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Model Theoretic Methods for Special Classes of (Finite) Structures

A Game-Oriented & Modal View of Fragments of FO

Martin Otto, Technische Universität Darmstadt
www.mathematik.tu-darmstadt.de/~otto

Overview

Part I: Transition Systems and Bisimulation

- Back & Forth, modal and FO Ehrenfeucht-Fraïssé
- Locality and modularity of games, model constructions
- Trees and locally tree-like models
- Special classes of transition systems

Part II: From Graphs to Hypergraphs

- Back & Forth on hypergraphs, guardedness
- Bounded treewidth and tree-like models
- Extension theorems for partial automorphisms
- Applications/transfer: fmp for CGF; BDD(neg $\forall^*$)

Part III: Special Classes of (Finite) Structures

- Bounded treewidth
- Wide and almost wide structures
- New proofs for classical results

Part I: Transition Systems and Bisimulation

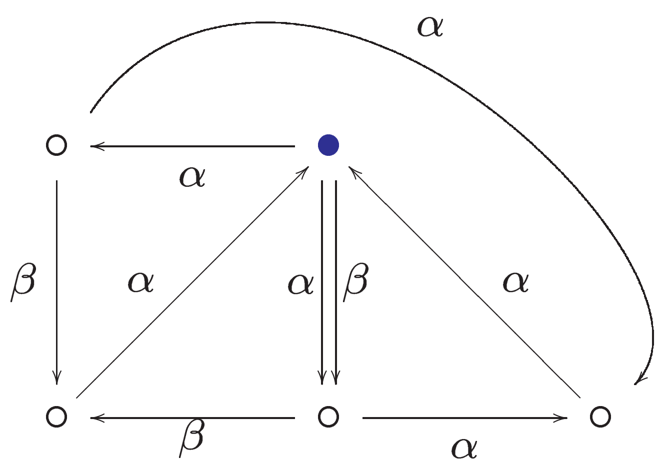
the structures:

transition systems Kripke structures game graphs	}	**edge- and vertex-coloured directed graphs** $\mathfrak{A} = \big(A, (E_\alpha), (P_i)\big)$

ubiquitous format: Kripke semantics for modal logic
representation of processes/protocols/etc

various terminolgies:

states	transitions
possible worlds	accessibility
positions	moves
nodes	edges



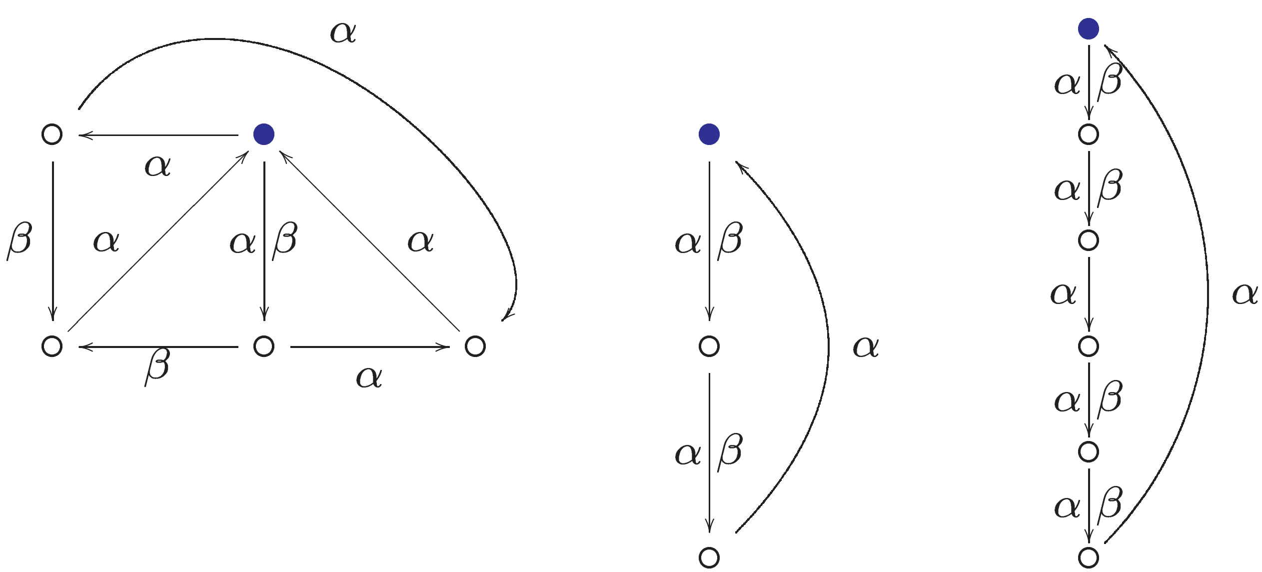
bisimulation

behavioural equivalence between states in transition systems

move-by-move equivalence between positions in game graphs

- **equivalence in modal Ehrenfeucht-Fraïssé game**
- **modal version of partial isomorphism**

different traditions: bisimulation: Hennessy, Milner, Park
zig-zag equivalence: van Benthem



bisimulation game $\mathsf{G}_\infty(\mathfrak{A};\mathfrak{B})$

between players **I** and **II** on $\mathfrak{A}$, $\mathfrak{B}$

configurations: $(\mathfrak{A}, a; \mathfrak{B}, b)$, a, b with matching vertex colours
"pebbles on locally isomorphic states"

single round: challenge/response pair of moves
along designated edge type

termination: players lose when stuck for a move
G_∞: **II** wins infinite play
G_ℓ: **II** wins after ℓ rounds

$\mathfrak{A}, a \sim \mathfrak{B}, b$: **II** has winning strategy in $\mathsf{G}_\infty(\mathfrak{A}, a; \mathfrak{B}, b)$

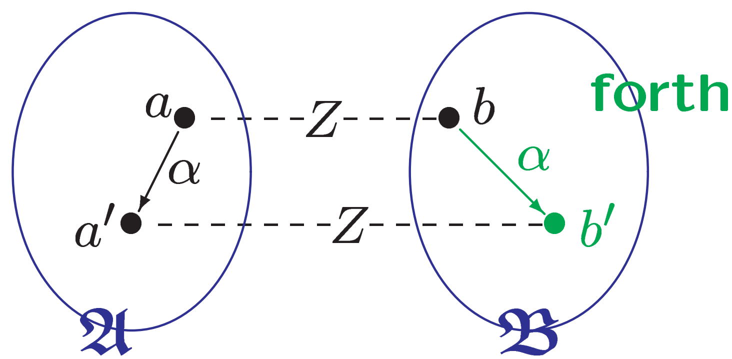
bisimulation: back & forth along edges

$\mathfrak{A}, a \sim \mathfrak{B}, b$ iff

- $a \simeq b$ (same colours)
- for all $a \xrightarrow{\alpha} a'$ in $\mathfrak{A}$ there is $b \xrightarrow{\alpha} b'$ in $\mathfrak{B}$: $\mathfrak{A}, a' \sim \mathfrak{B}, b'$
- for all $b \xrightarrow{\alpha} b'$ in $\mathfrak{B}$ there is $a \xrightarrow{\alpha} a'$ in $\mathfrak{A}$: $\mathfrak{A}, a' \sim \mathfrak{B}, b'$

back & forth system $Z \subseteq A \times B$:

non-det. winning strategy for **II**
witnessing bisimulation equivalence



largest bisimulation

greatest fixed point Z^∞ w.r.t. the back & forth conditions

$\mathfrak{A}, a \sim \mathfrak{B}, b$ iff $(a, b) \in Z^\infty$

variations: global / two-way bisimulation

global equivalence $\mathfrak{A}, a \sim_\forall \mathfrak{B}, b$ or $\mathfrak{A} \sim_\forall \mathfrak{B}$

in the game, also allow rounds of unconstrained relocation

global back & forth systems $Z \subseteq A \times B$ s.t. $\pi_1(Z) = A$, $\pi_2(Z) = B$

$(\forall a \in A)(\exists b \in B)\ \mathfrak{A}, a \sim \mathfrak{B}, b$

and $(\forall b \in B)(\exists a \in A)\ \mathfrak{A}, a \sim \mathfrak{B}, b$

two-way equivalence $\mathfrak{A}, a \sim_- \mathfrak{B}, b$

in the game, also allow backward traversal of edges

back & forth property w.r.t. R_α and their converses R_α^{-1}

global two-way equivalence $\mathfrak{A}, a \approx \mathfrak{B}, b$ or $\mathfrak{A} \approx \mathfrak{B}$

simultaneous refinement w.r.t. both

finite approximations

ℓ-bisimulation equivalence $\mathfrak{A}, a \sim^{\ell} \mathfrak{B}, b$

winning strategy for **II** in $\mathsf{G}_{\ell}(\mathfrak{A}, a; \mathfrak{B}, b)$ (ℓ rounds)

back & forth system $(Z_i)_{i \leqslant \ell}$, where
$(a, b) \in Z_{i+1}$ has back & forth responses $(a', b') \in Z_i$

bisimulation ℓ-types

inductively: $\theta^0(a)$: set of colours true at a in $\mathfrak{A}$

$$\theta^{n+1}(a) = \theta^n(a) \cup \left\{ \Big(\alpha, \theta^n(a')\Big) : a \xrightarrow{\alpha} a' \right\}$$

then $\mathfrak{A}, a \sim^{\ell} \mathfrak{B}, b$ iff $\theta^{\ell}(a) = \theta^{\ell}(b)$

similarly for $\sim_{-}$, $\sim_{\forall}$ and $\approx$

bisimulation and Ehrenfeucht-Fraïssé games

connected in both directions:

(A) bisimulation as modal E-F:

the restriction of the familiar E-F game to moves that capture the restricted quantification available in modal logic

(B) bisimulation view of FO E-F games:

from the FO E-F game on $\mathfrak{A}, \mathfrak{B}$
to game graphs G($\mathfrak{A}$) and G($\mathfrak{B}$) for game semantics of FO

from E-F equivalence between $\mathfrak{A}$ and $\mathfrak{B}$
to bisimulation equivalence between G($\mathfrak{A}$) and G($\mathfrak{B}$)

(B) bisimulation view of FO E-F games

put $\mathsf{G}(\mathfrak{A}) := \Big(A^{<\omega}, \{(\boldsymbol{a}, \boldsymbol{a}a) \colon \boldsymbol{a} \in A^{<\omega}, a \in A\}, (P_\theta)_{\text{qfr-free } \theta}\Big)$

then $\mathfrak{A}, \boldsymbol{a} \simeq_q \mathfrak{B}, \boldsymbol{b}$ iff $\mathsf{G}(\mathfrak{A}), \boldsymbol{a} \sim^q \mathsf{G}(\mathfrak{B}), \boldsymbol{b}$

. . . correspondences/translations, e.g.:

convergence of finite to partial isomorphism in ω-saturated structures	$\longleftrightarrow$	Hennessy-Milner property of modally saturated game graphs

. . . restricted FO games

k-pebble games: $\mathsf{G}^k(\mathfrak{A})$ based on A^k, $(\boldsymbol{a}, \boldsymbol{a}\frac{a}{i}) \in E_i$
guarded FO games: $\mathsf{G}^{\mathrm{g}}(\mathfrak{A})$ based on guarded tuples in $\mathfrak{A}$
modal E-F game: $\mathsf{G}(\mathfrak{A}) = \mathfrak{A}$

other forms of localised/modular games?

