A few Smarandache Integer Sequences

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Abstract

This paper deals with the analysis of a few Smarandache Integer Sequences which first appeared in *__Properties of the Numbers, F. Smarandache, University of Craiova Archives, 1975.__* *The first four sequences are recurrence generated sequences while the last three are concatenation sequences.*

The Non-Arithmetic Progression: {a_i :a_i is the smallest integer such that $a_i > a_{i-1}$ and such that for k≤i there are at most t-1 equal differences $a_k - a_{k_1} = a_{k_1} - a_{k_2} = \ldots = a_{k_{t-2}} - a_{k_{t-1}}$ }

A strategy for building a t-term non-arithmetic progression is developed and computer implemented for 3≤t≤15 to find the first 100 terms. Results are given in tables and graphs together with some observations on the behaviour of these sequences.

The prime-Product Sequence: {t_n : t_n = p_n#+1, p_n is the n*th* prime number}, where p_n# denotes the product of all prime numbers which are less than or equal to p_n.

The number of primes q among the first 200 terms of the prime-product sequence is given by 6≤q≤9. The six confirmed primes are terms numero 1, 2, 3, 4, 5 and 11. The three terms which are either primes or pseudo primes (according to Fermat's little theorem) are terms numero 75, 171 and 172. The latter two are the terms 1019#+1 and 1021#+1.

The Square-Product Sequence: {t_n : $t_n = (n!)^2+1$}

As in the previous sequence the number of primes in the sequence is of particular interest. Complete prime factorization was carried out for the first 37 terms and the number of prime factors f was recorded. Terms 38 and 39 are composite but were not completely factorized. Complete factorization was obtained for term no 40. The terms of this sequence are in general much more time consuming to factorize than those of the prime-product sequence which accounts for the more limited results. Using the same method as for the prime-product sequence the terms t_n in the interval 40<n≤200 which may possible be primes were identified. There are only two of them, term #65: $N=(65!)^2+1$ which is a 182 digit number and term #76: $N=(76!)^2+1$ which has 223 digits.

The Prime-Digital Sub-Sequence: The prime-digital sub-sequence is the set {$M=a_0+a_1 \cdot 10+a_2 \cdot 10^2+\ldots a_k 10^k$:M is a prime and all digits a_0, a_1, $a_2 \ldots a_k$ are primes}

A proof is given for the theorem: __The Smarandache prime-digital sub sequence is infinite,__ which until now has been a conjecture.

Smarandache Concatenated Sequences: Let G={g_1, g_2, ... g_k, ... } be an ordered set of positive integers with a given property G. The corresponding concatenated S.G sequence is defined through $S.G = \{a_i : a_1 = g_1, a_k = a_{k-1} \cdot 10^{1+\log_{10} g_k} + g_k, k \geq 1\}$.

The S.Odd Sequence: Fermat's little theorem was used to find all primes/pseudo-primes among the first 200 terms. There are only five cases which all were confirmed to be primes using the elliptic curve prime factorization program, the largest being term 49:

13579111315171921232527293133353739414345474951535557596163656769717375777981838587899193959597

Term #201 is a 548 digit number.

The S.Even Sequence: The question how many terms are nth powers of a positive integer was investigated. It was found that there is not even a perfect square among the first 200 terms of the sequence. Are there terms in this sequence which are 2·p where p is a prime (or pseudo prime)? Strangely enough not a single term was found to be of the form 2·p.

The S.Prime Sequence: How many are primes? Again we apply the method of finding the number of primes/pseudo primes among the first 200 terms. Terms #2 and #4 are primes, namely 23 and 2357. There are only two other cases which are not proved to be composite numbers: term #128 which is a 355 digit number and term #174 which is a 499 digit number.

I. The Non-Arithmetic Progression

This integer sequence was defined in simple terms in the February 1997 issue of Personal Computer World. It originates from the collection of Smarandache Notions. We consider an ascending sequence of positive integers $a_1, a_2, \ldots a_n$ such that each element is as small as possible and no t-term arithmetic progression is in the sequence. In order to attack the problem of building such sequences we need a more operational definition.

