CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

K. A. HARDIE
K. H. KAMPS

Homotopy over B and under A

Cahiers de topologie et géométrie différentielle catégoriques, tome 28, nº 3 (1987), p. 183-196.

<http://www.numdam.org/item?id=CTGDC_1987__28_3_183_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

Vol. XXVIII-3 (1987)

HOMOTOPY OVER B AND UNDER A

by K.A. HARDIE and K.H. KAMPS

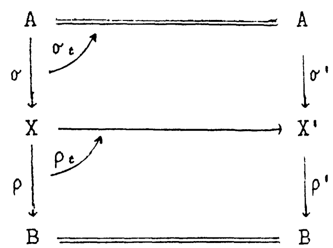
RÉSUMÉ. Dans cet article, on décrit une certaine catégorie d'homotopie cohérente $\mathbb{H}_B^A$ d'espaces au-dessus d'un espace B et sous un espace A. Le problème d'isomorphisme et le problème de classification sont résolus. On indique aussi les liens avec les classes de composition secondaires.

0. INTRODUCTION.

An object X of $\mathbb{H}_B^A$ is a diagram

(0.1) $$A \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

where X is a space and A and B are fixed spaces. An arrow from X to X' in $\mathbb{H}_B^A$ will be a certain equivalence class of homotopy commutative diagrams of the form



where u is a continuous map and σ_t and ρ_t are homotopies. Our first result (Theorem 1.2) is to the effect that such an arrow is an isomorphism in $\mathbb{H}_B^A$ whenever the map u is (an ordinary) homotopy equivalence. As an immediate corollary we obtain (Corollary 1.8) that allowing σ and ρ to vary up to homotopy does not change the isomorphism class of X.

We shall denote the set of morphisms in $\mathbb{H}_B^A$ from X to X' by $\pi(A/X,X'/B)$. If u: X → X' is a continuous map, let $\pi_1^X(X';u)$ denote the u-based track group. (See [3, 2].) Then $\pi_1^X(X';u)$ depends up to isomorphism only on the homotopy class $\{u\}$ of u. In Section 2 we classify the elements of $\pi(A/X,X'/B)$ in the sense that we exhibit a bijection between $\pi(A/X,X'/B)$ and the union over certain classes $\{u\}$ of sets of double cosets in $\pi_1^X(X';u)$. The result generalizes classifications obtained in [2] for $\pi(X,X'/B)$ (the case $A = \emptyset$) and for $\pi(A/X,X')$ (the case B is a singleton).

In view of Corollary 1.6 the set $\pi(A/X,X'/B)$ is determined by the spaces A,X,X',B and by the homotopy classes $\{\sigma\},\{\sigma'\},\{\rho\},\{\rho'\}$. Thus it is an invariant that is defined whenever we are given two factorizations of a homotopy class. In Section 3 we show, in a special case, that it is related to, and in a sense measures, the set of possible nontrival secondary homotopy compositions of a certain type.

1. THE EQUIVALENCE RELATION.

If h_t: $h_0 \simeq h_1$ and k_t: $k_0 \simeq k_1$ are homotopies from Y to Z with the property that $h_0 = k_0$ and $h_1 = k_1$ then h_t and k_t are *relatively homotopic*, denoted $h_t \equiv k_t$, if there exists a homotopy of homotopies $H_{t,s}$: Y → Z such that

$$h_t = H_{t,0},\quad k_t = H_{t,1},\ H_{0,s} = h_0 = k_0 \ (s \in I),\ \text{and}\ H_{1,s} = h_1 = k_1 \ (s \in I).$$

The *track* $\{h_t\}$ of a homotopy h_t is its relative homotopy class. The track of the constant homotopy $h_0 \simeq h_0$ is denoted by $\{h_0\}$.