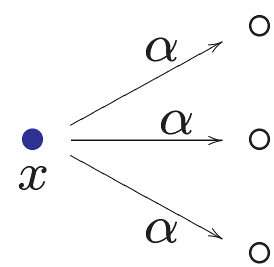
(A) bisimulation: Ehrenfeucht-Fraïssé for modal logic

basic modal logic ML

atomic formulae:	$\top$, $\bot$, monadic atoms (vertex colours)
boolean connectives:	$\vee$, $\wedge$, $\neg$, $\rightarrow$, ...
relativised quantification:	$\langle\alpha\rangle$, $[\alpha]$

$$\langle\alpha\rangle\psi : \quad \exists y\Big((x \xrightarrow{\alpha} y) \wedge \psi(y)\Big) \equiv \exists y\Big(E_\alpha xy \wedge \psi(y)\Big)$$

$$[\alpha]\psi : \quad \forall y\Big((x \xrightarrow{\alpha} y) \rightarrow \psi(y)\Big) \equiv \forall y\Big(E_\alpha xy \rightarrow \psi(y)\Big)$$



variations

ML[$\forall$] with (unary) global quantification $\forall/\exists$
universal modality

ML[$-$] with backward modal quantification $[\alpha^{-1}]/\langle\alpha^{-1}\rangle$
inverse (past) modalities

ML[$\forall, -$] with both, universal and inverse modalities

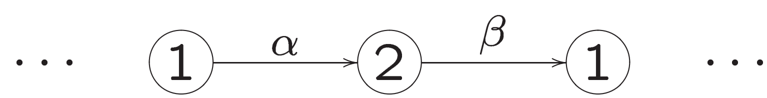
restricted quantification, restricted E-F moves

$\langle\alpha\rangle$:	move pebble along an α-edge
$\langle\alpha^{-1}\rangle$:	move pebble backwards along an α-edge
global modality:	unconstrained relocation of pebble

remark

modal moves as restricted moves in FO^2 game:

alternating pebble moves,



other pebble marking source vertex
for checking edge type

$\longrightarrow$ standard translation into FO^2

modal Ehrenfeucht-Fraïssé theorem (1)

the following are equivalent:

(i) $\mathfrak{A}, a \equiv^{\mathsf{ML}_\ell} \mathfrak{B}, b$
(indistinguishable in ML up to nesting depth ℓ)

(ii) $\mathfrak{A}, a \sim^\ell \mathfrak{B}, b$
(**II** wins $\mathsf{G}_\ell(\mathfrak{A}, a; \mathfrak{B}, b)$)

with obvious variations for ML[∀/−]

Hintikka formulae

as in FO Ehrenfeuch-Fraïssé analysis:
characteristic formulae for ℓ-types
inductively, with crucial back & forth step:

$$\chi^{\ell+1}_{\mathfrak{A},a} = \chi^{\ell}_{\mathfrak{A},a} \wedge \bigwedge_{\alpha} \bigwedge_{a \xrightarrow{\alpha} a'} \langle\alpha\rangle\, \chi^{\ell}_{\mathfrak{A},a'} \wedge \bigwedge_{\alpha} [\alpha] \bigvee_{a \xrightarrow{\alpha} a'} \chi^{\ell}_{\mathfrak{A},a'}$$

forth: responses for challenges in $\mathfrak{A}$

back: responses for challenges in $\mathfrak{B}$

modal Ehrenfeucht-Fraïssé theorem (2)

the following are equivalent:

(i) $\mathfrak{A}, a \equiv^{\mathsf{ML}} \mathfrak{B}, b$
(indistinguishable in ML)

(ii) $\mathfrak{A}, a \sim^{\ell} \mathfrak{B}, b$ for all $\ell \in \mathbb{N}$
($\mathbf{II}$ wins $\mathsf{G}_{\ell}(\mathfrak{A}, a; \mathfrak{B}, b)$ for all $\ell \in \mathbb{N}$)

$\boxed{\mathfrak{A}, a \sim^{\omega} \mathfrak{B}, b}$

with obvious variations

$\sim^{\ell}$	$\equiv^{\mathsf{ML}_{\ell}}$	G_{ℓ}
$\sim^{\omega} := \bigcap_{\ell} \sim^{\ell}$	$\equiv^{\mathsf{ML}}$	G_{fin}
$\sim$		G_{∞}

infinitary modal logic ML_{∞}

closure under set-size $\bigwedge / \bigvee$
analogous to $\mathsf{L}_{\infty\omega}$

modal Ehrenfeucht-Fraïssé theorem (3)

Karp's theorem in modal logic

the following are equivalent:

(i) $\mathfrak{A}, a \equiv^{\mathsf{ML}_\infty} \mathfrak{B}, b$
(indistinguishable in ML_∞)

(ii) $\mathfrak{A}, a \sim \mathfrak{B}, b$
($\mathbf{II}$ wins $\mathsf{G}_\infty(\mathfrak{A}, a; \mathfrak{B}, b)$)

with obvious variations

summary: ML/FO Ehrenfeucht-Fraïssé

modal	first-order	
single pebble	m pebbles for m rounds	
move pebble along edge	put additional pebble	
maintain colour	maintain local isomorphy	
ℓ-round game: $\sim^{\ell}$ / $\equiv^{\mathsf{ML}_\ell}$	$\simeq_\ell$ / $\equiv_\ell$	E-F
infinite game: $\sim$ / $\equiv^{\mathsf{ML}_\infty}$	$\simeq_{\text{part}}$ / $\equiv^{\mathsf{L}_{\infty\omega}}$	Karp

preservation/invariance from Ehrenfeucht-Fraïssé

e.g., modal E-F: $\sim^{\ell}$ implies $\equiv^{\mathsf{ML}_\ell}$: **$\mathsf{ML}_\ell$ $\sim^{\ell}$ invariant**

$\Rightarrow$ **ML $\sim$ invariant**

bisimulation invariance and special models

tree model property (tmp)

- any model is bisimilar to its tree unfolding (unravelling)
- $\sim$ invariant logics have tree model property

examples: ML, PDL, CTL, L_μ, ...
tmp crucial for satisfiability, over and above fmp

variations for refined notions of bisimulation

e.g., ML[$-$], $\sim_-$ invariant, admits two-way tree models
ML[$\forall$], $\sim_\forall$ invariant, admits forest models
guarded fragments admit tree-like models ($\rightarrow$ later)

how far into FO?

in arbitrary relational structures $\mathfrak{A} = (A, R^{\mathfrak{A}}, \dots)$

Gaifman graph $G(\mathfrak{A})$ of $\mathfrak{A}$

$G(\mathfrak{A}) = (A, E)$ where $(a, a') \in E$ if $a \neq a'$ and
$a, a' \in [\boldsymbol{a}]$ for some $\boldsymbol{a} \in R^{\mathfrak{A}}$

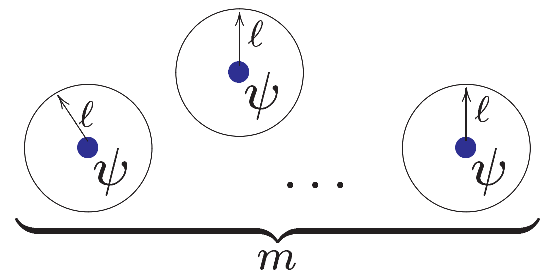
Gaifman distance $d(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{N} \cup \{\infty\}$
Gaifman neighbourhoods $N^\ell(a) = \{b \in A \colon d(a, b) \leqslant \ell\}$
$N^\ell(\boldsymbol{a}) = \bigcup_i N^\ell(a_i)$

$\rightarrow$ **modularity of FO E-F game w.r.t. Gaifman locality**

Gaifman locality

local FO formulae: $\varphi^\ell(\boldsymbol{x}) := [\varphi(\boldsymbol{x})]^{N^\ell(\boldsymbol{x})}$ (relativisation)

basic local FO sentences:
asserting existence of m elements
of pairwise distance $> 2\ell$
within some $\psi^\ell[\mathfrak{A}]$
ℓ-scattered sets/tuples



Gaifman-equivalence

$\mathfrak{A}, \boldsymbol{a} \equiv^{\ell}_{q,m} \mathfrak{B}, \boldsymbol{b}$ agreement w.r.t. local properties / scattered sets;

parameters/bounds: $\begin{cases} \ell & \text{– radius} \\ q & \text{– qfr rank} \\ m & \text{– set size} \end{cases}$

local similarity & global condition via scattered sets; finite index

Gaifman's theorem

- each FO-formula $\varphi(\boldsymbol{x})$ preserved under $\equiv^{\ell}_{q,m}$ for sufficiently large ℓ, q, m
- each FO-formula $\varphi(\boldsymbol{x})$ logically equivalent to boolean combination of local formulae and basic local sentences

proof: modularity of strategies

back & forth for systems of $(\boldsymbol{a} \mapsto \boldsymbol{b})$

where $\mathfrak{A}, \boldsymbol{a} \equiv^{\ell_i}_{q_i} \mathfrak{B}, \boldsymbol{b}$ (local compatibility)

in $\mathfrak{A} \equiv^{L}_{Q,M} \mathfrak{B}$ (global condition)

special locality in modal quantification: modal view

- modal quantification $\langle \alpha \rangle$ is local
- $\varphi \in$ ML of nesting depth ℓ is ℓ-local

$\sim^\ell$ is ℓ-local

and even

$\mathfrak{A}, a \sim^\ell \mathfrak{B}, b \;\Leftrightarrow$
$\mathfrak{A}{\restriction}N^\ell(a), a \sim \mathfrak{B}{\restriction}N^\ell(b), b$
in tree structures

consequences:

ML has fmp

from tree models to finite tree models for $\varphi \in \mathrm{ML}_\ell$:

cut-off at depth ℓ
pruning to degree $\leqslant$ index of $\sim^\ell$

ML$[\forall, -]$ has fmp

from forest models to finite models for $\varphi \in \mathrm{ML}_\ell[\forall, -]$:

finite submodel realising same $\sim^\ell_-$-types
plus simple surgery for joining loose ends

Gaifman for special local properties: FO view

$$\text{closure under} \begin{cases} \text{(two-way) bisimulation} \\ \text{global (two-way) bisimulation} \end{cases}$$

$$\text{implies closure under} \begin{cases} \text{disjoint unions (with other structures)} \\ \text{disjoint copies} \end{cases}$$

Gaifman equivalence

$$\text{reduces to} \begin{cases} \equiv^{\ell}_{q,0} & \text{no global constraints} \\ \equiv^{\ell}_{q,1} & \text{local element types} \end{cases}$$

in FMT as well as classically: (examples)

for any $\varphi(x) \in$ FO:

$\varphi \sim$ invariant iff $\varphi \equiv \varphi' \in$ ML

$\varphi \sim_\forall$ invariant iff $\varphi \equiv \varphi' \in$ ML[$\forall$]

$\varphi \approx$ invariant iff $\varphi \equiv \varphi' \in$ ML[$\forall, -$]

crux of proofs: **expressive completeness**
invariance $\longrightarrow$ definability

expressive completeness: invariance $\longrightarrow$ definability

proofs crucially establish expressive completeness via

compactness property
for back & forth $\leftrightharpoons$ invariance of FO properties:

$$\varphi \leftrightharpoons \textbf{ invariant } \Rightarrow \varphi \leftrightharpoons^{\ell} \textbf{ invariant for some } \ell \in \mathbb{N}$$

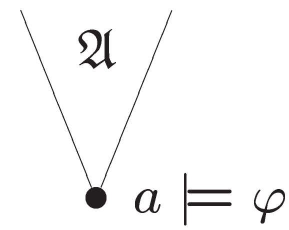
classically: compactness and ω-saturated models

classically and in FMT: upgrading $\leftrightharpoons^{\ell}$ to some $\equiv^{k}_{q,m}$ in game-oriented model constructions

