f'}{f}; z_0\right) = m$. Then the statement follows from Proposition 7.1.5. □

Proposition 7.1.7 has an analogue in algebraic geometry: if f is a rational function on an algebraic curve C, the formal sum $(f) = \sum m_j a_j$, where the a_j and m_j are defined as above, is called the *principal divisor* associated to f and its *degree* is $\deg(f) = \sum m_j$. Then one can prove that $\deg(f) = 0$.

Continuing in the complex setting, let f be an elliptic function and let $a_1, \ldots, a_r$ be the poles and zeroes of f in the fundamental domain of Λ. Write $\mathrm{ord}_{a_i} f = m_i$ if a_i is a pole of order $-m_i$ or if a_i is a zero of multiplicity m_i. The sum $\mathrm{ord}(f) = \sum_{i=1}^r m_i$ is called the *order* of f. Then Corollary 7.1.6 says that there are no elliptic functions of order 1. We will show that the field of elliptic functions with period lattice Λ is generated by an order 2 and an order 3 function.

Let f be elliptic and $z_0 \in \mathbb{C}$ with $\mathrm{ord}_{z_0} f = m$. Then for any $\ell \in \Lambda$, $\mathrm{ord}_{z_0+\ell} f = m$ as well. Indeed, if z_0 is a zero then

$$0 = f(z_0) = f(z_0) = \ldots = f^{(m-1)}(z_0)$$

but $f^{(k)}(z)$ is also elliptic for all $k \geq 1$. If z_0 is a pole of f, the same result can be obtained using $\frac{1}{f}$ instead of f.

If Φ_1 and Φ_2 are any two fundamental domains for Λ, then for all $a_1 \in \Phi_1$, there is a unique $a_2 \in \Phi_2$ such that $a_2 = a_1 + \ell$ for some $\ell \in \Lambda$. Thus Propositions 7.1.5 and 7.1.7 hold for any fundamental domain of Λ, so it follows that $\mathrm{ord}(f)$ is well-defined on the quotient $\mathbb{C}/\Lambda$.

Now given any meromorphic function $f(z)$ on $\mathbb{C}$, we would like to construct an elliptic function $F(z)$ with lattice Λ. Put

$$F(z) = \sum_{\ell \in \Lambda} f(z + \ell).$$

There are obvious problems of convergence and (in a related sense) the order of summation. It turns out we can do this construction with $f(z) = \frac{1}{z^m}, m \geq 3$ though. First, we need the following result, which can be proven using Cauchy's integral formula (3.5.1) and Morera's theorem (3.6.1).

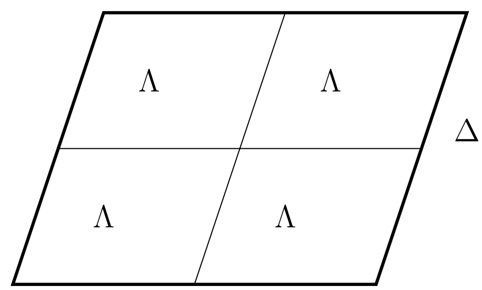
Lemma 7.1.8. *Let $U \subseteq \mathbb{C}$ be an open set and suppose (f_n) is a sequence of holomorphic functions on U such that $f_n \to f$ uniformly on every compact subset of U. Then f is holomorphic on U and $f_n' \to f'$ uniformly on every compact subset of U.*

Proposition 7.1.9. *Let Λ be a lattice with basis $[\omega_1, \omega_2]$. Then the sum*

$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^s}$$

converges for all $s > 2$.

Proof. Extend the fundamental domain by translation by the vectors ω_1, ω_2 and $\omega_1 + \omega_2$, and call the boundary of the resulting region Δ:



Then Δ is compact, so there exists $c > 0$ such that $|z| \geq c$ for all $z \in \Delta$. We claim that for all $m, n \in \mathbb{Z}$,

$$|m\omega_1 + n\omega_2| \geq c \cdot \max\{|m|, |n|\}.$$

The cases when $m = 0$ or $n = 0$ are trivial, so without loss of generality assume $m \geq n > 0$. Then

$$|m\omega_1 + n\omega_2| = |m| \left|\omega_1 + \frac{n}{m}\omega_2\right| \geq |m|c.$$

Hence the claim holds. Set $M = \max\{|m|, |n|\}$ and arrange the sum in question so that the $\frac{1}{|\omega|^s}$ are added in order of increasing M values. Then the sum can be estimated by

$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^s} \leq \sum_{M=1}^{\infty} \frac{8M}{c^s M^s} \sim \sum_{M=1}^{\infty} \frac{1}{M^{s-1}}.$$

This converges for $s > 2$ by p-series. □

Proposition 7.1.10. *Let $n \geq 3$ and define*

$$F_n(z) = \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^n}.$$

Then $F_n(z)$ is holomorphic on $\mathbb{C} \setminus \Lambda$ and has poles of order n at the points of Λ. Moreover, F_n is doubly periodic and hence elliptic.

Proof. Fix $r > 0$ and let $B_r = B_r(0)$ be the open complex r-ball centered at the origin in $\mathbb{C}$. Let $\Lambda_r = \Lambda \cap \overline{B}_r$ be the lattice points contained in the closed r-ball. Then the function

$$F_{n,r}(z) = \sum_{\omega \in \Lambda \smallsetminus \Lambda_r} \frac{1}{(z-\omega)^n}$$

is holomorphic on B_r. To see this, one has $\frac{1}{|z-\omega|^n} \leq \frac{C}{|\omega|^n}$ for some constant C and for all $z \in B_r, \omega \in \Lambda \smallsetminus \Lambda_r$. Then $\frac{C}{|\omega|^n}$ converges by Proposition 7.1.9, so by the Weierstrass M-test, $\frac{1}{|z-\omega|^n}$ converges uniformly and hence $F_{n,r}(z)$ is holomorphic. It follows from the definition that F_n has a pole of order n at each $\omega \in \Lambda$. Finally, for $\ell \in \Lambda$, we have

$$F_n(z+\ell) = \sum_{\omega \in \Lambda} \frac{1}{(z+\ell-\omega)^n} = \sum_{\eta \in \Lambda} \frac{1}{(z-\eta)^n} = F_n(z)$$

since the series is absolutely convergent and we can rearrange the terms. □

This shows that elliptic functions exist and more specifically that for each $n \geq 3$, there is at least one elliptic function of order n. Unfortunately the previous proof won't work to construct an elliptic function of order 3. However, Weierstrass discovered the following elliptic function.