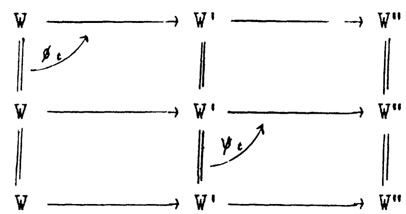
For each homotopy commutative diagram 0.2 there is a triple $(\{\rho_t\},u,\{\sigma_t\})$, where $\{\sigma_t\}$ and $\{\rho_t\}$ are tracks. We obtain an equivalence relation in the set of such triples if, whenever u_t: X → X' is a homotopy with $u_0 = u$, we define

$$(1.1)\qquad (\{\rho_t + \rho' u_t\},\ u_1,\ \{u_{1-t}\sigma + \sigma_t\}) \sim (\{\rho_t\},u,\{\sigma_t\}).$$

In 1.1 the + refers to the usual track addition of homotopies. Denoting the set of equivalence classes by $\pi(A/X,X'/B)$ we obtain a category with composition induced by juxtaposition of diagrams. Formally we set

$$(\{\rho'_t\},u',\{\sigma'_t\})\circ(\{\rho_t\},u,\{\sigma_t\}) = (\{\rho_t + \rho'_t u\},\ u'u,\ \{u'\sigma_t + \sigma'_t\}).$$

To check that composition respects the equivalence relation may at first glance present a problem, but we may bear in mind that a diagram



represents equally well the tracks $\{\psi_0\phi_t + \psi_t\phi_1\}$ and $\{\psi_t\phi_0 + \psi_1\phi_t\}$, for we have

$$\psi_0\phi_t + \psi_t\phi_1 \equiv \psi_t\phi_0 + \psi_1\phi_t.$$

The required verifications are now easily done. The identity morphism $X \to X$ in $\underline{H}_B{}^A$ is the equivalence class of the triple $(\{\rho\},1_X,\{\sigma\})$. The equivalence class of a triple $(\{\rho_t\},u,\{\sigma_t\})$ will be denoted simply by $\{\rho_t,u,\sigma_t\}$.

THEOREM 1.2. *The arrow* $\{\rho_t,u,\sigma_t\}$ *is an isomorphism in* $\underline{H}_B{}^A$ *if and only if the map* $u\colon X \to X'$ *is a homotopy equivalence.*

PROOF. If $\{\rho_t,u,\sigma_t\}$ is an isomorphism then it is obvious that u is a homotopy equivalence. Suppose conversely that u is a homotopy equivalence. Then by [6] there exists a homotopy inverse u' of u and homotopies

$$\phi_t\colon uu' \simeq 1_{X'},\ \psi_t\colon u'u \simeq 1_X \text{ such that}$$