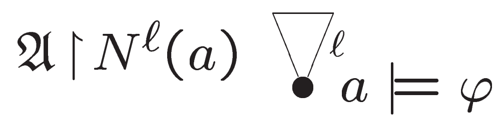
van Benthem–Rosen reproved   Ehrenfeucht-Fraïssé

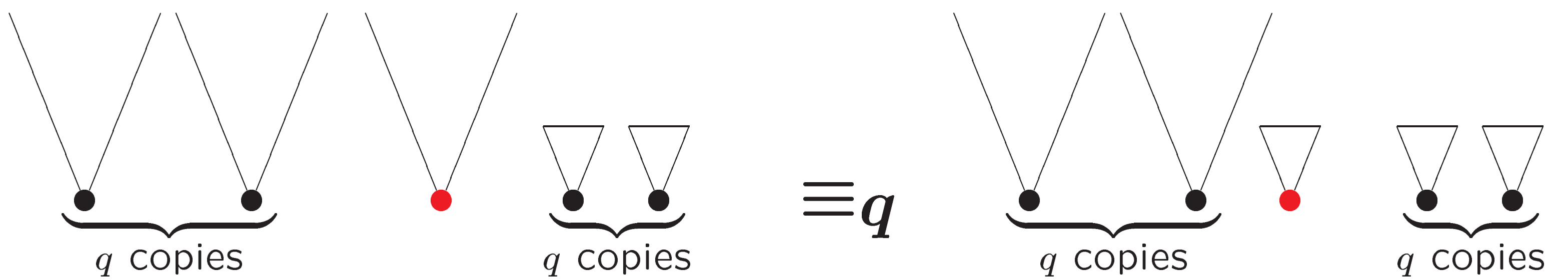
$\varphi \sim$ **invariant** $\Rightarrow \varphi$ ℓ-local for $\ell = 2^{\mathrm{qr}(\varphi)}$

$\Rightarrow \varphi \sim^{\ell}$ **invariant for** $\ell = 2^{\mathrm{qr}(\varphi)}$



show $\quad \mathfrak{A}, a \models \varphi \quad$ **iff** $\quad \mathfrak{A} \restriction N^{\ell}(a), a \models \varphi$





II can play q rounds (‘locally or isomorphically’)
respecting critical distance $d_m = 2^{q-m}$ in round m

expressive completeness w.r.t. global equivalences

Ehrenfeucht-Fraïssé + Gaifman: local modularity via $\equiv^{\ell}_{q,m}$

generic idea: upgrading $\leftrightharpoons^{\ell}$ to $\equiv^{\ell}_{q,m}$

$$\begin{array}{ccc} \mathfrak{A}, a & \overset{\leftrightharpoons^{\ell}}{\longrightarrow} & \mathfrak{B}, b \\ \Big| \leftrightharpoons & & \Big| \leftrightharpoons \\ \tilde{\mathfrak{A}}, \tilde{a} & \overset{\equiv^{\ell}_{q,m}}{\longrightarrow} & \tilde{\mathfrak{B}}, \tilde{b} \end{array} \qquad \Downarrow$$

local control over FO
up to quantifier rank q

globally

$$\left. \begin{array}{r} \leftrightharpoons \text{ invariance} \\ \text{preservation under } \equiv^{\ell}_{q,m} \end{array} \right\} \Rightarrow \quad \leftrightharpoons^{\ell} \text{ invariance}$$

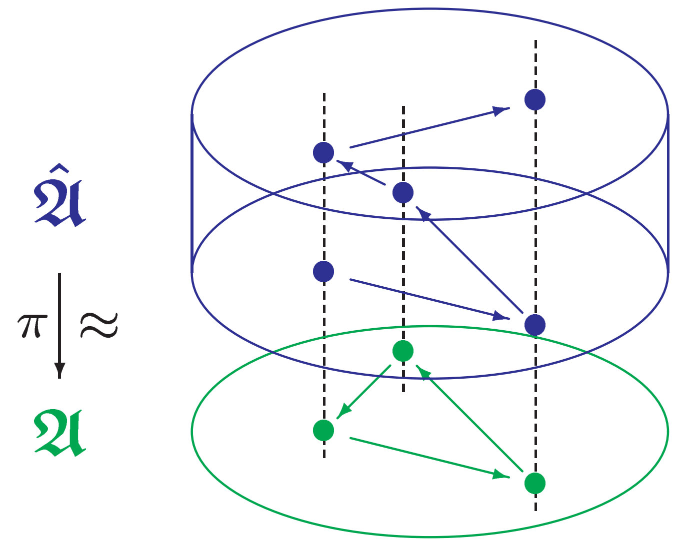
e.g., global two-way bisimulation invariance

upgrading $\approx^{\ell}$ to $\equiv^{\ell}_{q,m}$ in ℓ-locally acyclic covers

bisimilar covers

homomorphism $\pi : \hat{\mathfrak{A}} \rightarrow \mathfrak{A}$
whose graph induces a
two-way global bisimulation

faithful if it preserves
in- and out-degrees



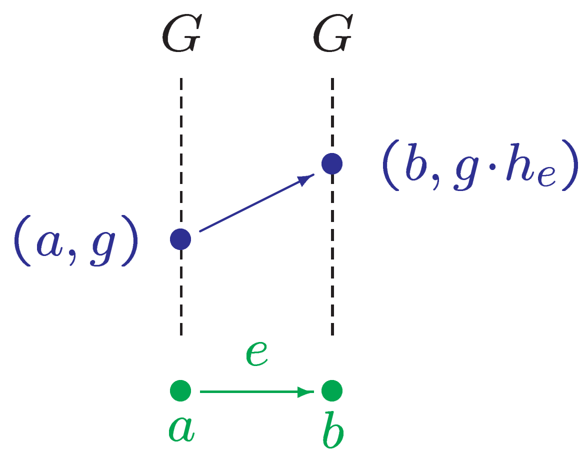
- two-way tree unfoldings are (infinite) acyclic covers
- locally acyclic covers can be kept finite

locally acyclic covers

theorem

any [finite] transition system admits a faithful cover
by a [finite] ℓ-locally acyclic system

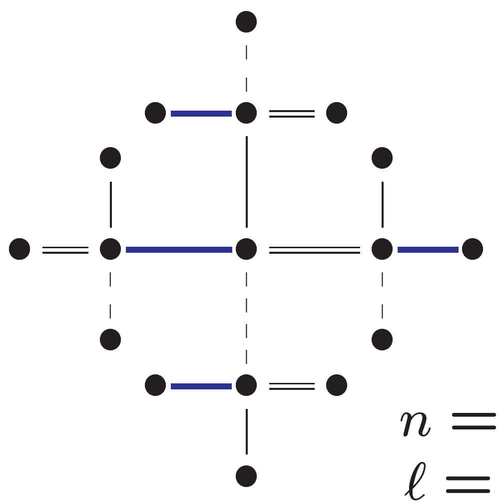
construction:
a product with group G with
Cayley graph of large girth



variations/ramifications:

similar results now also for **S5 frames** (equivalence relations)
rooted frames (connectivity)
other metrics

aside: Biggs' construction of Cayley graphs of large girth



for $n := |E|$ consider tree T:
full n-regular tree of depth ℓ,
regularly n-coloured with $e \in E$

$\boldsymbol{e \in E}$ induces permutation $e \in \text{Sym}(T)$: **flip e-coloured edges**

$$\boxed{G := \langle e \colon e \in E \rangle \subseteq \text{Sym}(T)}$$

Cayley graph of G has girth $> 4\ell$

refinement, polynomial in n for fixed ℓ $\longrightarrow$ Margulis, Imrich

from $\approx^\ell$ to $\equiv^\ell_{q,m}$

locally in acyclic $N^\ell(x)$
win q rounds of FO game if
all multiplicities are at least q

globally need
ℓ-scattered sets of size m
for each available $\approx^\ell$ type

use **faithful covers**
after boosting multiplicities
in trivial products and sums

remark can also upgrade $\sim^{2\ell}_\forall$ to $\approx^\ell$
in $\sim_\forall$ equivalent structures

variations: other non-elementary classes

examples

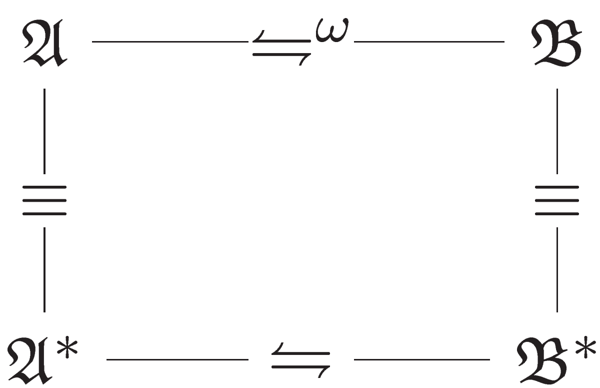
theorem Dawar O_ 05

ML[∀] expressively complete also for $\sim_\forall$-invariance
in restriction to the class of all

- **(finite) rooted frames**
- **(finite) S5 frames** (several equivalence relations)
- **(finite) transitive frames** (one transitive relation)
  based on decomposition techniques rather than locality

observation: orthogonal approaches in expressive completeness proofs

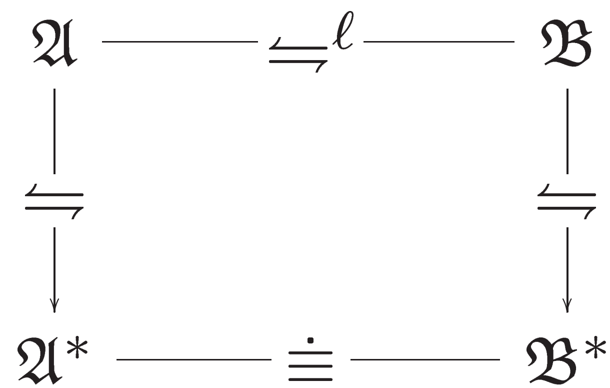
via full $\equiv$ to full $\leftrightharpoons$



prep: $(\leftrightharpoons^{\ell})_{\ell \in \omega} \longrightarrow \leftrightharpoons^{\omega}$

upgrading via ω-saturation

via full $\leftrightharpoons$ to approximate $\equiv$



direct upgrading

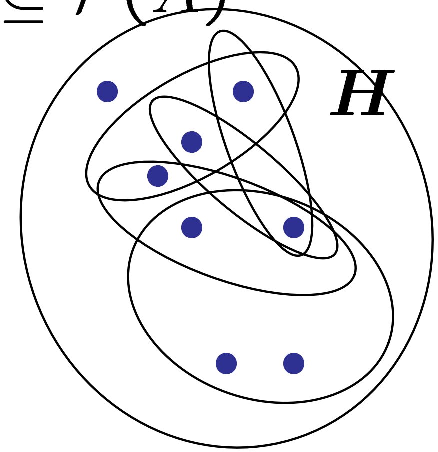
Part II: From Graphs to Hypergraphs

aim: to extend 'modal' analysis to richer structures/fragments

hypergraph $H = (A, S),\ S \subseteq \mathcal{P}(A)$

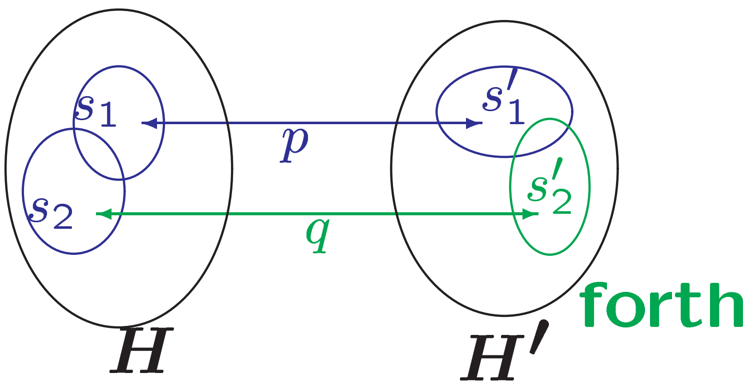
hyperedges $s \in S$

width: max $|s|$



back & forth on hypergraphs: hypergraph bisimulation

$Z\colon H \approx H'$



- $\emptyset \neq Z \subseteq \mathrm{Part}(A, A')$
- local bijections between hyperedges
- back & forth respecting overlaps