Definition. *The* **Weierstrass $\wp$-function** *for a lattice Λ is defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{\omega^2}\right].$$

Theorem 7.1.11. *For any lattice Λ, $\wp(z)$ is an elliptic function with poles of order 2 at the points of Λ and no other poles. Moreover, $\wp(-z) = \wp(z)$ and $\wp'(z) = -2F_3(z)$.*

Proof. (Sketch) To show $\wp(z)$ is meromorphic, one estimates the summands by

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| \leq \frac{D}{|\omega|^3}$$

for some constant D and all $z \in B_r, \omega \in \Lambda \smallsetminus \Lambda_r$ as in the previous proof.

Next, $\wp(z)$ can be differentiated term-by-term to obtain the expression $\wp'(z) = -2F_3(z)$. And proving that $\wp(z)$ is odd is straightforward:

$$\begin{aligned}\wp(-z) &= \frac{1}{(-z)^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2}\right] \\ &= \frac{1}{z^2} + \sum_{-\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(z-(-\omega))^2} - \frac{1}{(-\omega)^2}\right] = \wp(z)\end{aligned}$$

after switching the order of summation.

Finally, proving $\wp(z)$ is doubly periodic is difficult since we don't necessarily have absolute convergence. However, one can reduce to proving $\wp(z+\omega_1) = \wp(z) = \wp(z+\omega_2)$. Then using the formula for $\wp'(z)$, we have

$$\begin{aligned}\frac{d}{dz}[\wp(z+\omega_1) - \wp(z)] &= -2F_3(z+\omega_1) + 2F_3(z) \\ &= -2F_3(z) + 2F_3(z) = 0\end{aligned}$$

since $F_3(z)$ is elliptic by Proposition 7.1.10. Hence $\wp(z+\omega_1) - \wp(z) = c$ is constant. Evaluating at $z = -\frac{\omega_1}{2}$, we see that $c = \wp\left(\frac{\omega_1}{2}\right) - \wp\left(-\frac{\omega_1}{2}\right) = 0$ since $\wp(z)$ is odd. Hence $c = 0$, so it follows that $\wp(z)$ is doubly periodic and therefore elliptic. □

Lemma 7.1.12. *Let $\wp(z)$ be the Weierstrass $\wp$-function for a lattice $\Lambda \subseteq \mathbb{C}$ and let Π be the fundamental domain of Λ. Then*

(1) For any $u \in \mathbb{C}$, the function $\wp(z) - u$ has either two simple roots or one double root in Π.

(2) The zeroes of $\wp'(z)$ in Π are simple and they only occur at $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$.

(3) The numbers $u_1 = \wp\left(\frac{\omega_1}{2}\right), u_2 = \wp\left(\frac{\omega_2}{2}\right)$ and $u_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ are precisely those u for which $\wp(z) - u$ has a double root.

Proof. (1) follows from Corollary 7.1.6.

(2) By Theorem 7.1.11, $\deg \wp'(z) = 3$ so it suffices to show that $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ are all roots. For $z = \frac{\omega_1}{2}$, we have

$$\wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(\frac{\omega_1}{2} - \omega_1\right) = -\wp'\left(\frac{\omega_1}{2}\right)$$

since $\wp'(z)$ is elliptic. Thus $\wp'\left(\frac{\omega_1}{2}\right) = 0$. The others are similar.

(3) The double roots occur exactly when $\wp'(u) = 0$, so use (2). □

We now prove that any elliptic function can be written in terms of $\wp(z)$ and $\wp'(z)$.

Theorem 7.1.13. *Fix a lattice $\Lambda \subseteq \mathbb{C}$ and let $\mathcal{E}(\Lambda)$ be the field of all elliptic functions with lattice of periods Λ. Then $\mathcal{E}(\Lambda) = \mathbb{C}(\wp, \wp')$.*

Proof. Take $f(z) \in \mathcal{E}(\Lambda)$. Then $f(-z) \in \mathcal{E}(\Lambda)$ as well and thus we can write $f(z)$ as the sum of an even and an odd elliptic function:

$$f(z) = f_{even}(z) + f_{odd}(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

We will prove that every even elliptic function is rational in $\wp(z)$, but this will imply the theorem, since then $f_{even}(z) = \varphi(\wp(z))$ and $\frac{f_{odd}(z)}{\wp'(z)} = \psi(\wp(z))$ for some $\varphi, \psi \in \mathbb{C}(\wp(z))$ and we can then write $f(z) = \varphi(\wp(z)) + \wp'(z)\psi(\wp(z))$.

Assume $f(z)$ is an even elliptic function. It's enough to construct $\varphi(\wp(z))$ such that $\frac{f(z)}{\varphi(\wp(z))}$ only has (potential) zeroes and poles at $z = 0$ in the fundamental parallelogram for Λ, since then by Corollary 7.1.6, $\frac{f(z)}{\varphi(\wp(z))}$ is holomorphic and then by Proposition 7.1.4 it is constant. Suppose $f(a) = 0$ for a some zero of order m. Consider $\wp(z) = u$. If $u \neq \wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ then $\wp(z) = u$ has precisely two solutions in the fundamental parallelogram, $z = a$ and $z = a^*$ where

$$a^* = \begin{cases} \omega_1 + \omega_2 - a & \text{if } a \in \mathrm{Int}(\Pi) \\ \omega_1 - a & \text{if } a \text{ is parallel to } \omega_1 \\ \omega_2 - a & \text{if } a \text{ is parallel to } \omega_2. \end{cases}$$

(Notice that since f is even, $f(a) = 0$ implies $f(a^*) = 0$ as well.) Moreover, if $\mathrm{ord}_a f = 0$ then $\mathrm{ord}_{a^*} f = m$. Note that $a = a^*$ holds precisely when a is in the set $\Theta := \left\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\right\}$.