$$\phi_t u \equiv u\,\psi_t. \tag{1.3}$$

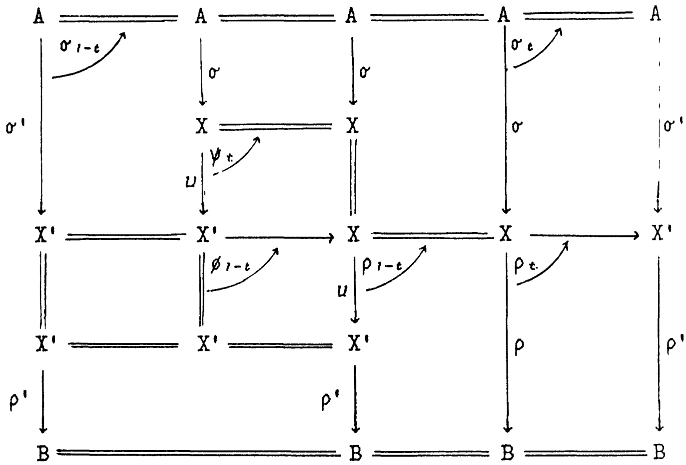
It is claimed that

$$\{\rho'\phi_{1-t} + \rho_{1-t}u',u',\ u\sigma'_{1-t} + \psi_t\sigma\}$$

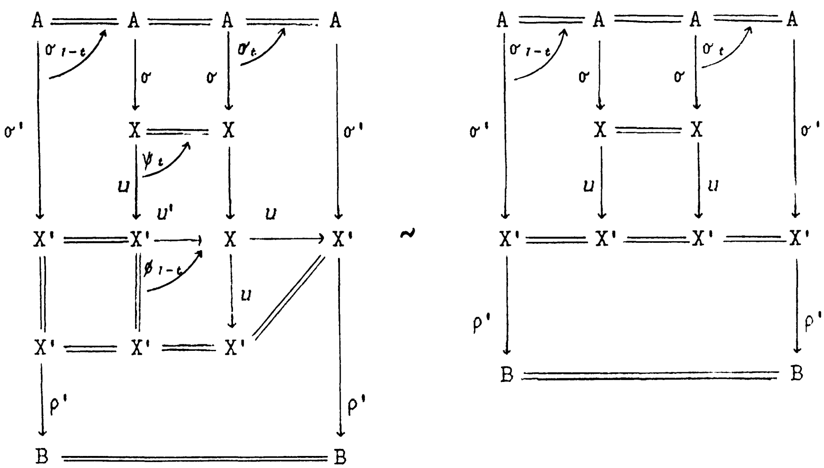
is inverse to $\{\rho_t,u,\sigma_t\}$ in $\underline{H}_B{}^A$, for firstly

$$\{\rho_t,u,\sigma_t\}\circ\{\rho'\phi_{1-t} + \rho_{1-t}u',u',\ u\sigma'_{1-t} + \psi_t\sigma\}$$

is represented by the following composite diagram



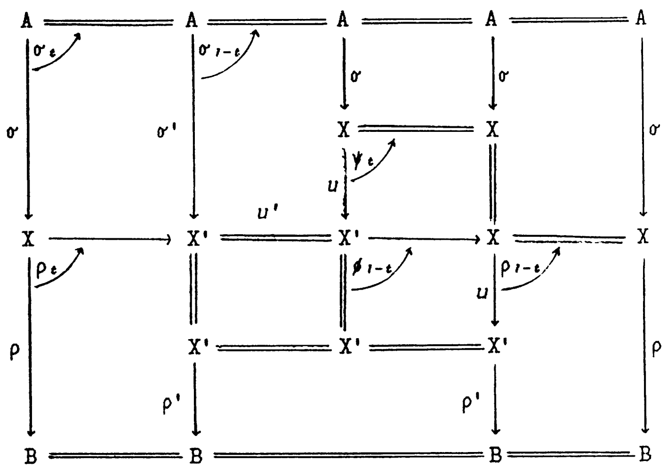
whose associated triple is equal to that of



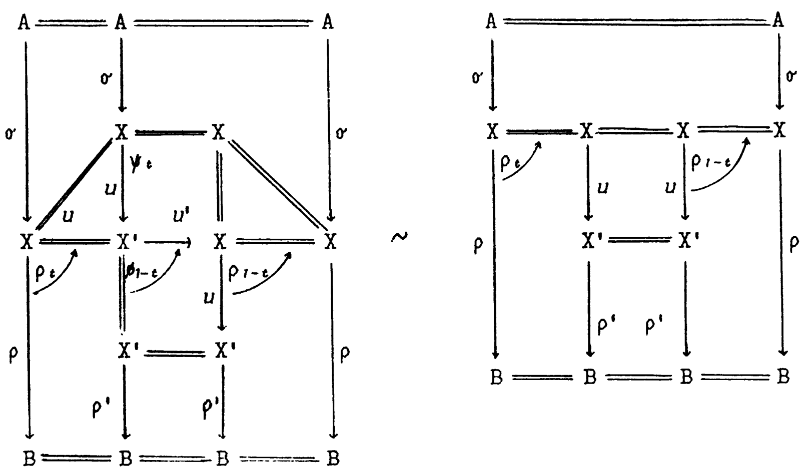
(using 1.3) which is equal to $(\{\rho'\}, 1_{X'}, \{\sigma'\})$. Secondly the composite