- **game formulation**
- **generalisation of bisimulation on undirected graphs**
- **an Ehrenfeucht-Fraïssé notion?**
  $\rightarrow$ hypergraphs associated with relational structures

guardedness: the hypergraph of guarded subsets

for any relational $\mathfrak{A} = (A, R^{\mathfrak{A}}, \ldots)$:

$$H(\mathfrak{A}) = (A, S)$$

$$S = \{\{a\} : a \in A\} \cup \{[\boldsymbol{a}] : \boldsymbol{a} \text{ in } R^{\mathfrak{A}}\}$$

hypergraph of guarded sets

$\text{width}(H(\mathfrak{A})) = \text{width}(\tau)$

$H(\mathfrak{A})$ and $G(\mathfrak{A})$

with any hypergraph associate graph:

$$G(H) = (A, E)$$

$$E = \bigcup_{s \in S} K[s] \qquad K[s]: \text{ clique on } s$$

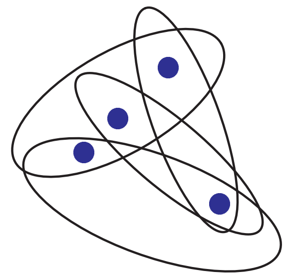
associated (Gaifman) **graph**

Then $G(\mathfrak{A}) = G(H(\mathfrak{A}))$

graphs and hypergraphs

hyperedges of H give rise to cliques in $G(H)$, but not every clique in $G(H)$ is covered by a hyperedge:

unguarded clique in $G(H)$:



$\longrightarrow$ hypergraph $G_{\text{cl}}(\mathfrak{A})$
whose hyperedges are the cliques in $G(\mathfrak{A})$

conformal hypergraphs: no unguarded cliques

acyclicity of hypergraphs: chordality and conformality

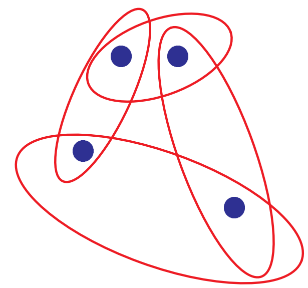
equivalent (for finite hypergraphs):

H fully reducible via deletions of

- $a \in A$ that are not multiply covered by S
- $s \in S$ if $s \subseteq s'$ for some $s' \in S$

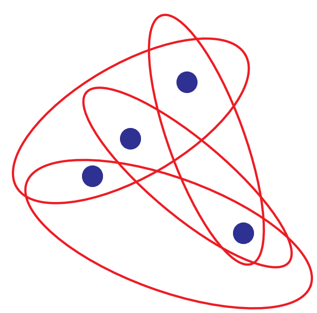
$G(H)$ has neither

irreducible cycles
chordal: no bad cycles



nor

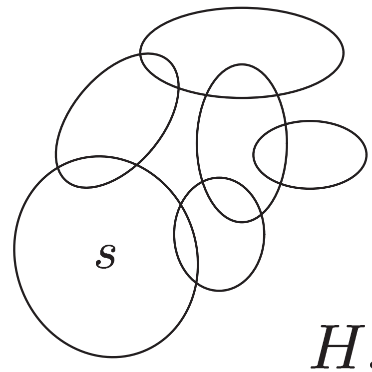
unguarded cliques
conformal: no bad cliques



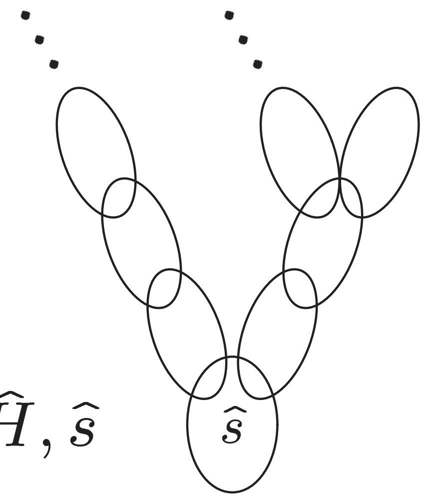
both obstacles, cycles and unguarded cliques
broken up in tree-like unfoldings

tree unfoldings

bisimilar hypergraph covers
built from tree of paths
of overlapping hyperedges



$H, s \approx \hat{H}, \hat{s}$



$\hat{H}$ acyclic

construction:
use tree unfolding of derived transition system with state set S
and edge labels for overlap identification

- tree decompositions of $G(\hat{H})$, of treewidth width$(H) - 1$
- locality disturbed!

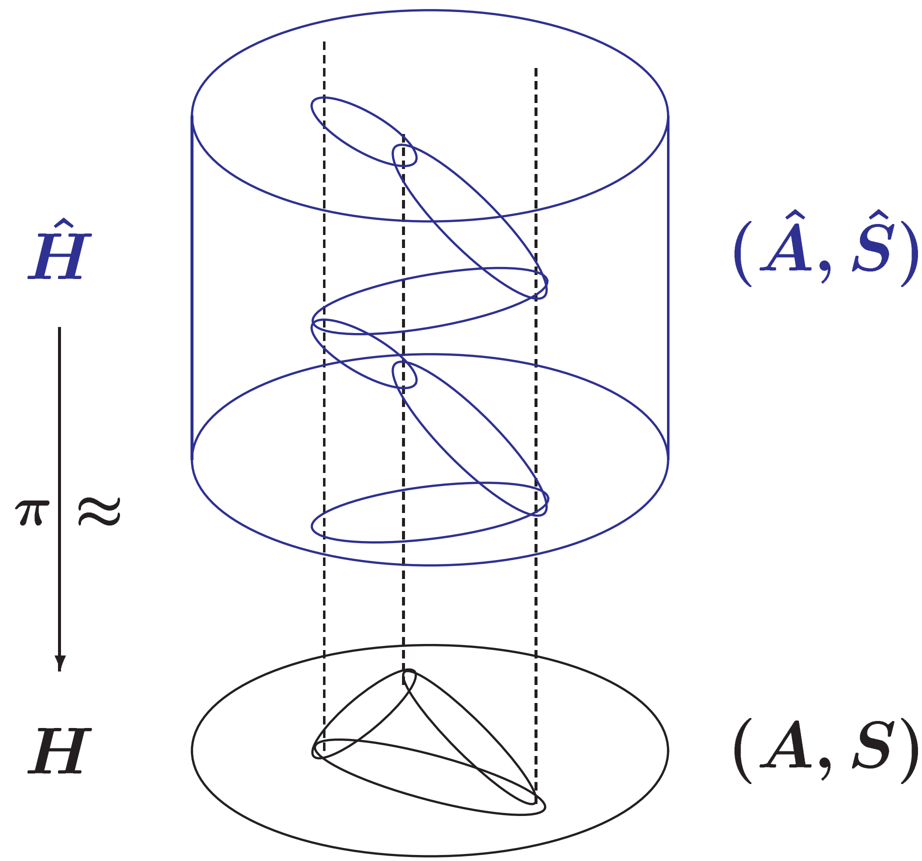
bisimilar hypergraph cover

of $H = (A, S)$ by $\hat{H} = (\hat{A}, \hat{S})$

$$\pi : \hat{H} \to H$$
$$\hat{H} \approx H$$

induced bisimulation

locally 1-1



unfoldings provide
(infinite) **acyclic covers**

finite conformal hypergraph covers

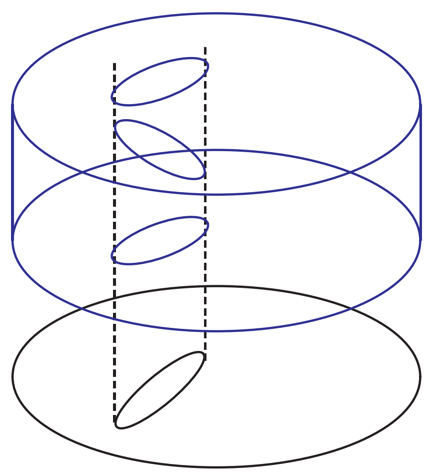
theorem Hodkinson O_ 03

every finite hypergraph H admits a cover

$\pi \colon \hat{H} \xrightarrow{\approx} H$ by a **finite conformal $\hat{H}$**

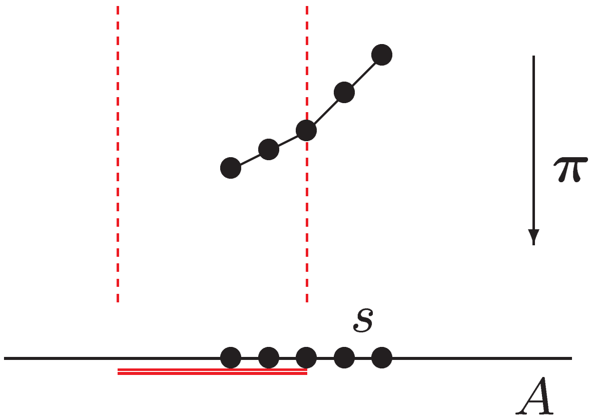
construction canonical and **symmetry preserving**:

- $\mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(\hat{H})$
- $\mathrm{Aut}_\pi(\hat{H})$ **transitive on** $\{\hat{s} \in \hat{S} \colon \pi(\hat{s}) = s\} \subseteq \hat{S}$



finite conformal covers: the construction

$$\pi\colon \hat{H} = (\hat{A}, \hat{S}) \xrightarrow{\approx} H = (A, S)$$



geometric intuition:
local “generic” cross-sections
in fibres over base set

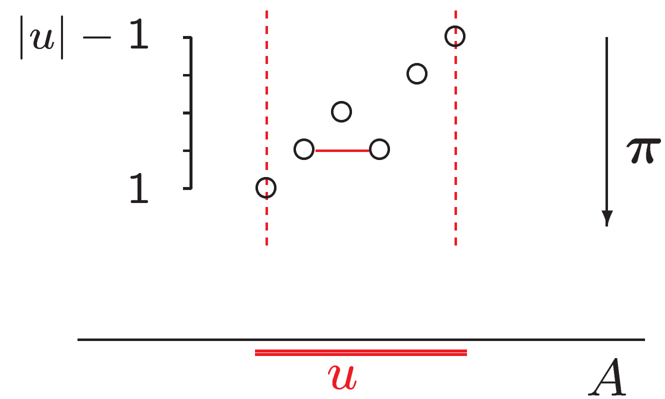
forbidden subsets: $U := \{ u \subseteq A \colon u \text{ unguarded} \}$

$u \in U$ must not be covered by clique

construction of conformal covers ctd.

the universe: $\hat{\mathrm{A}}: \begin{cases} (a,\chi)\ ,\ \chi\colon U \to \mathbb{N} \\ 0 \leqslant \chi(u) \leqslant |u|-1 \\ \chi(u) \neq 0 \text{ for all } u \text{ s.t. } a \in u \end{cases}$

generic subsets $\hat{s} \subseteq \hat{A}$: **local cross sections** $\hat{s}\colon a \mapsto (a, \chi_a)$
s.t. $a \mapsto \chi_a(u)$ **is 1–1 above every** u