Definition: The t-term non-arithmetic progression is defined as the set :
$\{a_i : a_i$ is the smallest integer such that $a_i > a_{i-1}$ and such that for $k \le i$ there are at most t-1 equal differences $a_k - a_{k_1} = a_{k_1} - a_{k_2} = \ldots = a_{k_{t-2}} - a_{k_{t-1}}\}$

From this definition we can easily formulate the starting set of a t-term non-arithmetic progression:

$\{1, 2, 3 \ldots\ldots t-1, t+1\}$ or $\{a_i : a_i = i$ for $i \le t-1$ and $a_t = t+1$ where $t \ge 3\}$

It may seem clumsy to bother to express these simple definitions in stringent terms but it is in fact absolutely necessary in order to formulate a computer algorithm to generate the terms of these sequences.

Question: How does the density of a t-term non arithmetic progression vary with t. i.e. how does the fraction a_k/k behave for $t \ge 3$?[1]

Strategy for building a t-term non-arithmetic progression: Given the terms $a_1, a_2, \ldots a_k$ we will examine in turn the following candidates for the term a_{k+1}:

$a_{k+1} = a_k + d$, $d = 1, 2, 3, \ldots$

Our solution is the smallest d for which none of the sets

$\{a_1, a_2, \ldots a_k, a_k+d, a_k+d-e, a_k+d-2e, \ldots a_k+d-(t-1)\cdot e : e \ge d\}$

contains a t-term arithmetic progression.

We are certain that a_{k+1} exists because in the worst case we may have to continue constructing sets until the term $a_k+d-(t-1)e$ is less than 1 in which case all possibilities have been tried with no t terms in arithmetic progression. The method is illustrated with an example in diagram 1.

In the computer application of the above method the known terms of a no t-term arithmetic progression were stored in an array. The trial terms were in each case added to this array. In the example we have for d=1, e=1 the array: 1,2,3,5,6,8,9,10,11,10,9,8. The terms are arranged in ascending order: 1,2,3,5,6,8,8,9,9,10,10,11. Three terms 8,9 and 10 are duplicated and 11 therefore has to be rejected. For d=3, e=3 we have 1,2,3,5,6,8,9,10,13,10,7,4 or in ascending order: 1,2,3,4,5,6,7,8,9,10,10,13 this is acceptable but we have to check for all values of e that produce terms

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[1] This question is slightly different from the one posed in the Personal Computer World where also a wider definition of a t-term non arithmetic progression is used in that it allows $a_2 > a_1$ to be chosen arbitrarily.

which may form a 4-term arithmetic progression and as we can see from diagram 1 this happens for d=3, e=4, so 13 has to be rejected. However, for d=5, e=5 no 4-term arithmetic progression is formed and e=6 does not produce terms that need to be checked, hence a_9 = 15.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Known terms		1	2	3		5	6		8	9	10						
Trials																	
d=1	e=1								8	9	10	11					reject 11
d=2	e=2						6		8		10		12				reject 12
d=3	e=3				4			7			10			13			try next e
	e=4	1				5				9				13			reject 13
d=4	e=3		2				6				10				14		reject 14
d=5	e=5					5					10					15	accept 15

Diagram 1. To find the 9th term of the 4-term non-arithmetic progression.

Routines for ordering an array in ascending order and checking for duplication of terms were included in a *QBASIC* program to implement the above strategy.