$$\{\rho'\phi_{1-t} + \rho_{1-t}u',\ u',\ u\sigma_{1-t} + \psi_t\sigma\} \circ \{\rho_t, u, \sigma_t\}$$

is represented by the composite diagram



whose associated triple is equal to



(using 1.3) which is equal to $(\langle\rho\rangle, 1_X, \langle\sigma\rangle)$, completing the proof.

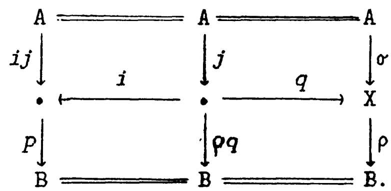
COROLLARY 1.4. *Each object* X *in* $\mathbb{H}_B^A$ *is isomorphic to an object* X' *such that* σ' *is a closed cofibration and* ρ' *a Hurewicz fibration.*

PROOF. Let

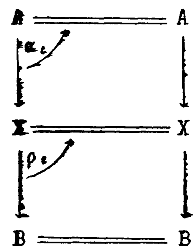
$$A \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

be the given object. By the mapping cylinder construction we can factor σ as qj where j is a closed cofibration and q is a homotopy equivalence. By [5], Proposition 2 (see also [4], Remark (d)) we can factor ρq as pi with p a fibration and i a closed cofibration and a homotopy equivalence. Now apply Theorem 1.2 twice to the diagram

(1.5)



Now suppose that α_t: A → X and ρ_t: X → B are homotopies. Applying Theorem 1.2 to the diagram



we obtain the following corollary.

COROLLARY 1.6. *The isomorphism type of an object* X *of* $\mathbb{H}_B{}^A$ *depends only on the homotopy classes of* α *and* ρ.

Let f: A → B be a fixed map and consider the class of all objects X of $\mathbb{H}_B{}^A$ such that $\rho\sigma = f$. Let $\mathbb{HR}_B{}^A$ be the category with these objects whose arrows are equivalence classes under the relation 1.1 of triples $(\{\rho_t\}, u, \{\sigma_t\})$ for which the tracks $\{\rho_t\}$ and $\{\sigma_t\}$ satisfy the additional condition

(1.7) $$\{\rho_t\sigma\} + \{\rho'\sigma_t\} = \{f\},$$

where $\{f\}$ denotes the track of the constant homotopy f. Note that the relation 1.1 still makes sense for such triples and that the inclusion $\mathbb{HR}_B{}^A \rightsquigarrow \mathbb{H}_B{}^A$ is an embedding.

COROLLARY 1.8. *An arrow* $\{\rho_t, u, \sigma_t\}$ *of* $\underline{HR}_B{}^A$ *is an isomorphism if and only if* u *is a homotopy equivalence.*

PROOF. If $(\{\rho_t\}, u, \{\sigma_t\})$ satisfies (1.7) then it can be checked that the triple

$$(\{\rho'\phi_{1-t} + \rho_{1-t}u'\},\ u', \{u'\sigma_{1-t} + \psi_t\sigma\})$$

as constructed in the proof of Theorem 1.2 also satisfies condition 1.7. •

REMARK 1.9. Since the diagram 1.5 is strictly commutative, a result corresponding to Corollary 1.4 also holds in $\underline{HR}_B{}^A$.