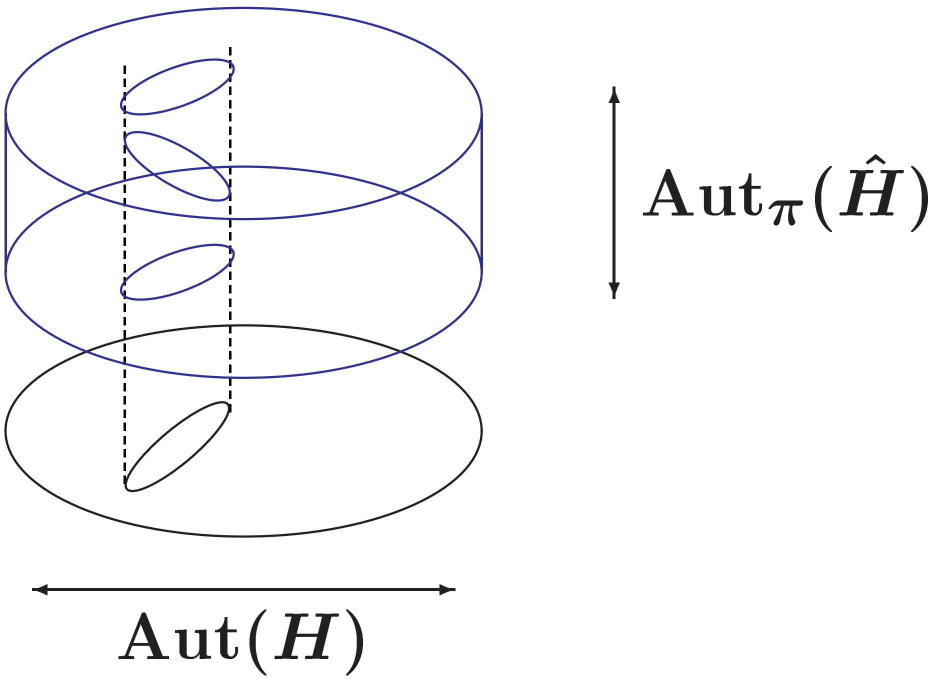


- for cardinality reasons $\pi(\hat{s}) \notin U$: cannot cover forbidden subsets

- generic sets sufficient for cover: **back** & forth property

automorphism properties of $\pi\colon \hat{H} \xrightarrow{\approx} H$

- $\mathbf{Aut}(H) \hookrightarrow \mathbf{Aut}(\hat{H})$
- $\mathbf{Aut}_{\pi}(\hat{H})$ **transitive on** $\{\hat{s} \in \hat{S}\colon \pi(\hat{s}) = s\}$



guarded fragment GF

Andréka, van Benthem, Németi

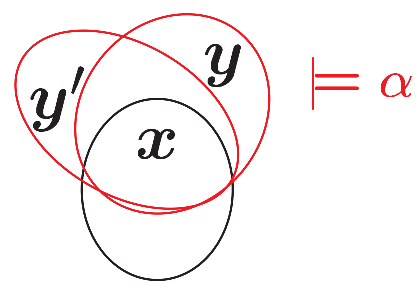
quantification relativised to relational (hyper-)edges

$$\forall \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \rightarrow \varphi(\boldsymbol{x},\boldsymbol{y})\big)$$

guard atom: free(φ) $\subseteq$ var(α)

$$\exists \boldsymbol{y}\big(\alpha(\boldsymbol{x},\boldsymbol{y}) \wedge \varphi(\boldsymbol{x},\boldsymbol{y})\big)$$

guarded subsets $[\boldsymbol{a}]$ for $\boldsymbol{a} \in R^{\mathfrak{A}}$
—hyperedges of $H(\mathfrak{A})$—
as "visible patches"



clique guarded fragment CGF

quantification relativised to clique guarded tuples

clique guarded subsets $[\boldsymbol{a}]$
—cliques in $G(\mathfrak{A})$—
as "visible patches"

ML$[\forall, -]$ $\subseteq$ GF $\subseteq$ CGF $\subseteq$ FO

some key algorithmic properties of guarded logics

satisfiability testing

GF and CGF $\left\{ \begin{array}{l} \text{2-ExpTime complete} \\ \text{ExpTime complete for fixed width} \end{array} \right.$ Grädel

robustness: same for fixpoint extensions Grädel,
no fmp Walukiewicz

methods: tree-like models ($\rightarrow$ below)
logical reductions, **tree automata**

key algorithmic properties of guarded logics ctd.

model checking complexity

GF: check $\mathfrak{A} \models \varphi$ for GF in $O(||\mathfrak{A}|| \cdot |\varphi|)$ Grädel, Berwanger
as good as for ML

CGF: $||\mathfrak{A}|| \longrightarrow$ bound on number of guarded cliques of corresponding width

fixpoint extensions: exponential dependency on alternation depth

methods: **model checking games**

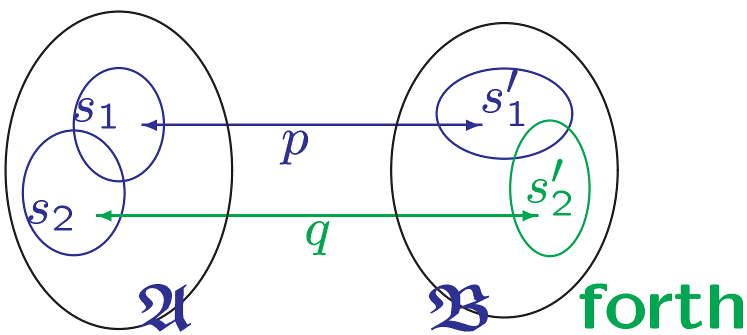
$|A|$ size of universe
$||A||$ length of succinct encoding

GF and guarded bisimulation

back & forth: **hypergraph bisimulation + local isomorphism**
= Ehrenfeucht-Fraïssé for guarded quantification

guarded bisimulation $Z\colon \mathfrak{A} \sim_{\mathrm{g}} \mathfrak{B}$

hypergraph bisimulation $Z\colon H(\mathfrak{A}) \approx H(\mathfrak{B})$ where $Z \subseteq$ Part$(\mathfrak{A}, \mathfrak{B})$



- GF preserved under $\sim_{\mathrm{g}}$
- tree-like models via hypergraph unfoldings
- finite models?

classical analysis

for $\varphi(\boldsymbol{x}) \in \mathrm{FO}$: $\sim_{\mathrm{g}}$ invariance $\Rightarrow$ $\sim_{\mathrm{g}}^{\ell}$ invariance for some $\ell \in \mathbb{N}$

theorem Andréka, van Benthem, Németi

for any $\varphi(\boldsymbol{x}) \in \mathrm{FO}$: $\boldsymbol{\varphi \sim_{\mathrm{g}}}$ **invariant iff** $\boldsymbol{\varphi \equiv \varphi' \in}$ **GF**

status in FMT open!

- can $\sim_{\mathrm{g}}^{\ell}$ be upgraded to $\equiv_{q,m}^{\ell}$ in finite structures?
- which degree of hypergraph acyclicity in finite covers?

  clear: acyclicity in N^{ℓ} too strong

generalised tree model property

theorem Grädel

any satisfiable GF formula has a model $\mathfrak{A}$ with acyclic $H(\mathfrak{A})$ of treewidth width$(\tau) - 1$

$\rightarrow$ tree decompositions, bounded treewidth (later)

review

modal analayis, locality and modularity of games

- **bisimulation**
- **expressive completeness proofs**
  upgrading to levels of $\equiv$ based on locality
- **from graphs to hypergraphs**
  hypergraph bisimulation
  guardedness as another form of locality

next: two connections/applications

- **extension properties for partial automorphisms**
  finite model properties for guarded fragments
- **a decidable case of FO boundedness**
  via guardedness and ML

aside: modal view of guarded bisimulation

relational structure $\mathfrak{A}$	guarded bisimulation
$\downarrow$	$\downarrow$
hypergraph $H(\mathfrak{A})$	hypergraph bisimulation by partial isomorphisms
$\downarrow$	$\downarrow$
derived transition system	modal bisimulation

derived transition system:

states: guarded subsets
transitions: identifications in overlaps

$\longrightarrow$ guarded tree unfolding via
modal tree unfolding and reconstruction

reductions to model theory of trees

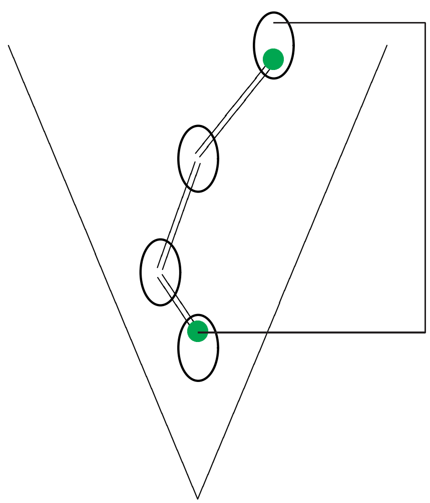
guarded bisimulation and finite models

no simple adaptation of modal fmp arguments:

non-locality of guarded unravelling

derived transition systems ‘carry’ elements along paths

$\rightarrow$ distance in tree does *not* reflect Gaifman distance



problem:
potential conflict
of identifications

solution: **extension properties** for partial isomorphisms
instead of identification

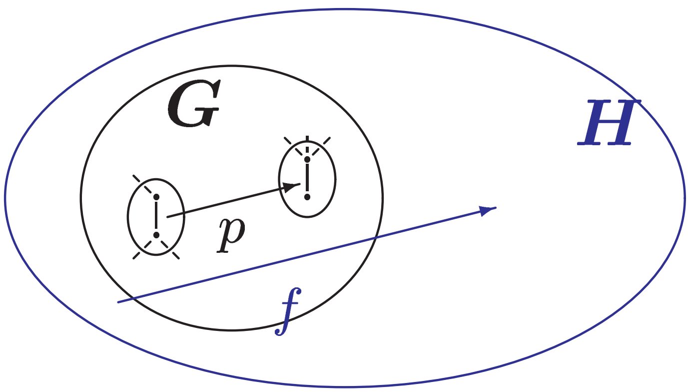
extensions of partial automorphisms, EPPA

Truss, Hrushovski, Herwig, Lascar, . . .

extension theorem

basic version, for finite undirected graphs

given **finite G**, $p \in \mathrm{Part}(G)$ — find **finite $H \supseteq G$**, $p \subseteq f \in \mathrm{Aut}(H)$



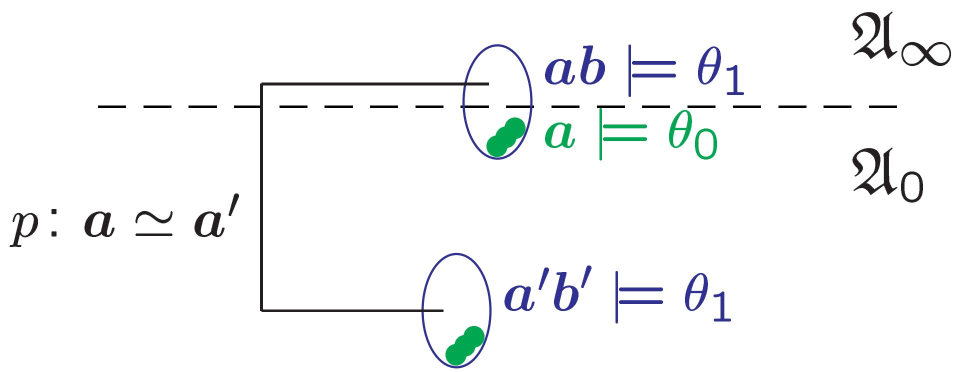
finiteness!
reduced product with free group
gives infinite solution

with variations: extend one p / extend all p
preserve other features

after Skolemization, need to provide witnesses for

$$\boldsymbol{\theta_0(x) \to \exists y \theta_1(x, y)}$$

θ_i qfr-free, with (clique) guarded $\boldsymbol{xy}$



in suitable EPPA extension $\mathfrak{B} \supseteq \mathfrak{A}_0$:

find automorphic image of $\boldsymbol{a'b'}$ at $\boldsymbol{a}$
without creating "new" instances

aside: Lascar's combinatorial proof for graphs

$G = (V, E)$ finite, undirected, w.l.o.g. regular of degree n

else extend with new vertices

embed G into incidence graph $\widehat{G}$ of regular hypergraph H

$$
\begin{aligned}
H &= (E, S) \\
S &= [E]^n \qquad \text{(n-element subsets)} \\
\widehat{G} &= \big(S, I) \\
I &= \{(s, s') : s \cap s' \neq \emptyset\}
\end{aligned}
$$