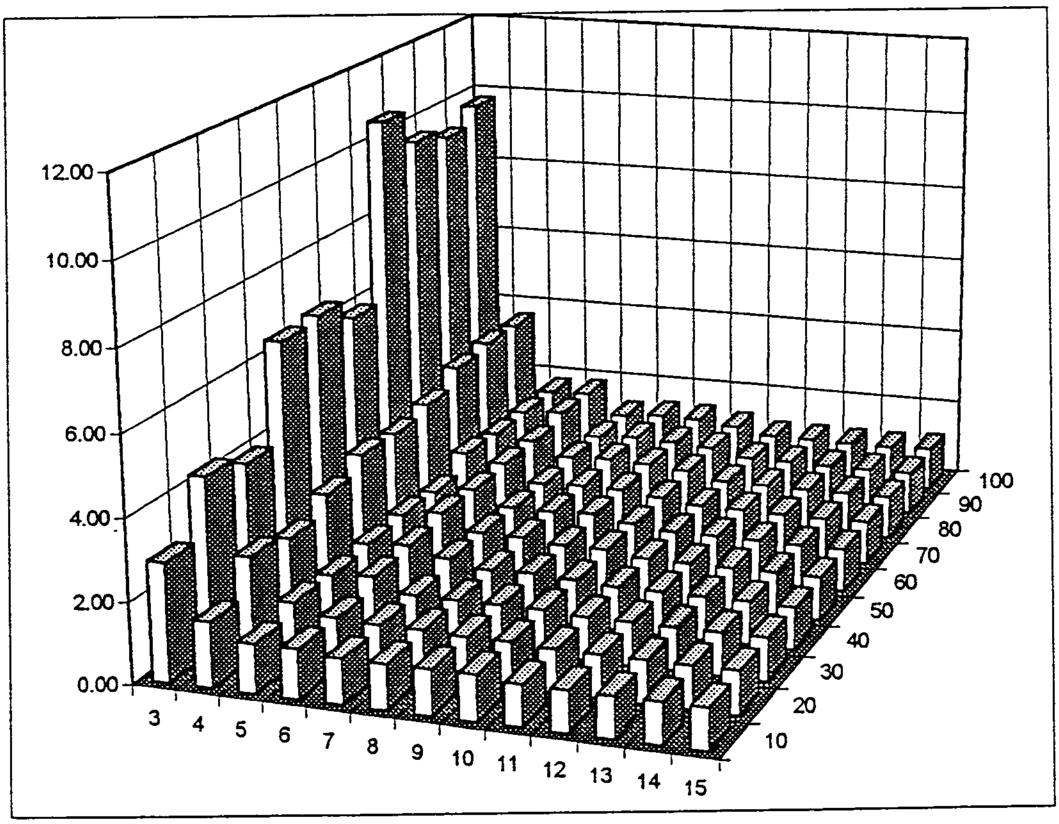

Diagram 2. a_k/k for non-arithmetic progressions with t=3, 4, 5, ... 15. Bars are shown for k = multiples of 10.

Results and observations: Calculations were carried out for $3 \le t \le 15$ to find the first 100 terms of each sequence. The first 65 terms and the 100th term are shown in table 1. In diagram 2 the fractions a_k/k has been chosen as a measure of the density of these sequences. The looser the terms are packed the larger is a_k/k. In fact for t>100 the value of a_k/k = 1 for the first 100 terms.

In table 1 there is an interesting leap for t=3 between the 64th and the 65th terms in that a_{64} = 365 and a_{65} = 730. Looking a little closer at such leaps we find that:

Table 1. The 65 first terms of the non-arithmetic progressions for t=3 to 15.