2. THE KERVAIRE DIAGRAM.

Let

$$(2.1)\qquad \begin{array}{ccc} A & = & A \\ \sigma\downarrow & \overset{\chi_t}{\nearrow} & \downarrow\sigma' \\ X & \xrightarrow{u} & X' \\ \rho\downarrow & \overset{\mu_t}{\nearrow} & \downarrow\rho' \\ B & = & B \end{array}$$

be a fixed homotopy commutative diagram over B and under A. Let

$$U = (\mu_t, u) = \begin{array}{ccc} X & \xrightarrow{u} & X' \\ \rho\downarrow & \overset{\mu_t}{\nearrow} & \downarrow\rho' \\ B & = & B \end{array}$$

and

$$V = (u, \chi_t) = \begin{array}{ccc} A & = & A \\ \sigma\downarrow & \overset{\chi_t}{\nearrow} & \downarrow\sigma' \\ X & \xrightarrow{u} & X' \end{array}$$

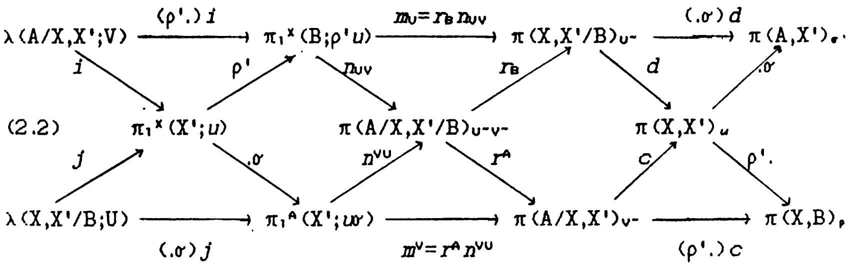
be the induced homotopy commutative diagrams over B and under A whose equivalence classes are

$$U^{\sim} \in \pi(X, X'/B) \quad \text{and} \quad V^{\sim} \in \pi(A/X, X')$$

respectively. Let us also denote by $U^{\sim}V^{\sim}$ the element of $\pi(A/X,X'/B)$ represented by 2.1.

Consider the following interlocking (Kervaire) diagram of groups and pointed sets with base points as indicated.

(2.2)

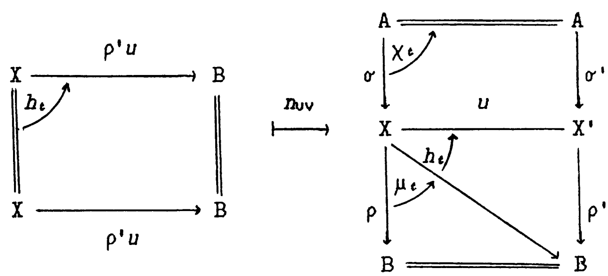


Here, $\lambda(A/X,X';V)$ and $\lambda(X,X'/B;U)$ are subgroups of $\pi_1{}^X(X';u)$ defined as the kernel of the induced group homomorphism

$$.\sigma\colon \pi_1{}^X(X';u) \to \pi_1{}^A(X';u\sigma), \text{ respectively } \rho'.\colon \pi_1{}^X(X';u) \to \pi_1{}^X(B;\rho'u);$$

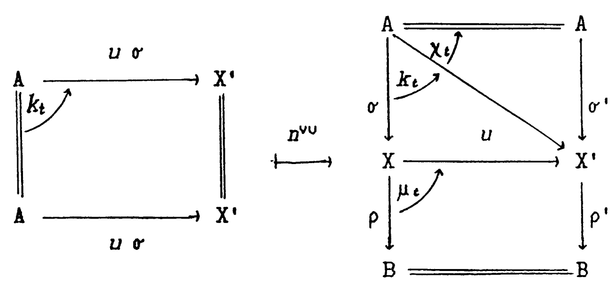
the maps i and j are inclusions, r_B, r^A, d and c are the obvious restriction operators and n_{UV}, n^{VU} are defined by the rules

(2.3)



,

(2.4)