$G \hookrightarrow \widehat{G}$ via $\quad \rho \colon v \longmapsto \{e \in E : e \text{ incident with } v\}$

$$
\Rightarrow \quad \boxed{\begin{array}{rcl} \text{for all } u, v \in G\text{: } |\rho(u) \cap \rho(v)| & \leqslant & 1 \\ (u, v) \in E \text{ iff } |\rho(u) \cap \rho(v)| & = & 1 \end{array}}
$$

- $\mathbf{Aut}(\hat{G}) = \mathbf{Aut}(\hat{H}) = \mathbf{Sym}(E)$
- $\hat{G} \supseteq G$ **has extensions for all** $p \in \mathrm{Part}(G)$

EPPA for arbitrary relational structures

theorem Herwig 95

any finite relational $\mathfrak{A}$ has finite extension $\mathfrak{B} \supseteq \mathfrak{A}$
such that every partial automorphism of $\mathfrak{A}$
extends to automorphism of $\mathfrak{B}$

omitting configurations

interest here: extensions without 'incidental' guarded sets
or clique guarded sets

e.g., in EPPA extension $\mathfrak{B} \supseteq \mathfrak{A}$,
passage from $R^{\mathfrak{B}}$ to $R^{\mathfrak{B}} \cap \langle R^{\mathfrak{A}} \rangle^{\mathrm{Aut}(\mathfrak{B})}$
eliminates incidental guarded sets

key EPPA results

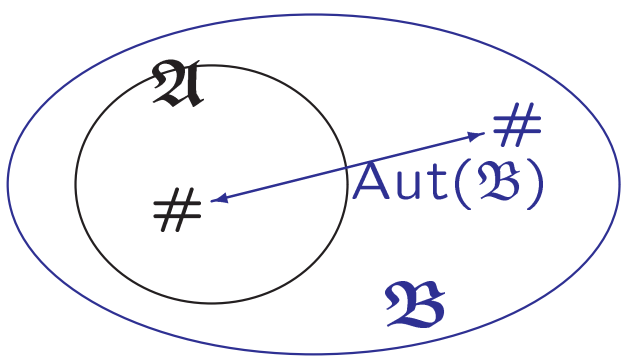
Truss 92	graphs, one p
Hrushovski 92	graphs, all p
Herwig 95	arbitrary relational structures
Herwig 98	omitting configurations e.g.: K_n-free graphs Henson digraphs
Herwig/Lascar 00	connections with theory of free groups and simpler combinatorial proofs

omitting configurations, tight EPPA

a class of configurations #

guarded subsets,
clique guarded subsets (= Gaifman cliques)

EPPA extension $\mathfrak{B} \supseteq \mathfrak{A}$ **tight w.r.t. #**
if $\mathfrak{B}$ realises no "new" #
all # in $\mathfrak{B}$ automorphic images of # from $\mathfrak{A}$



- tight w.r.t. guarded subsets: immediate by restriction
- tight w.r.t. Gaifman cliques: conformal hypergraph covers

Hodkinson O_ 03

finite conformal covers for clique tight EPPA

from ordinary to clique tight EPPA extensions:

(1) $\mathfrak{B} \supseteq \mathfrak{A}$ ordinary EPPA extension

(2) consider hypergraph $H := (B, S)$, $S = \{ f[A] \colon f \in \mathrm{Aut}(\mathfrak{B}) \}$

(3) conformal cover $\pi \colon \hat{H} \xrightarrow{\approx} H$

induces $\pi \colon \hat{\mathfrak{B}} \xrightarrow{\sim_{\mathrm{g}}} \mathfrak{B}$

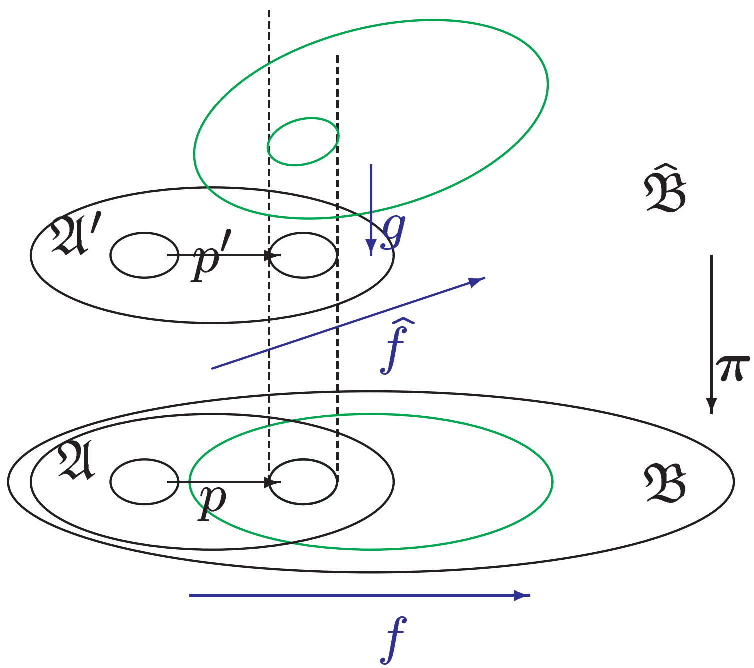
(4) $\mathfrak{A} = \mathfrak{B} \restriction A$ embeds as $\mathfrak{A} \simeq \mathfrak{A}' \subseteq \hat{\mathfrak{B}}$

$\Rightarrow$ **$\mathfrak{A}' \subseteq \hat{\mathfrak{B}}$ EPPA extension, tight w.r.t. Gaifman cliques**

tightness w.r.t. Gaifman cliques: obvious by conformality

crux of proof: $\mathrm{Aut}(\mathfrak{B}) \hookrightarrow \mathrm{Aut}(\hat{\mathfrak{B}})$
special automorphism properties
of conformal cover contruction

- $p \in \mathrm{Part}(\mathfrak{A})$ induces $p' \in \mathrm{Part}(\mathfrak{A}')$
- find extension $f \in \mathrm{Aut}(\mathfrak{B})$: $f \supseteq p$
- f has lift $\hat{f}$ to cover: $\hat{f} \in \mathrm{Aut}(\hat{\mathfrak{B}})$
- in general $\hat{f} \not\supseteq p'$ but $g \circ \hat{f} \supseteq p'$ for a 'vertical' automorphism g



clique tight extension thm

EPPA extensions without “new” Gaifman cliques

corollaries

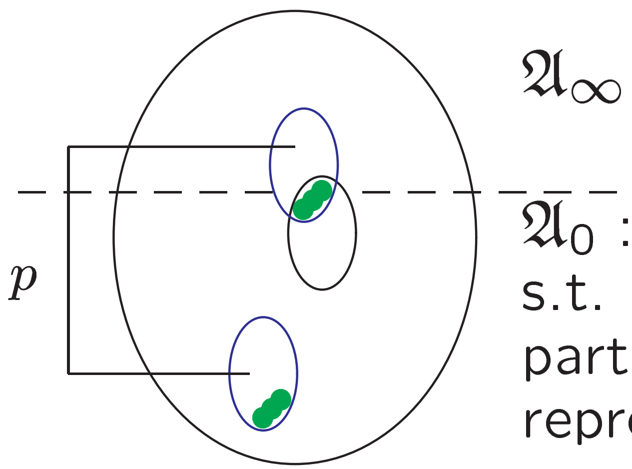
EPPA for

- **K_n-free graphs** Herwig/Lascar
- **Henson digraphs** Herwig/Lascar
- **conformal relational structures**

application: fmp for guarded logics

EPPA and the finite model property for GF, CGF

finite closure of partial model (after Skolemization)



$\mathfrak{A}_\infty \models \varphi$

$\mathfrak{A}_0$: a large finite segment s.t. all missing witnesses have partially isomorphic representatives in $\mathfrak{A}_0$

φ holds in EPPA extensions of $\mathfrak{A}_0$ that are tight w.r.t.

- guarded sets [for GF] Grädel
- clique guarded sets [for CGF] Hodkinson O_

guarded analysis of boundedness

monadic least fixed point induction on $\varphi(X, x)$, positive in X

least fixpoint $\boxed{(\mu_X \varphi)[\mathfrak{A}] = \bigcap \{ P \subseteq A^r : \varphi[\mathfrak{A}, P] \subseteq P \}}$

inductive generation: $(\mu_X \varphi)[\mathfrak{A}] = \bigcup_\alpha X^\alpha[\mathfrak{A}]$

$$\text{where} \quad \begin{aligned} X^0[\mathfrak{A}] &= \emptyset \\ X^{\alpha+1}[\mathfrak{A}] &= \varphi[\mathfrak{A}, X^\alpha[\mathfrak{A}]] \\ X^\lambda[\mathfrak{A}] &= \bigcup_{\alpha<\lambda} X^\alpha[\mathfrak{A}] \end{aligned}$$

closure ordinal $\gamma_\varphi(\mathfrak{A}) = \min_\alpha \big(X^{\alpha+1}[\mathfrak{A}] = X^\alpha[\mathfrak{A}] \big)$

$\mu_X \varphi(X, x)$ **bounded** if, for some $n \in \mathbb{N}$, $\gamma_\varphi(\mathfrak{A}) < n$ for all $\mathfrak{A}$

Barwise–Moschovakis theorem

for any FO formula $\varphi(X, \boldsymbol{x})$, postive in X,
the following are equivalent:

(i) $\boldsymbol{\mu_X \varphi}$ **bounded**

(ii) $\boldsymbol{\mu_X \varphi}$ **uniformly FO definable**

(iii) $\boldsymbol{\mu_X \varphi[\mathfrak{A}]}$ **FO definable in each** $\boldsymbol{\mathfrak{A}}$

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) obvious

(iii) $\Rightarrow$ (i): compactness argument in ω-saturated $\mathfrak{A}$

boundedness as a decision problem

for a class $\mathcal{F}$ of FO formulae: $\mathbf{BDD}(\mathcal{F})$

given $\varphi(X, \boldsymbol{x}) \in \mathcal{F}$
decide whether $\mu_X \varphi$ is bounded

- BDD a generalised SAT problem
- natural $\mathcal{F}$ for BDD: closed under iteration
- few decidable classes

undecidability results for (monadic) BDD

BDD(mon $\exists^*$) undecidable (even pos $\exists^*$ with $\neq$) Gaifman et al 87

BDD(mon FO^2) undecidable Kolaitis O_ 98

decidability results for monadic BDD

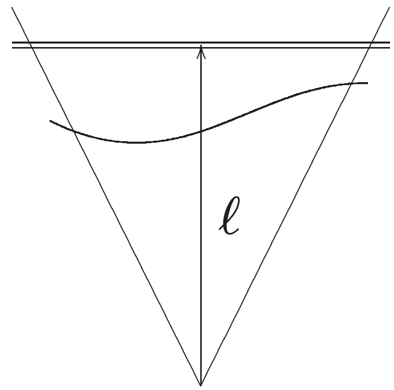
BDD(pos mon $\exists^*$) decidable (Datalog) Cosmadakis et al