Let Z (resp. P) be the set of zeroes (resp. poles) of $f(z)$ in Π. Then the assignment $a \mapsto a^*$ is in fact an involution on Z and P, so we can write

$$Z = Z_1' \cup \cdots \cup Z_r' \cup Z_1'' \cup \cdots \cup Z_s''$$
$$P = P_1' \cup \cdots \cup P_u' \cup P_1'' \cup \cdots \cup P_v''$$

where the Z_i' and P_i' are the 2-element orbits of the involution and the Z_j'' and P_j'' are the 1-element orbits. Of course then $s, v \leq 3$. For $a_i' \in Z_i'$, set $\mathrm{ord}_{a_i'} f = m_i'$ and for $a_j'' \in Z_j''$, set $\mathrm{ord}_{a_i''} f = m_i''$, which is even. Likewise, for $b_i' \in P_i'$, set $\mathrm{ord}_{b_i'} f = n_i'$ and for $b_j'' \in P_j''$, set $\mathrm{ord}_{b_i''} f = n_i''$ which is even. Then we define $\varphi(\wp(z))$ by

$$\varphi(\wp(z)) = \frac{\prod_{i=1}^{r}(\wp(z) - \wp(a_i'))^{m_i'} \prod_{j=1}^{s}(\wp(z) - \wp(a_j''))^{m_j''/2}}{\prod_{i=1}^{u}(\wp(z) - \wp(b_i'))^{n_i'} \prod_{j=1}^{v}(\wp(z) - \wp(b_j''))^{n_j}}.$$

Then $\varphi(\wp(z))$ has only potential zeroes/poles at $z = 0$ in the fundamental parallelogram, so we are done. □

7.2 Elliptic Curves

Let $\Lambda \subseteq \mathbb{C}$ be a lattice. There is a canonical way to associate to the complex torus $\mathbb{C}/\Lambda$ an elliptic curve E such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$. We would also like to reverse this process, i.e. given an elliptic curve E, define a lattice $\Lambda \subseteq \mathbb{C}$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$. This procedure generalizes for a curve C of genus $g > 1$ and produces its Jacobian, $C \hookrightarrow \mathbb{C}^g/\Lambda = J(C)$.

We need the following lemma.

Lemma 7.2.1. *Suppose $f_0, f_1, f_2, \ldots$ is a sequence of analytic functions on the ball $B_r(z_0)$ with Taylor expansions*

$$f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)}(z - z_0)^k.$$

Then if $F(z) = \sum_{n=0}^{\infty} f_n(z)$ converges uniformly on $B_\rho(z_0)$ for all $\rho < r$, each series $A_k = \sum_{n=0}^{\infty} a_k^{(n)}$ converges and $F(z)$ has Taylor expansion

$$F(z) = \sum_{k=0}^{\infty} A_k(z - z_0^k).$$

Let $\wp(z)$ be the Weierstrass $\wp$-function for Λ. Then $\wp'(z)^2$ is an even elliptic function, so by Theorem 7.1.13, $\wp'(z)^2 \in \mathbb{C}(\wp)$. On a small enough neighborhood around $z_0 = 0$,

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

is analytic. Moreover, for each $\omega \in \Lambda \setminus \{0\}$ we have

$$\begin{aligned} \frac{1}{(z-\omega)^2} &= \frac{1}{\omega^2} + \frac{2z}{\omega^3} + \frac{3z^2}{\omega^4} + \dots \\ \implies \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{2z}{\omega^2} + \frac{3z^2}{\omega^4} + \dots \end{aligned}$$

which is uniformly convergent. Hence Lemma 7.2.1 shows that

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \sum_{k=1}^{\infty} \frac{k+1}{\omega^{k+2}} z^k = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k$$

where $G_m = G_m(\Lambda) := \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^m}$. These G_m are examples of *modular forms.*

Definition. *The series* $G_m(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^m}$ *is called the* **Eisenstein series** *for* Λ *of weight* m.

From the above work, we obtain the following formulas:

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots \\ \wp(z)^2 &= \frac{1}{z^4} + 6G_4 + \dots \\ \wp(z)^3 &= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \dots \\ \wp'(z) &= -\frac{2}{z^3} + 6G_4 z + \dots \\ \wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 - \dots \end{aligned}$$

This implies:

Proposition 7.2.2. *The functions* $\wp$ *and* $\wp'$ *satisfy the following relation:*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4$ *and* $g_3 = 140G_6$.

Consider the polynomial $p(x) = 4x^3 - g_2 x - g_3$, where the g_n are defined for the lattice $\Lambda \subseteq \mathbb{C}$.

Proposition 7.2.3. $p(x) = 4(x-u_1)(x-u_2)(x-u_3)$ *where* $u_1 = \wp\left(\frac{\omega_1}{2}\right), u_2 = \wp\left(\frac{\omega_2}{2}\right)$ *and* $u_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ *are distinct roots.*

Thus $(x, y) = (\wp(z), \wp'(z))$ determine an equation $y^2 = 4x^3 - g_2x - g_3$ which is the defining equation for an elliptic curve E_0 over $\mathbb{C}$. Let $E = E_0 \cup \{[0,1,0]\} \subseteq \mathbb{P}^2$ be the projective closure of E_0. The point $[0,1,0]$ is sometimes denoted ∞.

Theorem 7.2.4. *The map*

$$\begin{aligned} \varphi : \mathbb{C}/\Lambda &\longrightarrow E(\mathbb{C}) \\ z + \Lambda &\longmapsto \varphi(z+\Lambda) = \begin{cases} [\wp(z), \wp'(z), 1], & z \notin \Lambda \\ [0,1,0], & z \in \Lambda \end{cases} \end{aligned}$$

is a bijective, biholomorphic map.

Proof. Assume $z_1, z_2 \in \mathbb{C}$ are such that $z_1 + \Lambda \neq z_2 + \Lambda$. Without loss of generality we may assume $z_1, z_2 \in \Pi$, the fundamental domain of Λ (otherwise, translate). If $\wp(z_1) = \wp(z_2)$ and $\wp'(z_1) = \wp'(z_2)$, then with the notation of Theorem 7.1.13, we must have $z_2 = z_1^* \neq z_1$ and thus $z_1, z_2 \notin \Theta = \left\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\right\}$. Since $\wp'(z)$ is odd, we get $\wp'(z_1) = \wp'(z_2) = -\wp'(-z_2) = -\wp'(z_1)$, but this implies $\wp(z_1) = 0$, contradicting $z_1 \notin \Theta$. Therefore φ is one-to-one.