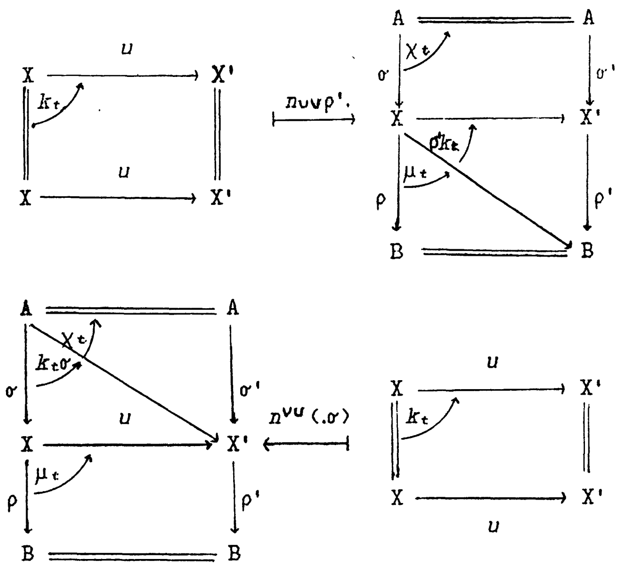
.

THEOREM 2.5. *The Kervaire diagram 2.2 is commutative and its four interlocking sequences of homomorphisms and pointed maps are exact. Moreover*

$$\bullet \xrightarrow{(.\sigma)j} \bullet \xrightarrow{n^{\vee\cup}} \bullet$$

is exact of type E3 *(see* [2], 1.A) *at* $\pi_1{}^A(X';u\sigma)$.

PROOF. We prove commutativity from $\pi_1{}^X(X';u)$. We have that



The two sequences passing through $\pi_1{}^X(X';u)$ are exact by definition of $\lambda(A/X,X';V)$, $\lambda(X,X'/B;U)$ and by [2], Theorem 4.3 and its dual. Thus the following proposition remains to be proved.

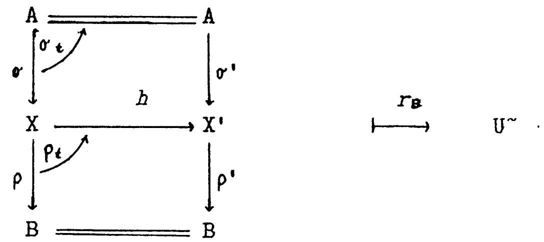
PROPOSITION 2.6. *The sequence*

$$\lambda(X,X'/B;U) \xrightarrow{(.\sigma)j} \pi_1{}^A(X';u\sigma) \xrightarrow{n^{\vee\cup}} \pi(A/X,X'/B)_{\cup\text{-}\vee\text{-}} \xrightarrow{r_B} \pi(X,X'/B)_{\cup\text{-}} \xrightarrow[(.\sigma)d]{} \pi(A,X')_{\sigma}.$$

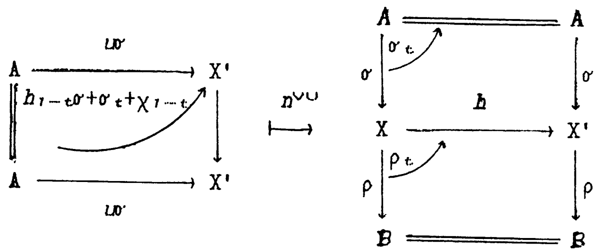
is exact. At $\pi_1{}^A(X';u\sigma)$ *it is exact of type* E3.

PROOF. The exactness at $\pi(X,X'/B)_{\cup\text{-}}$ is obvious.

Exactness at $\pi(A/X,X'/B)_{\upsilon\text{-}\nu\text{-}}$: $n^{\nu\upsilon}$ is given by 2.4. Clearly r_B applied to the composite square on the right of 2.4 yields $U^{\sim}$. Now suppose that



Then there exists h_t: X → X' with $h_0 = h$ such that $h_1 = u$ and $\rho_t + \rho' h_t \equiv \mu_t$ and it can be checked that

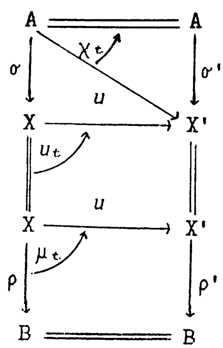


as required.