#	t-3	t=4	t=5	t=6	t=7	t=8	t=9	t=10	t=11	t=12	t=13	t=14	t=15
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	4	3	3	3	3	3	3	3	3	3	3	3	3
4	5	5	4	4	4	4	4	4	4	4	4	4	4
5	10	6	6	5	5	5	5	5	5	5	5	5	5
6	11	8	7	7	6	6	6	6	6	6	6	6	6
7	13	9	8	8	8	7	7	7	7	7	7	7	7
8	14	10	9	9	9	9	8	8	8	8	8	8	8
9	28	15	11	10	10	10	10	9	9	9	9	9	9
10	29	16	12	12	11	11	11	11	10	10	10	10	10
11	31	17	13	13	12	12	12	12	12	11	11	11	11
12	32	19	14	14	13	13	13	13	13	13	12	12	12
13	37	26	16	15	15	14	14	14	14	14	14	13	13
14	38	27	17	17	16	16	15	15	15	15	15	15	14
15	40	29	18	18	17	17	16	16	16	16	16	16	16
16	41	30	19	19	18	18	17	17	17	17	17	17	17
17	82	31	26	20	19	19	19	18	18	18	18	18	18
18	83	34	27	22	20	20	20	20	19	19	19	19	19
19	85	37	28	23	22	21	21	21	20	20	20	20	20
20	86	49	29	24	23	23	22	22	21	21	21	21	21
21	91	50	31	25	24	24	23	23	23	22	22	22	22
22	92	51	32	26	25	25	24	24	24	24	23	23	23
23	94	53	33	33	26	26	27	25	25	25	24	24	24
24	95	54	34	34	27	27	28	26	26	26	25	25	25
25	109	56	36	35	29	28	29	27	27	27	27	26	26
26	110	57	37	36	30	30	30	28	28	28	28	28	27
27	112	58	38	37	31	31	31	31	29	29	29	29	28
28	113	63	39	39	32	32	32	32	30	30	30	30	29
29	118	65	41	43	33	33	33	33	31	31	31	31	31
30	119	66	42	44	34	34	34	34	32	32	32	32	32
31	121	67	43	45	36	35	37	35	34	33	33	33	33
32	122	80	44	46	37	37	38	36	35	35	34	34	34
33	244	87	51	47	38	38	39	37	36	36	35	35	35
34	245	88	52	49	39	39	40	38	37	37	36	36	36
35	247	89	53	50	40	40	41	39	38	38	37	37	37
36	248	91	54	51	41	41	43	41	39	39	38	38	38
37	253	94	56	52	50	42	44	42	40	40	40	39	39
38	254	99	57	59	51	44	45	43	41	41	41	41	40
39	256	102	58	60	52	45	46	44	42	42	42	42	41
40	257	105	59	62	53	46	47	45	43	43	43	43	42
41	271	106	61	63	54	47	48	49	45	44	44	44	45
42	272	109	62	64	55	48	49	50	46	46	45	45	46
43	274	110	63	65	57	49	50	51	47	47	46	46	47
44	275	111	64	66	58	50	53	52	48	48	47	47	48
45	280	122	66	68	59	59	55	53	49	49	48	48	49
46	281	126	67	69	60	60	56	54	50	50	49	49	50
47	283	136	68	71	61	61	57	55	51	51	50	50	51
48	284	145	69	73	62	62	58	58	52	52	51	51	52
49	325	149	76	77	64	63	59	59	53	53	53	52	53
50	326	151	77	85	65	64	60	60	54	54	54	54	54
51	328	152	78	87	66	65	64	61	56	55	55	55	55
52	329	160	79	88	67	67	65	62	57	57	56	56	56
53	334	163	81	89	68	69	66	63	58	58	57	57	58
54	335	167	82	90	69	70	67	64	59	59	58	58	59
55	337	169	83	91	71	71	68	65	60	60	59	59	60
56	338	170	84	93	72	72	69	66	61	61	60	60	61
57	352	171	86	96	73	74	70	68	62	62	61	61	62
58	353	174	87	97	74	75	71	69	63	63	62	62	63
59	355	176	88	98	75	76	78	70	64	64	63	63	64
60	356	177	89	99	76	77	79	71	65	65	64	64	65
61	361	183	91	100	78	78	80	72	67	66	66	65	66
62	362	187	92	103	79	79	81	73	68	68	67	67	67
63	364	188	93	104	80	81	82	74	69	69	68	68	68
64	365	194	94	107	81	84	83	75	70	70	69	69	69
65	730	196	126	111	82	85	84	77	71	71	70	70	70
...													
100	977	360	179	183	130	139	138	126	109	109	108	108	113

Leap starts at		Leap finishes at	
5		10	
14	$=3\cdot5-1$	28	$=2\cdot14$
41	$=3\cdot14-1$	82	$=2\cdot41$
122	$=3\cdot41-1$	244	$=2\cdot122$
365	$=3\cdot122-1$	730	$=2\cdot365$

Does this chain of regularity continue indefinitely?

Sometimes it is easier to look at what is missing than to look at what we have. Here are some observations on the only excluded integers when forming the first 100 terms for t=11, 12 , 13 and 14.

For t=11: 11, 22, 33, 44, 55, 66, 77, 88, 99 — The *nth* missing integer is $11\cdot n$

For t=12: 12, 23, 34, 45, 56, 67, 78, 89, 100 — The *nth* missing integer is $11\cdot n+1$

For t=13: 13, 26, 39, 52, 65, 78, 91, 104 — The *nth* missing integer is $13\cdot n$

For t=14: 14, 27, 40, 53, 66, 79, 92, 105 — The *nth* missing integer is $13\cdot n+1$

Do these regularities of missing integers continue indefinitely? What about similar observations for other values of t?

II. The Prime-Product Sequence

The prime-product sequence originates from Smarandache Notions. It was presented to readers of the Personal Computer World's Numbers Count Column in February 1997.

Definition: The terms of the prime-product sequence are defined through $\{t_n : t_n = p_n\#+1, p_n$ is the *nth* prime number$\}$, where $p_n\#$ denotes the product of all prime numbers which are less than or equal to p_n.