Next, we must show that for any $(x_0, y_0) \in E(\mathbb{C})$, $x_0 = \wp(z)$ and $y_0 = \wp'(z)$ for some $z \in \mathbb{C}$. If $\wp(z_1) = x_0$, then it's clear that $\wp'(z_1) = y_0$ or $-y_0$. Now one shows as in the previous paragraph that we must have $\wp'(z_1) = y_0$.

Now consider $F(x, y) = y^2 - p(x)$, where $p(x) = 4x^3 - g_2x - g_3$. If (x_0, y_0) satisfies $F(x_0, y_0) = 0$ and $y_0 \neq 0$, then $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ and thus the assignment $(x, y) \mapsto x$ is a local chart about (x_0, y_0). Likewise, $(x, y) \mapsto y$ defines a local chart about (x_0, y_0) when $x_0 \neq 0$. Finally, we conclude by observing that a locally biholomorphic map is biholomorphic. □

In general, an elliptic curve can be defined by a *Weierstrass equation*

$$E : y^2 = f(x) = ax^3 + bx^2 + cx + d.$$

This embeds into projective space via $(x, y) \mapsto [x, y, 1]$. Setting $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, we also obtain a homogeneous equation for the curve:

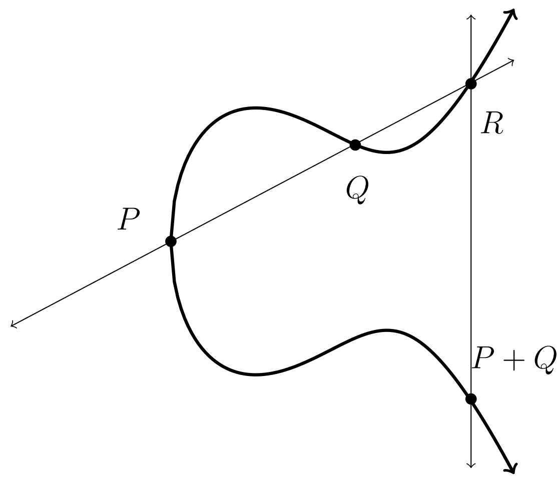
$$E : ZY^2 = aX^3 + bX^2Z + cXZ^2 + dZ^3.$$

The single point at infinity, $[0,1,0]$, can be studied by dehomogenizing via the coordinates $\tilde{z} = \frac{Z}{Y}$ and $\tilde{x} = \frac{X}{Y}$, which yield

$$E : \tilde{z} = a\tilde{x}^3 + b\tilde{x}^2\tilde{z} + a\tilde{x}\tilde{z}^2 + d\tilde{z}^3.$$

We have shown that a lattice $\Lambda \subseteq \mathbb{C}$ determines elliptic functions $\wp(z)$ and $\wp'(z)$ that satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ and that this polynomial expression has no multiple roots. Therefore the mapping $z \mapsto (\wp(z), \wp'(z))$ determines a bijective correspondence $\mathbb{C}/\Lambda \smallsetminus \{0\} \to E(\mathbb{C}) \smallsetminus \{\infty\}$ which can be extended to all of $\mathbb{C}/\Lambda \to E(\mathbb{C})$ (this is Theorem 7.2.4). There is a natural group structure on $\mathbb{C}/\Lambda$ induced from $\mathbb{C}$, but what is not so obvious is that $E(\mathbb{C})$ also possesses a group structure, the so-called "chord-and-tangent method".

Let E be an elliptic curve over an arbitrary field k, let $O \in E(k)$ be the point at infinity and fix two points $P, Q \in E(k)$. In the plane $\mathbb{P}^2_k$, there is a unique line containing P and Q; call it L. (If $P = Q$, then take L to be the tangent line to E at P.) By Bézout's theorem, $E \cap L = \{P, Q, R\}$ for some third point $R \in E(k)$, which may not be distinct from P and Q if multiplicity is counted. Let L' be the line through R and O and call its third point R'.



Addition of two points $P, Q \in E(k)$ is defined by $P + Q = R'$, where R' is the unique point lying on the line through R and O. If $R = O$, we set $R' = O$.

Proposition 7.2.5. *Let E be an elliptic curve with $O \in E(k)$. Then*

(a) If L is a line in $\mathbb{P}^2$ such that $E \cap L = \{P, Q, R\}$, then $(P + Q) + R = O$.

(b) For all $P \in E(k)$, $P + O = P$.

(c) For all $P, Q \in E(k)$, $P + Q = Q + P$.

(d) For all $P \in E(k)$, there exists a point $-P \in E(k)$ satisfying $P + (-P) = O$.

(e) For all $P, Q, R \in E(k)$, $(P + Q) + R = P + (Q + R)$.

Together, (b) – (e) say that chord-and-tangent addition of points defines an associative, commutative group law on $E(k)$. The proofs of (a) – (d) are rather routine using the definition of this addition law, whereas verifying associativity is notoriously difficult. There are formulas for the coordinates of $P + Q$ that make this possible though (see Silverman).

Theorem 7.2.6. *The map $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C})$ is an isomorphism of abelian groups.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda & \xrightarrow{\varphi \times \varphi} & E(\mathbb{C}) \times E(\mathbb{C}) \\ \alpha \downarrow & & \downarrow \beta \\ \mathbb{C}/\Lambda & \xrightarrow[\varphi]{} & E(\mathbb{C}) \end{array}$$

where α and β are the respective group operations. Since $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ is a topological group, it's enough to show the diagram commutes on a dense subset of $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$. Consider

$$\widetilde{X} = \{(u_1, u_2) \in \mathbb{C}^2 \mid u_1, u_2, u_1 \pm u_2, 2u_1 + u_2, u_1 + 2u_2 \notin \Lambda\}.$$

Then $\widetilde{X} \cong \mathbb{C}^2$ so $X = \widetilde{X} \mod \Lambda \times \Lambda$ is dense in $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$. Take $(u_1 + \Lambda, u_2 + \Lambda) \in X$ and set $u_3 = -(u_1 + u_2)$. Then $u_1 + u_2 + u_3 = 0$ in $\mathbb{C}/\Lambda$. Set $P = \varphi(u_1), Q = \varphi(u_2)$ and $R = \varphi(u_3) \in E(\mathbb{C})$. By the assumptions on X, the points P, Q, R are distinct. We want to show $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2) = P + Q$. Since $\wp(z)$ is even and $\wp'(z)$ is odd, we see that $\varphi(-z) = -\varphi(z)$ for all $z \in \mathbb{C}/\Lambda$. Thus $\varphi(u_1 + u_2) = -\varphi(-(u_1 + u_2)) = -R$ so we need to show $P + Q + R = O$, i.e. P, Q, R are colinear. Since $u_1 \neq u_2$, the line $\overline{PQ}$ is not vertical, so there exist a, b such that $\wp'(u_i) = a\wp(u_i) + b$ for $i = 1, 2$. Consider the elliptic function