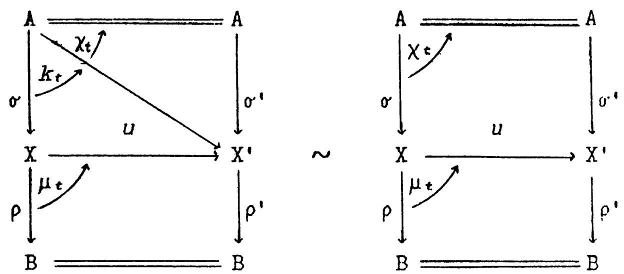
Exactness at $\pi_1{}^A(X';u\sigma)$: Let $\{u_t\} \in \lambda(X,X'/B;U)$, i.e., $\{u_t\}$ is in$\pi_1{}^X(X';u)$ such that $\rho' u_t \equiv \rho' u$, whence

(2.7) $$\mu_t + \rho' u_t \equiv \mu_t .$$

Now $n^{\nu\upsilon}(.\sigma)\{u_t\}$ is represented by the diagram



and in view of the relation 2.7, we have that $n^{VU}(.\sigma)\{u_t\} = U^{\sim}V^{\sim}$. Conversely, if $\{k_t\} \in \pi_1^A(X';u\sigma)$ is such that



then there exists a homotopy u_t: X → X' with $u_0 = u_1 = u$ such that

$$\mu_t + \rho' u_t \equiv \mu_t \tag{2.8}$$

and

$$u_{1-t}\sigma + k_t + \chi_t \equiv \chi_t \,. \tag{2.9}$$

By 2.8 $\{u_t\}$ belongs to $\lambda(X,X'/B;U)$ and by 2.9 we have that

$$(.\sigma)\{u_t\} = \{k_t\}.$$

Exactness of type E3 is proved by checking that

$$n^{VU}\{k_t\} = n^{VU}\{k'_t\} \quad \text{iff} \quad \{k'_t\}\{k_t\}^{-1} \in \mathrm{Im}(.\sigma)\,;$$

and applying ordinary exactness at $\pi_1^A(X';u\sigma)$. •

Applying [2], Theorem 2.13 and Lemma 2.11, we obtain

COROLLARY 2.10. *The sequence*

$$\lambda(A/X,X';V)\times\lambda(X,X'/B;U) \xrightarrow{\theta} \pi_1^X(X';u) \xrightarrow{\Delta} \pi(A/X,X'/B)_{U^{\sim}V^{\sim}}$$
$$\longrightarrow \pi(A/X,X')_{V^{\sim}} \times \pi(X,X'/B)_{U^{\sim}}$$

is exact, where $\Delta = n^{VU}(.\sigma)$ *and* $\theta(\alpha,\beta) = \beta^{-1}\alpha$.

Moreover the images of two elements under Δ *coincide iff they belong to the same double coset of the subgroups* $\lambda(X,X'/B;U)$ *and* $\lambda(A/X,X';V)$.

Let $K(u,U,V)$ denote the set of double cosets in $\pi_1^X(X';u)$ of the subgroups $\lambda(X,X'/B;U)$ and $\lambda(A/X,X';V)$.

COROLLARY 2.11. *There is a bijection*

$$\pi(A/X,X'/B) \longleftrightarrow \bigcup_S K(u,U,V),$$

where

$$S = \{(u,U,V) \mid U^{\sim} \in d^{-1}\{u\},\ V^{\sim} \in c^{-1}\{u\},\ \{u\} \in (.\sigma)^{-1}\{\sigma'\} \cap (\rho'.)^{-1}\{\rho\}\}.$$

REMARK 2.12. The homomorphisms

$$\rho'.: \pi_1{}^X(X';u) \longrightarrow \pi_1{}^X(B;\rho'u) \quad \text{and} \quad .\sigma: \pi_1{}^X(X';u) \longrightarrow \pi_1{}^A(X';u\sigma)$$

can, in certain special cases, be computed as discussed by Rutter [3].