The sequence begins {3, 7, 31, 211, 2311, 30031, ... }. In the initial definition of this sequence t_1 was defined to be equal to 2. However, there seems to be no reason for this exception.

Question: How many members of this sequence are prime numbers?

The question is in the same category as questions like *'How many prime twins are there?, How many Carmichael numbers are there?, etc.'* So we may have to contend ourselves by finding how frequently we find prime numbers when examining a fairly large number of terms of this sequence.

From the definition it is clear that the smallest prime number which divides t_n is larger than p_n. The terms of this sequence grow rapidly. The prime number functions *prmdiv(n)* and *nxtprm(n)* built into the *Ubasic* programming language were used to construct a prime factorization program for $n<10^{19}$. This program was used to factorize the 18 first terms of the sequence. An elliptic curve factorization program, ECM.UB, conceived by Y. Kida was adapted to generate and factorize further terms up to and including the 49*th* term. The result is shown in table 2. All terms analysed were found to be square free. A scatter diagram, Diagram 3, illustrates how many prime factors there are in each term .

The 50*th* term presented a problem. $t_{50}=126173\cdot n$, where n has at least two factors. At this point prime factorization begins to be too time consuming and after a few more terms the numbers will be too large to handle with the above mentioned program. To obtain more information the method of factorizing was given up in favor of using Fermat's theorem to eliminate terms which are definitely not prime numbers. We recall Fermat's little theorem:

If p is a prime number and (a, p)=1 then $a^{p-1} \equiv 1 \pmod p$.

$a^{n-1} \equiv 1 \pmod n$ is therefore a necessary but not sufficient condition for n to be a prime number. If n fills the congruence without being a prime number then n is called a pseudo prime to the base a, psp(a). We will proceed to find all terms in the sequence which fill the congruence

$$a^{t_n - 1} \equiv 1(\bmod t_n)$$

for $50 \leq n \leq 200$. t_{200} is a 513 digit number so we need to reduce the powers of a to the modulus t_n gradually as we go along. For this purpose we write t_n-1 to the base 2:

$$t_n - 1 = \sum_{k=1}^{m} \delta(k) \cdot 2^k \text{ , where } \delta(k) \varepsilon \{0,1\}$$

From this we have

$$a^{t_n - 1} = \prod_{k=1}^{m} a^{\delta(k) \cdot 2^k}$$

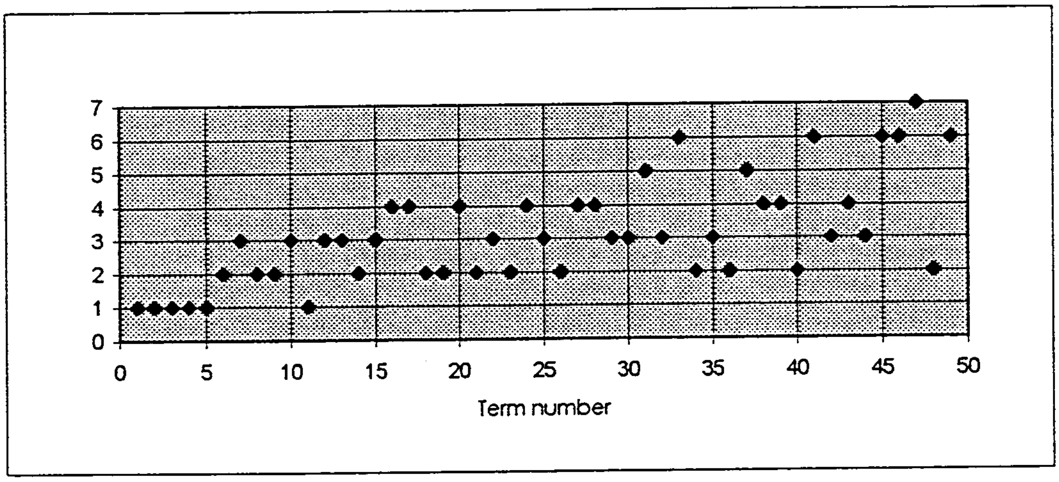

Diagram 3. The number of prime factors in the first 49 terms of the prime-product sequence.