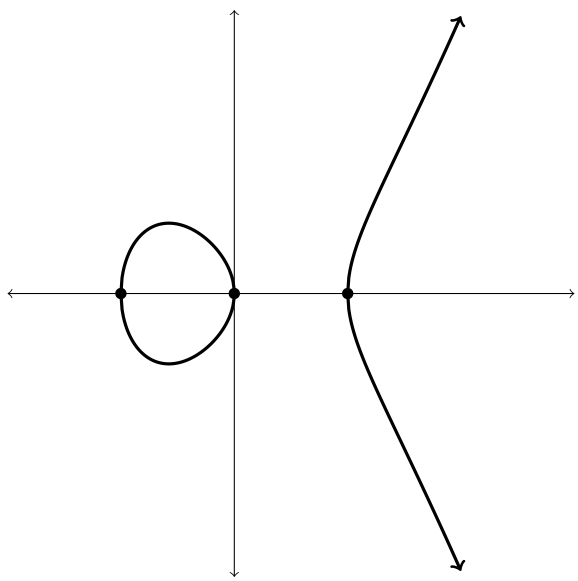
$$f(z) = \wp'(z) - (a\wp(z) + b).$$

Then on the fundamental domain Π, f only has a pole at 0, so $\mathrm{ord}_0 f = -3$. Also, u_1 and u_2 are distinct zeroes of f, so there is a third point $\omega \in \Pi$ such that $\deg(f) = u_1 + u_2 + \omega - 3 \cdot 0 = 0$, i.e. $u_1 + u_2 + \omega = 0$. Solving for ω, we get $\omega = -(u_1 + u_2) = u_3$. It follows that $R = \varphi(u_3)$ is on the same line as P and Q, so we are done. □

The compatibility of the group operations of $\mathbb{C}/\Lambda$ and $E(\mathbb{C})$ is highly useful. For example, fix $N \in \mathbb{N}$ and let

$$E[N] = \{P \in E(\mathbb{C}) \mid [N]P = O\},$$

where $[N]P = \underbrace{P + \ldots + P}_{N}$. The points of $E[N]$ are called the N-torsion points of E. For $N = 2$, the points P such that $P = -P$ are exactly the intersection points of E with the x-axis along with $O = [0, 1, 0]$:



In general, one can show that $\#E[N] = N^2$. This is hard to see from the geometric picture, but working with the isomorphism $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ from Theorem 7.2.6, we see that since $\mathbb{C}/\Lambda = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as an abelian group, the N-torsion is given by $(\mathbb{C}/\Lambda)[N] = \frac{1}{N}\mathbb{Z}/\mathbb{Z} \times \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. This is a group of order N^2.

A morphism in the category of elliptic curves is called an *isogeny*. Explicitly, $\varphi : E_1 \to E_2$ is an isogeny between two elliptic curves if it is a (nonconstant) morphism of schemes that takes the basepoint $O_1 \in E_1$ to the basepoint $O_2 \in E_2$.

Proposition 7.2.7. *Suppose $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$ are lattices and $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is a holomorphic map. Then there exist $a, b \in \mathbb{C}$ such that $a\Lambda_1 \subseteq \Lambda_2$ and*

$$f(z \mod \Lambda_1) = az + b \mod \Lambda_2.$$

Proof. As topological spaces, $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are complex tori with the same universal covering space $\mathbb{C}$, so any $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ lifts to $F : \mathbb{C} \to \mathbb{C}$ making the diagram commute:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{C}/\Lambda_1 & \xrightarrow[f]{} & \mathbb{C}/\Lambda_2 \end{array}$$

Since covers are local homeomorphisms, it follows that F is holomorphic as well. Thus for any $z \in \mathbb{C}, \ell \in \Lambda_1$,

$$\pi_2(F(z+\ell) - F(z)) = f(\pi_1(z+\ell) - \pi_1(z)) = f(\pi_1(z) - \pi_1(z)) = f(0) = 0.$$

So $F(z+\ell) - F(z) \in \Lambda_1$ for any $\ell \in \Lambda_1$ and the function $L(z) = F(z+\ell) - F(z)$ is constant. It follows that $F'(z+\ell) = F'(z)$, so F' is holomorphic and elliptic, but this means by Proposition 7.1.4 that $F'(z) = a$ for some constant a. Hence $F(z) = az + b$ as claimed. □

Corollary 7.2.8. *Any holomorphic map $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is, up to translation, a group homomorphism. In particular, if $f(0) = 0$ then f is a homomorphism.*

Corollary 7.2.9. *For any elliptic curve E, the group of endomorphisms* $\mathrm{End}(E)$ *has rank at most* 2.

Proof. Viewing $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for some $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, we get

$$\begin{aligned} \mathrm{End}(E) &= \{f : E \to E \mid f \text{ is an isogeny}\} \\ &= \{f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \mid f \text{ is holomorphic and } f(0) = 0\} \quad \text{by Corollary 7.2.8} \\ &= \{z \in \mathbb{C} \mid z\Lambda \subseteq \Lambda\} \\ &= \{z \in \mathbb{C} \mid z(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau)\} \\ &\subseteq \mathbb{Z} + \mathbb{Z}\tau. \end{aligned}$$

Hence $\operatorname{rank} \mathrm{End}(E) \le 2$. □

It turns out that there are two possible cases for the rank of $\mathrm{End}(E)$, breaking down as follows:

- $\mathrm{End}(E) = \mathbb{Z}$.
- $\mathrm{End}(E)$ is an *order* $\mathcal{O}$ in some imaginary quadratic number field $K/\mathbb{Q}$. In this case, E is said to have *complex multiplication*.

7.3 The Classical Jacobian

For the isomorphism $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C})$ in Theorem 7.2.6, let $\psi = \varphi^{-1} : E(\mathbb{C}) \to \mathbb{C}/\Lambda$ be the inverse map. To understand this map explicitly, we will show how to construct a torus for every elliptic curve, i.e. find a lattice $\Lambda \subseteq \mathbb{C}$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$.