3. SECONDARY COMPOSITION CLASSES.

In this final section we examine in a special case some interactions between elements of the set $\pi(A/X,X'/B)$ and secondary composition classes.

We consider pointed topological spaces A,B,X,X' and the case in which $\rho: X \to B$ and $\sigma': A \to X'$ are the trivial maps (denoted by $*$). The following operators may be defined.

$$R: \pi(A/X,X'/B) \longrightarrow \pi(X,X'). \tag{3.1}$$

Set $R\{\rho_t,u,\sigma_t\} = \{u\}$.

$$M: \pi(A/X,X'/B) \longrightarrow \pi(\sigma,\rho'). \tag{3.2}$$

Here $\pi(\sigma,\rho')$ refers to the homotopy pair set, for details see [1]. Set

$$M\{\rho_t,u,\sigma_t\} = \{*,\ *,\ \rho_t\sigma + \rho'\sigma_t\}.$$

Let $\alpha = \{\rho_t,u,\sigma_t\}$ denote an arbitrary element of $\pi(A/X,X'/B)$. Let

$$\{\{\rho'\},\{R\alpha\},\{\sigma\}\} \subset \pi(\Sigma A,B)$$

denote the Toda bracket coset of $\{\sigma\}$, $\{\rho'\}$ and $\{R\alpha\}$.

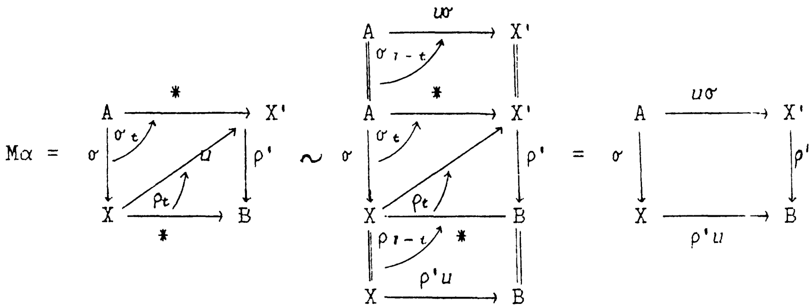
We have the following result.

THEOREM 3.3. *The following are equivalent.*

(i) $\quad 0 \in \{\{\rho'\},\{R\alpha\},\{\sigma\}\}.$

(ii) $M\alpha = 0.$

PROOF. Allowing diagrams to represent elements we have



By [2], Proposition 3.14 it follows that (i) and (ii) are equivalent. •

REFERENCES.

1. K.A. HARDIE & A.V. JANSEN, Toda brackets and the category of homotopy pairs, *Quaestiones Math.* 6 (1983), 107-128.
2. K.A. HARDIE & K.H. KAMPS, Exact sequence interlocking and free homotopy theory, *Cahiers Top. et Géom. Diff.* XXVI-1 (1985), 3-31.
3. J.W. RUTTER, A homotopy classification of maps into an induced fibre space, *Topology* 6 (1967), 379-403.
4. R. SCHÖN, The Brownian classification of fiber spaces, *Arch. Math.* 39 (1982), 359-365.
5. A. STRØM, The homotopy category is a homotopy category, *Arch. Math.* 23 (1972), 435-441.
6. R.M. VOGT, A note on homotopy equivalences, *Proc. A.M.S.* 32 (1972), 627-629.

The first author acknowledges grants to the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

K.A. HARDIE:
Department of Mathematics
University of Cape Town
RONDEBOSCH 7700
SOUTH AFRICA

K.H. KAMPS:
Fachbereich Mathematik u. Informatik
Fernuniversität
Postfach 940
D-5800 HAGEN, Fed. Rep. of GERMANY