BDD(ML) decidable O_ 98

deciding BDD(ML)

- **Barwise–Moschovakis** (modal version):
  $\mu_X\varphi(X)$ bounded iff $\mu_X\varphi$ definable in ML
- **locality and tree models**:
  $\mu_X\varphi$ definable in ML iff $\mu_X\varphi$ ℓ-local in all trees for some $\ell \in \mathbb{N}$

locality of $\mu_X\varphi$ in all trees	reduces to	**MSO theory of the binary tree**



interesting technicality:
finite-depth-property
directly MSO definable
only in ranked trees

deciding $\mathrm{BDD}(\text{neg}\,\forall^*)$

Moschovakis normal form

variant for X-positive $\forall^*$ formulae $\varphi(X,x)$

$$\varphi(X,\boldsymbol{x}) \equiv \forall \boldsymbol{y}\Big(\theta(\boldsymbol{x},\boldsymbol{y}) \to \rho(X,\boldsymbol{y})\Big) \qquad \theta \text{ qfr-free, no } X$$
$$\rho = X\boldsymbol{y}_1 \vee \ldots \vee X\boldsymbol{y}_n$$

moreover: φ negative in all $R \in \sigma$ $\Rightarrow$ θ positive in all $R \in \sigma$

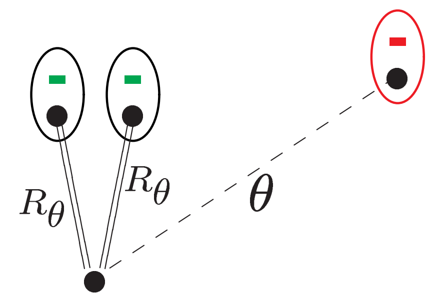
$\theta := \bigvee \theta_i$ (DNF)

reduction idea (monadic case):

- interpret the θ_i **as guards** (w.r.t. new R_{θ_i})
- stages of $\mu_X\varphi$ invariant under guarded unfolding $\mathfrak{A} \longmapsto \mathfrak{A}^*$
- in tree representations $T(\mathfrak{A}^*)$ φ translates into modal φ^* s.t.

$$\boxed{\gamma_\varphi[\mathfrak{A}] = \gamma_{\varphi^*}[\boldsymbol{T}(\mathfrak{A}^*)]}$$

problem: the class of $T(\mathfrak{A}^*)$ is not MSO definable arbitrary θ_i-trees T encode tree-decompositions of σ-structures $\mathfrak{A}(T)$



problem: **remote θ-witnesses**

accessible in MSO
inaccessible in ML

$$\varphi^* \text{ may overestimate } \begin{cases} \text{fixed point } (\mu_X\varphi)[\mathfrak{A}(T)] & (1) \\ \text{stages } X^i[\mathfrak{A}(T)] & (2) \end{cases}$$

(1) forbidden by MSO condition: **admissible trees**
(2) reduces to (harmless) $\gamma_{\varphi^*}[T] \leqslant \gamma_\varphi[\mathfrak{A}(T)]$

$\mu_X\varphi^*$ bounded over admissible trees $\Leftrightarrow$ $\mu_X\varphi$ bounded

some open problems

- (guarded) $\exists^*$-quantification similarly reducible?
- $\mathrm{BDD}(\mathbf{GF})$
- **$\mathrm{BDD}(\textbf{monadic FO})$ in bounded treewidth**

$\rightarrow$ uniform explanation for BDD decidability?

Part III: Special Classes

tree decompositions and bounded treewidth

- algorithmic benefits: model checking
- connections with locality
- generalisations and variations, CSP

locality related restrictions: wideness criteria

- wideness and connections with bounded treewidth
- preservation theorems classical and FMT
- homomorphism preservation and BDD
- extension preservation

bounded treewidth; tree decompositions; generalisations

tree decomposition of $\mathfrak{A}$:

tree T with labelling $\lambda\colon T \longrightarrow \mathcal{P}(A)$

$v \longmapsto \lambda(v) \subseteq A$ s.t.

(i) $s \subseteq A$ guarded $\Rightarrow$ $\exists v\,(s \subseteq \lambda(v))$

(ii) for every $a \in A$: $\{v \in T \colon a \in \lambda(v)\} \subseteq T$ connected

equivalent: the acyclic hypergraph $H = (A, \{\lambda(v) \colon v \in T\})$ majorises hypergraph $H(\mathfrak{A})$ of guarded sets

NB: guarded tree decompositions require $H \subseteq H(\mathfrak{A})$

width of tree decomposition: $\max_{v \in T}(|\lambda(v)|) - 1$

treewidth $\mathrm{tw}(\mathfrak{A})$: minimal width of any tree decomposition

$\mathcal{C}_k := \{\mathfrak{A} \colon \mathrm{tw}(\mathfrak{A}) \leqslant k\}$: treewidth k structures

bounded treewidth: algorithmic relevance

model checking problems

on input: φ, $\mathfrak{A}$
decide whether $\mathfrak{A} \models \varphi$

MC($\mathcal{L}, \mathcal{C}$): $\varphi \in \mathcal{L}$, $\mathfrak{A} \in \mathcal{C}$

combined complexity

uniform algorithms in $\mathfrak{A}, \varphi$
both $\mathfrak{A}$ and φ vary
complexity in terms of $||\mathfrak{A}||, |\varphi|$
considered for guarded logics above

data complexity

fix φ, non-uniform w.r.t. φ
complexity in terms of $||\mathfrak{A}||$

$|A|$ size of universe
$||A||$ length of succinct encoding

examples: combined complexities

- $\mathrm{MC}(\mathbf{FO}, \mathbf{all})$ complete for **PSpace**
- $\mathrm{MC}(\mathbf{FO}^{k}, \mathbf{all})$ complete for **PTime**
- $\mathrm{MC}(\mathbf{GF}, \mathbf{all})$ complete for **PTime**, in $O(|\varphi| \cdot ||\mathfrak{A}||)$, linear

compare FO data complexity:

in LogSpace (hence Ptime)
but with syntactic parameters of φ in the exponent

two orthogonal sources of
lower complexity:

arbitrary formulae over **tree-like structures** (data compl.)
tree-like formulae over **arbitrary structures** (combined)

model checking in bounded treewidth structures

$\mathrm{MC}(\mathbf{MSO}, \mathcal{C}_k)$ in linear time data complexity Courcelle

via tree representations and automata

generalisation for FO data complexity: Frick, Grohe 2001

locally bounded treewidth: $\mathfrak{A}$ with $\mathrm{tw}(N^\ell(a)) \leqslant f(\ell)$

much more general than $\mathcal{C}_k$: e.g. bounded genus graphs
bounded degree graphs

model checking via **locality**: Gaifman representation of φ
local evaluation: ‘neighbourhood covers’

non-elementary $f(\varphi)$
in fixed parameter style combined complexity
under complexity assumptions

model checking tree-like formulae

conjunctive queries

$$\varphi = \exists \boldsymbol{x} \underbrace{\bigwedge_i \boldsymbol{\theta_i(x)}}_{\text{conjunction of relational atoms}}$$

boolean conjunctive query model checking = homomorphism problem (CSP)

conjunctive query containment

model checking	vs.	homomorphism problem
$\mathfrak{A} \models \varphi$?		$\mathfrak{M}[\varphi] \xrightarrow{\text{hom}} \mathfrak{A}$?

$\mathfrak{M}[\varphi]$ the 'matrix' of φ, $M = [\boldsymbol{x}]$ with relations according to the θ_i

PTime for fixed φ, with $|\varphi| = ||\mathfrak{M}[\varphi]||$ in the exponent

model checking conjunctive queries: simulating guardedness

$$\varphi = \exists \boldsymbol{x} \bigwedge_i \boldsymbol{\theta}_i(\boldsymbol{x})$$

contrast exponential dependency on $|\varphi|$ or $|\mathrm{var}(\varphi)|$ with

- linear dependency on $|\varphi|$ in GF
- linear dependency on $|\varphi|$ for acyclic φ (acyclic $H(\mathfrak{M}[\varphi])$)
  tree decomposition of $H(\mathfrak{M}[\varphi])$ $\rightarrow$ guarded model checking
- exponential dependency on only $\mathrm{tw}(\mathfrak{M}[\varphi])$:
  acyclic hypergraph majorizing $H(\mathfrak{M}[\varphi])$

$\Rightarrow$ **combined complexity in PTime for bdd treewidth φ**

generalisations from
bounded treewidth to **bounded hypertreewidth**

$\rightarrow$ Gottlob, Leone, Scarcello

tractable case of homomorphism problem (uniform CSP)

$\mathrm{CSP}(\mathcal{C}_k, \_)$

given $\mathfrak{A} \in \mathcal{C}_k$ and arbitrary $\mathfrak{B}$ decide whether $\mathfrak{A} \xrightarrow{\text{hom}} \mathfrak{B}$

$\mathfrak{A} \xrightarrow{\text{hom}} \mathfrak{B}$ iff $\mathfrak{B} \models \eta_{\mathfrak{A}}$

$\eta_{\mathfrak{A}} \in$ pos $\exists^*$-FO (positive diagram)

in FO^{k+1} for $\mathfrak{A} \in \mathcal{C}_k$

$\mathfrak{A} \xrightarrow{\text{hom}} \mathfrak{B}$ iff $\mathfrak{A} \Rrightarrow_{\text{pos}\,\exists\text{-FO}^{k+1}} \mathfrak{B}$

Ptime by analysis of $(k+1)$-pebble game

back to logic and preservation

locality constraints for expressive comleteness (FMT)

Łos–Tarski extension preservation

equivalent for $\varphi \in$ FO: (i) φ **preserved in extensions**
(ii) $\varphi \equiv \varphi' \in \exists$-**FO**

long known to fail in FMT — Gurevich, Tait

valid over special classes in FMT — Atserias 05
in particular also $\mathcal{C}_k$ — Dawar, Grohe

Lyndon–Tarski homomorphism preservation

equivalent for $\varphi \in$ FO: (i) φ **preserved in homomorphisms**
(ii) $\varphi \equiv \varphi' \in$ **pos** $\exists$-**FO**

now proved in FMT throughout — Rossmann 05

over special classes in FMT — Atserias, Dawar 04
in particular also $\mathcal{C}_k$ — Kolaitis

extension preservation in special classes

$\mathcal{C}$ a $\subseteq$-closed class of finite structures
$\varphi \in$ FO preserved under extensions in $\mathcal{C}$

need: finitely many $\subseteq$-minimal elements in $\varphi[\mathcal{C}]$
then φ equivalent to disjunction over
$\exists$-closure of algebraic diagrams

homomorphism preservation in special classes

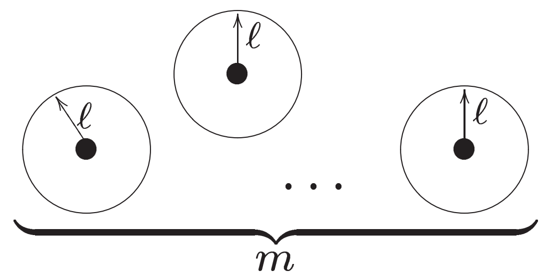
similarly need: finitely many $\subseteq_{\mathrm{w}}$-minimal elements in $\varphi[\mathcal{C}]$
homomorphism minimal elements: cores