Lemma 7.3.1. *Any lattice $\Lambda \subseteq \mathbb{C}$ can be written*

$$\Lambda = \left\{ \int_0^P dz : P \in \Lambda \right\}.$$

Notice that each differential form dz on $\mathbb{C}$ satisfies $d(z + \ell) = dz$ for all $\ell \in \Lambda$ by Lemma 7.3.1. Thus dz descends to a differential form on $\mathbb{C}/\Lambda$, which by abuse of notation we will also denote by dz. Formally, this is the pushforward of dz along the quotient $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. This implies:

Lemma 7.3.2. *Any lattice $\Lambda \subseteq \mathbb{C}$ can be written*

$$\Lambda = \left\{ \int_\gamma dz : \gamma \textit{ is a closed curve in } \mathbb{C}/\Lambda \textit{ passing through } 0 \right\}.$$

For an elliptic curve E defined by the equation $y^2 = f(x)$, fix a holomorphic differential form ω on $E(\mathbb{C})$. (In general, the space of holomorphic differential forms on a curve has dimension equal to the genus of the curve, so in the elliptic curve case, there is exactly one such ω, up to scaling.)

Definition. *The* **lattice of periods** *for an elliptic curve E is*

$$\Lambda = \left\{ \int_\gamma \omega : \gamma \textit{ is a closed curve in } E \textit{ passing through } P \right\}$$

where $P \in E(\mathbb{C})$ is fixed.

Example 7.3.3. Under the map $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C}), z \mapsto (x, y) = (\wp(z), \wp'(z))$, we see that

$$dx = \wp'(z)\,dz = y\,dz$$

so $\omega = \frac{dx}{y}$ is a differential form on $E(\mathbb{C})$. In fact, $\omega = \frac{dx}{f'(x)}$, where E is defined by $y^2 = f(x)$, is holomorphic because $f'(x) \not\equiv 0$. This differential form is also holomorphic at $O = [0, 1, 0]$, so up to scaling, this is the unique holomorphic form on E.

Historically, mathematicians were interested in studying solutions to *elliptic integrals*, or integrals of the form

$$\int \frac{dx}{\sqrt{ax^3 + bx + c}}.$$

When $f(x) = ax^3 + bx + c$, the expression $\omega = \frac{dx}{\sqrt{ax^3+bx+c}}$ is precisely the holomorphic differential form defining the lattice of periods of the elliptic curve $E : y^2 = f(x)$.

For a more functorial description, let $V_E = \Gamma(E, \Omega_E)$ be the space of all holomorphic differential forms on E. If γ is a curve in $E(\mathbb{C})$, there is an associated linear functional $\varphi_\gamma \in V_E^*$ defined by

$$\begin{aligned} \varphi_\gamma : V_E &\longrightarrow \mathbb{C} \\ \omega &\longmapsto \int_\gamma \omega. \end{aligned}$$

Fixing the basepoint $O \in E(\mathbb{C})$, the lattice of periods for E can be written

$$\Lambda = \{\varphi_\gamma : \gamma \in \pi_1(E(\mathbb{C}), O)\}.$$

In other words, this defines a map $\pi_1(E(\mathbb{C}), O) \to V_E^*, \gamma \mapsto \varphi_\gamma$.

Definition. *The* **Jacobian** *of an elliptic curve E is the quotient $J(E) = V_E^*/\Lambda$.*

For each point $P \in E(\mathbb{C})$, the coset $\varphi_\gamma + \Lambda$ is an element of the Jacobian, where γ is a path from O to P. This defines an injective map $i : E \hookrightarrow J(E)$.

Proposition 7.3.4. *Suppose $\sigma : E_1 \to E_2$ is an isogeny between elliptic curves, so that $\sigma(O_1) = O_2$. Then there is a map $\tau : J(E_1) \to J(E_2)$ making the following diagram commute:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\sigma} & E_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ J(E_1) & \xrightarrow[\tau]{} & J(E_2) \end{array}$$

Proof. The pullback gives a contravariant map $\sigma^* : V_{E_2} \to V_{E_1}, \omega \mapsto \sigma^*\omega = \omega \circ \sigma$. Taking the dual of this gives a linear map $\sigma^{**} : V_{E_1}^* \to V_{E_2}^*$ defined by $(\sigma^{**}\rho)(\omega) = \rho(\sigma^*\omega)$ for any $\rho \in V_{E_1}^*$ and $\omega \in V_{E_2}$. Taking $\rho = \varphi_{\gamma_1}$ for a path γ_1 in E_1 gives

$$\rho(\sigma^*\omega) = \varphi_{\gamma_1}(\sigma^*\omega) = \int_{\gamma_1} \sigma^*\omega = \int_{\sigma(\gamma_1)} \omega = \varphi_{\sigma(\gamma_1)}\omega.$$

Thus $\sigma^{**}\varphi_{\gamma_1} = \varphi_{\sigma(\gamma_1)}$. If γ_1 is a closed curve through O_1, then $\sigma(\gamma_1)$ is a closed curve passing through $O_2 = \sigma(O_1)$. Hence if $\Lambda_{E_1}, \Lambda_{E_2}$ are the lattices of periods for E_1, E_2, respectively, we have $\sigma^{**}(\lambda_{E_1}) \subseteq \Lambda_{E_2}$. So σ^{**} factors through the quotients, defining τ:

$$\tau = \overline{\sigma^{**}} : V_{E_1}^*/\Lambda_{E_1} \longrightarrow V_{E_2}^*/\Lambda_{E_2}.$$

It is immediate the diagram commutes. $\square$

Lemma 7.3.5. *For any elliptic curve E, the inclusion $i : E \hookrightarrow J(E)$ induces an isomorphism*

$$i^* : \pi_1(E, O) \longrightarrow \pi_1(J(E), i(O)).$$

Unfortunately, the construction of the Jacobian given so far is not algebraic so it would be hard to carry over to curves over an arbitrary ground field. To construct Jacobians algebraically, we will prove Abel's theorem:

Theorem 7.3.6 (Abel). *Suppose $\Lambda \subseteq \mathbb{C}$ is a lattice with fundamental domain Π and take any set $\{a_i\} \subset \Pi$ such that there are integers $m_i \in \mathbb{Z}$ satisfying $\sum m_i = 0$ and $\sum m_i a_i \in \Lambda$. Then there exists an elliptic function $f(z)$ whose set of zeroes and poles is $\{a_i\}$ and whose orders of vanishing/poles are $\operatorname{ord}_{a_i} f = m_i$.*

Given a lattice $\Lambda \subseteq \mathbb{C}$, we may assume $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$.