This product expression for $a^{t_n - 1}$ is used in the following *Ubasic* program to carry out the reduction of $a^{t_n - 1}$ modulus t_n. Terms for which $\delta(k)=0$ are ignored in the expansion were the exponents k are contained in the array E%(). The residue modulus t_n is stored in F. In the program below the reduction is done to base A=7.

```
100  dim E%(1000)
110  M=N-1:I%=0
120  T=1:J%=0
130  while (M-T)>=0
140  inc J%:T=2*T
150  wend
160  dec J%:M=M-T\2:inc I%:E%(I%)=J%
170  if M>0 then goto 120
180  F=1
190  for J%=1 to I%
200  A=7
210  for K%=1 to E%(J%)
240  A=(A^2)@N
250  next
260  F=F*A:F=F@N
270  next
```

Table 2. Prime factorization of prime-product terms

#	P	L	N=p#+1 and its factors
1	2	1	3 Prime number
2	3	1	7 Prime number
3	5	2	31 Prime number
4	7	3	211 Prime number
5	11	4	2311 Prime number
6	13	5	30031 = 59· 509
7	17	6	510511 =19 · 97 · 277
8	19	7	9699691 =347 · 27953
9	23	9	223092871 =317 · 703763
10	29	10	6469693231 =331 · 571 · 34231
11	31	12	200560490131 Prime number
12	37	13	7420738134811 =181 · 60611 · 676421
13	41	15	304250263527211 =61 · 450451 · 11072701
14	43	17	13082761331670031= 61 · 450451 · 11072701
15	47	18	614889782588491411 = 953 · 46727 · 13808181181
16	53	20	32589158477190044731 = 73 · 139 · 173 · 18564761860301
17	59	22	1922760350154212639071 =277 · 3467 · 105229 · 19026377261
18	61	24	117288381359406970983271 =223 · 525956867082542470777
19	67	25	7858321551080267055879091 =54730729297 · 143581524529603
20	71	27	557940830126698960967415391 = 1063 · 303049 · 598841 · 2892214489673
21	73	29	40729680599249024150621323471 = 2521 · 16156160491570418147806951
22	79	31	3217644767340672907899084554131 = 22093 · 1503181961 · 96888414202798247
23	83	33	267064515689275851355624017992791 = 265739 · 1004988035964897329167431269
24	89	35	23768741896345550770650537601358311 = 131 · 1039 · 2719 · 64225891884294373371806141
25	97	37	2305567963945518424753102147331756071 = 2336993 · 13848803 · 71237436024091007473549
26	101	39	232862364358497360900063316880507363071 = 960703 · 242387464553038099079594127301057
27	103	41	23984823528925228172706521638692258396211 = 2297 · 9700398839 · 179365737007 · 6001315443334531
28	107	43	2566376117594999414479597815340071648394471 = 149 · 13203797 · 30501264491063137 · 42767843651083711
29	109	45	279734996817854936178276161872067809674997231 = 334507 · 1290433 · 648046444234299714623177554034701
30	113	47	31610054640417607788145206291543662493274686991 = 5122427 · 2025436786007· 3046707595069540247157055819
31	127	49	4014476939333036189094441199026045136645885247731 = 1543 · 49999 · 552001 · 57900988201093 ·1628080529999073967231
32	131	51	525896479052627740771371797072411912900610967452631 = 1951 · 22993 · 11723231859473014144932345466415143728266617
33	137	53	72047817630210000485677936198920432067383702541010311 = 881 · 1657 · 32633677 · 160823938621 · 5330099340103 · 1764291759303233
34	139	56	10014646650599190067509233131649940057366334653200433091 = 678279959005528882498681487 · 14764768614544245139224580493
35	149	58	1492182350939279320058875736615841068547583863326864530411 = 87549524399 · 65018161573521013453 · 262140076844134219184937113
36	151	60	225319534991831177328890236228992001350685163362356544091911 = 23269086799180847 · 9683213481319911991636641541802024271084713
37	157	62	35375166993717494840635767087951744212057570647889977422429871 = 1381 · 1867 · 8311930927 · 38893867968570583 · 42440201875440880489113304753
38	163	64	5766152219975951659023630035336134306565384015606066319856068811 = 1361 · 214114727210560829 · 32267019267402210517 · 613228865630544238382107
39	167	66	962947420735983927056946215901134429196419130606213075415963491271 = 205590139 · 53252429177 · 7064576339566763 · 12450154709928940906197946067239
40	173	69	166589903787325219380851695350896256250980509594874862046961683989711 = 62614127 · 2660580156093611580352333193927566158528098772260689062181793
41	179	71	29819592777931214269172453467810429868925511217482600306406141434158091 = 601 · 1651781 · 8564177 · 358995947 · 1525310189119 · 6405328664096618954809029861252251
42	181	73	5397346292805549782720214077673687806275517530364350655459511599582614291 = 107453 · 5634838141 · 8914157280964101123344891396571257163632974628403174028667
43	191	76	1030893141925860008499560888835674370998623848299590975192766715520279329391 = 32999 · 175603474759 · 77148541513247 · 2305961466437323959598530415862423316227152033
44	193	78	198962376391690981640415251545285153602734402721821058212203976095413910572271 = 21639496447 · 7979125905967339495018877 · 1152307771625979758044020162101777453615909
45	197	80	39195588149163123383161804554421175259738677336198748467804183290796540382737191 = 521831 · 50257723 · 1601684368321 · 39081170243262541027 · 23875913958369977158572653160969521
46	199	82	7799922041683461553249199106329813876687996789903550945093032474868511536164700811 = 467 · 10723 · 57622771 · 5876645549 · 9458145520867 · 486325954430626096097192220405214947865503847
47	211	85	1645783550795210387735581011435590727981167322669649249414629852197255934130751870911 = 1051 · 2179 · 16333 · 43283699 · 75311908487 · 292812710684839 · 460965966728664692934303334044872907384889
48	223	87	367009731827331916465034565550136732339800312955331782619462457039988073311157667212931 = 13867889468159 · 26464714235716608676791598492896703564888100036053342930619468037572880509
49	227	89	83311209124804345037562846379881038241134671040860314654617977748077292641632790457335111 = 3187 · 31223 · 1737142793 · 11463039340315601 · 973104505470446969309113 · 43206785807567189232875099500379