$\rightarrow$ look for global structural properties
to bound size of minimal models
can use **Gaifman locality** in both scenarios

from Gaifman locality to size bounds on minimal models

key idea: couple existence of **scattered sets**
to **size of structures**

ℓ-scattered set of size m:
m elements of pairwise distance $> 2\ell$



recall **Gaifman's theorem:**

any FO sentence φ is equivalent to
boolean combination of basic local FO sentences

basic local FO sentence: existence of ℓ-scattered set
of certain size within certain $\psi^{\ell}[\mathfrak{A}]$

wideness

Atserias, Dawar, Grohe, Kolaitis 04/05
Ajtai, Gurevich 89

(ℓ, m)-wide structure $\mathfrak{A}$:

$\mathfrak{A}$ contains ℓ-scattered subset of size m — a property of $G(\mathfrak{A})$

wide class $\mathcal{C}$:

for all ℓ, m exists N: $\mathfrak{A} \in \mathcal{C}$, $|\mathfrak{A}| \geqslant N \Rightarrow \mathfrak{A}$ (ℓ, m)-wide

relax to **almost wide classes $\mathcal{C}$**

after removal of a constant number of elements (e.g., trees)

key theorems — Atserias, Dawar, Kolaitis 04

- **$\mathcal{C}_k$ almost wide** (remove one $\lambda(v)$)
- **any class of graphs with excluded minor is almost wide**

homomorphism preservation

Atserias, Dawar, Kolaitis 04
Rossmann 05

theorem Aijtai, Gurevich

$\mathcal{C}$ closed under substructures and disjoint unions

$\varphi \in$ FO preserved under homomorphisms on $\mathcal{C}$

$\Rightarrow$

minimal models of φ cannot be (ℓ, m)-wide (suitable ℓ, m)

similarly, even up to removal of any fixed number of elements

corollary

over almost wide $\mathcal{C}$:
$\rightarrow$ bound on size of minimal models
$\rightarrow$ finitely many minimal models
$\rightarrow$ positive $\exists^*$ definability

homomorphism preservation thm in restriction to $\mathcal{C}$

minimal models cannot be too wide — Ajtai, Gurevich

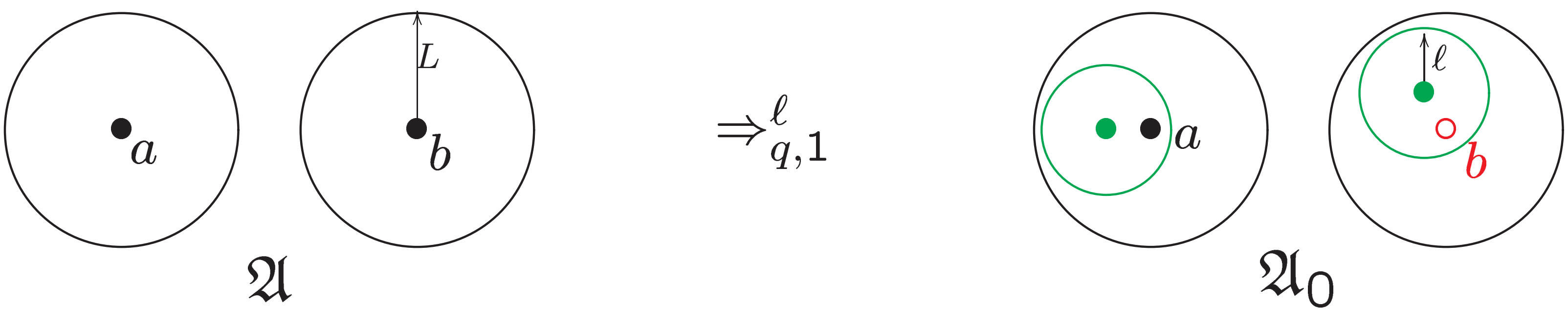
for $\varphi \in$ FO preserved under homomorphism and $\equiv^{\ell}_{q,m}$ $\quad (\Rightarrow^{\ell}_{q,m})$

exist L, M: s.t. no (L, M)-wide $\mathfrak{A} \models \varphi$ can be minimal

idea: within L-scattered set of size M in $\mathfrak{A} \models \varphi$

$L = L(\ell)$
$Q = Q(L, q)$
$M = M(Q, L)$

$$a \equiv^{L}_{Q,0} b \quad a \neq b \quad \Rightarrow \quad \mathfrak{A} \Rightarrow^{\ell}_{q,1} \mathfrak{A} \setminus \{b\}$$



then $\mathfrak{A}_0 := \mathfrak{A} \setminus \{b\} \models \varphi$:

$$\mathfrak{A} \xrightarrow{hom} \mathfrak{A} + m \otimes \mathfrak{A}_0 \quad \Rightarrow^{\ell}_{q,m} \quad m \otimes \mathfrak{A}_0 \xrightarrow{hom} \mathfrak{A}_0$$

homomorphism preservation,
boundedness and the Ajtai–Gurevich theorem

Atserias, Dawar, Kolaitis 04

Ajtai–Gurevich theorem

for positive existential least fixed points (Datalog programs):
FO over all finite structures $\Leftrightarrow$ bounded

for $\Rightarrow$:

- minimal models have bounded treewidth
- FO definable $\Rightarrow$ definable in positive $\exists^*$ (finite structures)
- FMT version of Barwise–Moschovakis theorem
  for $\exists^*$ definability of $\exists^*$ fixed point

  based on fmp for $\exists^*\forall^*$:

  $$\left.\begin{array}{rl} & \mu_X\varphi \equiv \psi \text{ in all finite structures} \\ \Rightarrow & \mu_X\varphi \equiv \psi \text{ in all structures} \end{array}\right\} \quad \text{for } \varphi, \psi \text{ in } \exists^*$$

  $\varphi^n(\boldsymbol{x}) \wedge \neg\psi(\boldsymbol{x})$; $\psi(\boldsymbol{x}) \wedge \neg P\boldsymbol{x} \wedge \forall\boldsymbol{x}(\varphi(P,\boldsymbol{x}) \rightarrow P\boldsymbol{x}) \in \exists^*\forall^*$

extension preservation

Atserias, Dawar, Grohe 05

can bound size of minimal models over:

- classes of structures with acyclic Gaifman graphs
- **all wide $\mathcal{C}$, e.g., bounded degree graphs**
- **$\mathcal{C}_k$ (almost wide)**

size bounds on minimal models via Gaifman:

in large $\mathfrak{A} \models \varphi$ find $\qquad \mathfrak{A}_0 \subsetneq \mathfrak{A} \subseteq \hat{\mathfrak{A}}$

finite chain construction! $\qquad \mathfrak{A}_0 \equiv^{\ell}_{q,m} \hat{\mathfrak{A}} \qquad \Rightarrow \mathfrak{A}_0 \models \varphi$

extension preservation theorem fails over:

- **planar finite graphs**

homomorphism preservation: new classical proof and FMT

homomomorphism preservation Rossmann 05

equivalent for any $\varphi \in \mathrm{FO}$:

classical

- (i) φ **preserved under homomorphisms**
- (ii) $\varphi \equiv \varphi' \in$ **pos** $\exists$**-FO** $\mathrm{qr}(\varphi') = \mathrm{qr}(\varphi)$ (!)

FMT

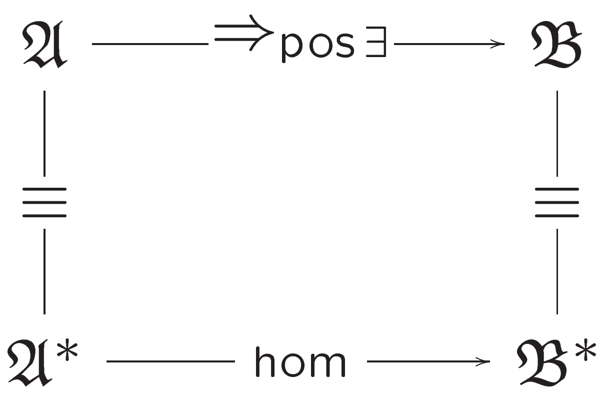
- (i) φ **preserved under homomorphisms**
- (ii) $\varphi \equiv \varphi' \in$ **pos** $\exists$**-FO**

qr(φ') non-elementary in qr(φ): Gurevich, Shelah

method: **existential positive types & saturation (chain)**

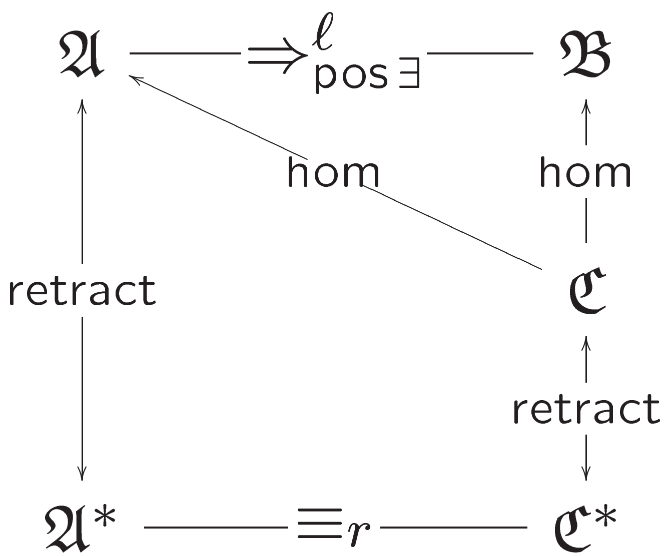
> compactness property in finite structures:
> large finite degree of saturation suffices

via full $\equiv$ to hom



upgrading via ω-saturation

via hom to approximate $\equiv$



finite $\mathfrak{A}^*$: $\ell(r)$ non-elementary

infinite $\mathfrak{A}^*$: $\ell = r$

key points of this particular view

from **all-out FMT** to **specific structural theories**
of well-behaved classes of models

- **the role of modularity & locality**
  games and finely graded logical equivalences

  how can classical model theory
  do without Gaifman locality?

- **old model theoretic techniques** (back & forth, chains, . . .)
  in increasingly involved combinatorial settings

  $\rightarrow$ more hierarchical view of model constructions
  new perspectives on classical results reproved

- **impact of combinatorial (hyper)graph theory**
  on algorithmic model theory of special classes

selected references

Modal logic over finite structures, Rosen, JLLC 1997

Modal and guarded characterisation theorems over finite transition systems, O_, APAL 2004

Bisimulation invariance and finite structures, O_, to appear in LNL (Logic Colloquium 2002)

Elementary proof of the van Benthem–Rosen characterisation theorem, O_, TUD online preprint 2342

Modal characterisation theorems over special classes of frames, Dawar&O_, LICS 2005

Model theory of modal logic, Goranko&O_, in: **Handbook of Modal Logic** (Blackburn, van Benthem, Wolter), to appear

Finite conformal hypergraph covers and Gaifman cliques in finite structures, Hodkinson&O_, BSL 2003

On the restraining power of guards, Grädel, JSL 1999

selected references ctd.

Games and model checking for guarded logics,
Berwanger&Grädel, LPAR 2001, LNCS 2250

Deciding FO properties of locally tree-decomposable structures,
Frick&Grohe, JACM 2001

The complexity of FO and MSO revisited, Frick&Grohe, APAL 2004

The complexity of acyclic conjunctive queries,
Gottlob,Leone&Scarcello, JACM 2001

Hypertree decompositions and tractable queries,
Gottlob,Leone&Scarcello, JCSS 2002

On presevation under homomorphisms and unions of conjunctive queries, Atserias,Dawar&Kolaitis, PODS 2004

Preservation under extensions on well-behaved finite structures,
Atserias,Dawar&Grohe, ICALP 2005, LNCS 3580

Existential positive types and preservation under homomorphisms,
Rossmann, LICS 2005