Definition. *The* **theta function** *for a lattice Λ is*

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i(n^2\tau + 2nz)}.$$

One has $|e^{\pi i(n^2\tau+2nz)}| = e^{-\pi(n^2 \operatorname{Im}\tau + 2n\operatorname{Im} z)}$ for any $z \in \mathbb{C}$, which implies that the above series converges absolutely.

Proposition 7.3.7. *Fix a theta function $\theta(z) = \theta(z,\tau)$. Then*

(1) $\theta(z) = \theta(-z)$.

(2) $\theta(z+1) = \theta(z)$.

(3) $\theta(z+\tau) = e^{-\pi i(\tau+2z)}\theta(z)$.

Properties (2) and (3) together say that $\theta(z)$ is what's known as a *semielliptic function*. For our purposes, this will be good enough. Notice that for $z = \frac{1+\tau}{2}$, we have

$$\begin{aligned}\theta\left(\frac{1+\tau}{2}\right) &= \theta\left(-\frac{1+\tau}{2} + (1+\tau)\right)\\ &= e^{\pi i\left(\tau+2\left(-\frac{1+\tau}{2}\right)\right)}\theta\left(-\frac{1+\tau}{2}\right)\\ &= e^{\pi i}\theta\left(-\frac{1+\tau}{2}\right) = -\theta\left(\frac{1+\tau}{2}\right).\end{aligned}$$

Thus $z = \frac{1+\tau}{2}$ is a zero of $\theta(z)$.

Lemma 7.3.8. *All zeroes of $\theta(z,t)$ are simple and are of the form $\frac{1+\tau}{2} + \ell$ for $\ell \in \Lambda$.*

Lemma 7.3.9. *For $x \in \mathbb{C}$, set $\theta^{(x)}(z,\tau) = \theta\left(z - \frac{1+\tau}{2} - x\right)$. Then $\theta^{(x)}(z) = \theta^{(x)}(z,\tau)$ satisfies:*

(1) $\theta^{(x)}(z+1) = \theta^{(x)}(z)$.

(2) $\theta^{(x)}(z+\tau) = e^{-\pi i(2(z-x)-1)}\theta^{(x)}(z)$.

We now prove Abel's theorem (7.3.6).

Proof. Given such a set $\{a_i\} \subset \Pi$, let $x_1, \ldots, x_n$ be the list of all a_i with $m_i > 0$, listed with repetitions corresponding to the number m_i. For example, if $m_1 = 2$ then $x_1 = x_2 = a_1$. Likewise, let $y_1, \ldots, y_n$ be the list of all a_i with $m_i < 0$, once again with repetitions. By the hypothesis $\sum m_i = 0$, there are indeed an equal number of each. Set

$$f(z) = \frac{\prod_{i=1}^n \theta^{(x_i)}(z)}{\prod_{i=1}^n \theta^{(y_i)}(z)}.$$

Then by Lemma 7.3.9, $f(z+1) = f(z)$. On the other hand, the lemma also gives

$$\begin{aligned} f(z+\tau) &= \frac{\prod_{i=1}^n \theta^{(x_i)}(z+\tau)}{\prod_{i=1}^n \theta^{(y_i)}(z)} \\ &= e^{2\pi i\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)} f(z) \\ &= e^{2\pi i \sum m_i a_i} f(z) \\ &= f(z) \quad \text{since } \sum m_i a_i = 0. \end{aligned}$$

Therefore $f(z)$ is elliptic. □

Note that $\theta(z)$ is a meromorphic function, so the integral

$$\frac{1}{2\pi i} \int_{\partial \Pi} \frac{\theta'(z)}{\theta(z)}\, dz$$

counts the number of zeroes of $\theta(z)$ in the fundamental domain Π, up to multiplicity. To ensure no zeroes lying on $\partial\Pi$ are missed, we may shift $\Pi \to \Pi_\alpha$ for an appropriate $\alpha \in \mathbb{C}$. Parametrize $\partial\Pi$ as in Proposition 7.1.5. Then once again the integrals along γ_2 and γ_4 cancel since $\theta(z+1) = \theta(z)$. On the other hand,

$$\begin{aligned} \theta(z+\tau) &= e^{-\pi i(\tau+2z)}\theta(z) \\ \Longrightarrow \theta'(z+\tau) &= e^{-\pi i(\tau+2z)}(-2\pi i\theta(z) + \theta'(z)) \\ \Longrightarrow \frac{\theta'(z+\tau)}{\theta(z+\tau)} &= -2\pi i + \frac{\theta'(z)}{\theta(z)}. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\partial\Pi} \frac{\theta'(z)}{\theta(z)}\, dz &= \int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_2} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_3} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_4} \frac{\theta'(z)}{\theta(z)}\, dz \\ &= \left(\int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_3} \frac{\theta'(z)}{\theta(z)}\, dz\right) + \left(\int_{\gamma_2} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_4} \frac{\theta'(z)}{\theta(z)}\, dz\right) \\ &= \left(\int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz - \int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + 2\pi i\right) + 0 \\ &= 2\pi i. \end{aligned}$$

It follows that $\theta(z)$ has exactly one zero in Π, and it must be $z = \frac{1+\tau}{2}$.

Definition. *For a curve E (need not be elliptic), define:*

- *A* **divisor** *on E is a formal sum $D = \sum n_P P$ over the points $P \in E$, with $n_P \in \mathbb{Z}$. The abelian group of all divisors is denoted* $\mathrm{Div}(E)$.
- *The* **degree** *of a divisor $D = \sum n_P P \in \mathrm{Div}(E)$ is $\deg(D) = \sum n_P$. The set of all* **degree 0 divisors** *is denoted* $\mathrm{Div}^0(E)$.
- *For a meromorphic function f on $E(\mathbb{C}) = \mathbb{C}/\Lambda$, the* **principal divisor** *associated to f is $(f) = \sum \deg_P P$ where $n_P = \mathrm{ord}_P f$. The group of all principal divisors is denoted* $\mathrm{PDiv}(E)$.
- *The* **Picard group** *of E is the quotient group $\mathrm{Pic}(E) = \mathrm{Div}(E)/\mathrm{PDiv}(E)$. The degree zero part of the Picard group is written $\mathrm{Pic}^0(E) = \mathrm{Div}^0(E)/\mathrm{PDiv}(E)$.*

The inverse map $\psi : E \to \mathbb{C}/\Lambda$ extends to the group of divisors on E:

$$\begin{aligned} \Psi : \mathrm{Div}(E) &\longrightarrow \mathbb{C}/\Lambda \\ \sum n_P P &\longmapsto \sum n_P \psi(P). \end{aligned}$$

Definition. *The map $\Psi : \mathrm{Div}(E) \to \mathbb{C}/\Lambda$ is called the* **Abel-Jacobi map**.

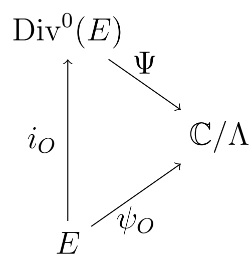
Recall that $\psi : P \mapsto \int_{\gamma_P} \omega + \Lambda \in \mathbb{C}/\Lambda$ where ω is a fixed holomorphic differential form on E and γ_P is a path connecting $O \in E(\mathbb{C})$ to P. If O' is another basepoint and ψ' is the corresponding map, we have $\psi(P) = \psi(O') + \psi'(P)$ for all $P \in E$. So it appears that Ψ is not well-defined. However, this issue vanishes when we restrict Ψ to $\mathrm{Div}^0(E)$: if $D = \sum n_P P$ is a degree 0 divisor, then

$$\begin{aligned} \Psi(D) &= \sum n_P \psi(P) \\ &= \sum n_P (\psi(O') + \psi'(P)) \\ &= \psi(O') \sum n_P + \sum n_P \psi'(P) \\ &= 0 + \sum n_P \psi'(P) = \Psi'(D). \end{aligned}$$

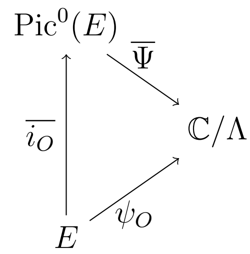
Corollary 7.3.10. *The map $\Psi : \mathrm{Div}^0(E) \to \mathbb{C}/\Lambda$ induces an isomorphism $\mathrm{Pic}^0(E) \cong \mathbb{C}/\Lambda$.*

Proof. One can prove that Ψ is a surjective group homomorphism. Moreover, Abel's theorem (7.3.6) implies that $\ker \Psi = \mathrm{PDiv}(E)$. □

Consider the map $i_O : E \to \mathrm{Div}^0(E)$ that sends $P \mapsto P - O$. This fits into a commutative diagram:



On the level of the Picard group, this diagram looks like



and every arrow is a bijection.

7.4 Jacobians of Higher Genus Curves

Let C be a complex curve of genus $g \geq 2$ and let $V = \Gamma(C, \Omega_C)$ be the vector space of holomorphic differential forms on C. Then $\dim_{\mathbb{C}} V = g$, so $V^* \cong \mathbb{C}^g$. As in the previous section, for any path ω in C the assignment $\varphi_\gamma : \omega \mapsto \int_\gamma \omega$ defines a functional $\varphi_\gamma \in V^*$. As for elliptic curves, we define:

Definition. *The* **lattice of periods** *for C is*

$$\Lambda = \{\varphi_\gamma \in V^* \mid \gamma \text{ is a closed curve in } C\}.$$

Lemma 7.4.1. *Λ is a lattice in V^*.*

Definition. *The* **Jacobian** *of C is the quotient space $J(C) = V^*/\Lambda$.*

As with elliptic curves, we have a map $\psi : C \to J(C)$ called the *Abel-Jacobi map*, which sends $P \mapsto \varphi_{\gamma_P} + \Lambda$, where γ_P is a curve through P. Also, ψ extends to the divisor group of C as a map

$$\Psi : \mathrm{Div}(C) \longrightarrow J(C)$$

which is canonical when restricted to $\mathrm{Div}^0(C)$. The Abel-Jacobi theorem generalizes Theorem 7.3.6 and Corollary 7.3.10.

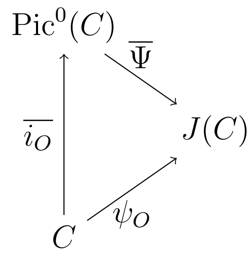
Theorem 7.4.2. *Let C be a curve of genus $g > 0$ and let $\Psi : \mathrm{Div}^0(C) \to J(C)$ be the Abel-Jacobi map. Then*

(1) (Abel) $\ker \Psi = \mathrm{PDiv}(C)$.

(2) (Jacobi) Ψ is surjective.

Therefore Ψ induces an isomorphism $\mathrm{Pic}^0(C) \cong J(C)$.

Just as with elliptic curves, if we fix a basepoint $O \in C$, the map $i_O : C \to \mathrm{Div}^0(C), P \mapsto P - O$ determines a commutative diagram



However, this time not every map is a bijection. In particular, $\dim C = 1 < g = \dim J(C)$. To remedy this, let C^g be the g-fold product of C and consider the map

$$\begin{aligned} \psi^g : C^g &\longrightarrow J(C) \\ (P_1, \ldots, P_g) &\longmapsto \psi(P_1) + \ldots + \psi(P_g) \end{aligned}$$

where $+$ denotes the group law on $J(C)$.

Theorem 7.4.3 (Jacobi)**.** *$\psi^g : C^g \longrightarrow J(C)$ is surjective.*

There is still work to do to show that the natural map $C^g \to \mathrm{Pic}^0(C)$ is surjective. It turns out that $J(C)$ is birationally equivalent to the *symmetric power* $C^{(g)} = C^g/\sim$, where $(P_1, \ldots, P_g) \sim (P_{\sigma(1)}, \ldots, P_{\sigma(g)})$ for any permutation $\sigma \in S_g$. Jacobi proved that this birational equivalence is enough to endow $\mathrm{Pic}^0(C) \cong J(C)$ with the structure of an algebraic group.

Theorem 7.4.4. *$J(C)$ is an abelian variety.*