This program revealed that there are at most three terms t_n of the sequence in the interval $50 \leq n \leq 200$ which could be prime numbers. These are:

Term #75. N=379#+1. N is a 154 digit number.
N=1719620105458406433483340568317543019584575635895742560438771105058321655238562613083979651479555788009994557822024565226932906295208262756822275663694111

Term #171. N=1019#+1. N is a 425 digit number.
N=204040689930163741945424641727746076956597971174231219132271310323390261691759299022444537574
104687288429298622716055678188216854906766619853898399586228024659868813761394041383761530961031
408346655636467401602797552123175013568630036386123906616684062354223117837423905105265872570265
003026968347932485267343058016341659487025063671767012332980646166635537169754290487515755971504
17381063934255689124486029492908966644747931

Term #172. N=1021#+1. N is a 428 digit number.
N=208325544418697180526278559204028744572686528568890074734049007840181457187286244301915872863
160885721486313893793092847430169408859808718870830265977538813177726058850383316252820523111213
067921935404833217036456300717761688853571267150232508655634427663661803312009807112476455894240
568090534683239067457957262234684834336252590008874119591973239736134883450319130587753586846905
76146066276875058596100236112260054944287636531

The last two primes or pseudo primes are remarkable in that they are generated by the prime twins 1019 and 1021.

Summary of results: The number of primes q among the first 200 terms of the prime-product sequence is given by $6 \le q \le 9$. The six confirmed primes are terms numero 1, 2, 3, 4, 5 and 11. The three terms which are either primes or pseudo primes are terms numero 75, 171 and 172. The latter two are the terms 1019#+1 and 1021#+1.

III. The Square-Product Sequence

Definition: The terms of the square-product sequence are defined through $\{t_n : t_n = (n!)^2+1\}$

This sequence has a structure which is similar to the prime-product sequence. The analysis is therefore carried out almost identically to the one done for the prime-product sequence. We merely have to state the results and compare them.

The sequence begins {2, 5, 37, 577, 14401, 518401, ... }

As for the prime-product sequence the question of how many are prime numbers has been raised and we may never know. There are similarities between these two sequences. There are quite a few primes among the first terms. After that they become more and more rare. Complete factorization of the 37 first terms of the square-product sequence was obtained and has been used in diagram 4 which should be compared with the corresponding diagram 3 for the prime-product sequence.

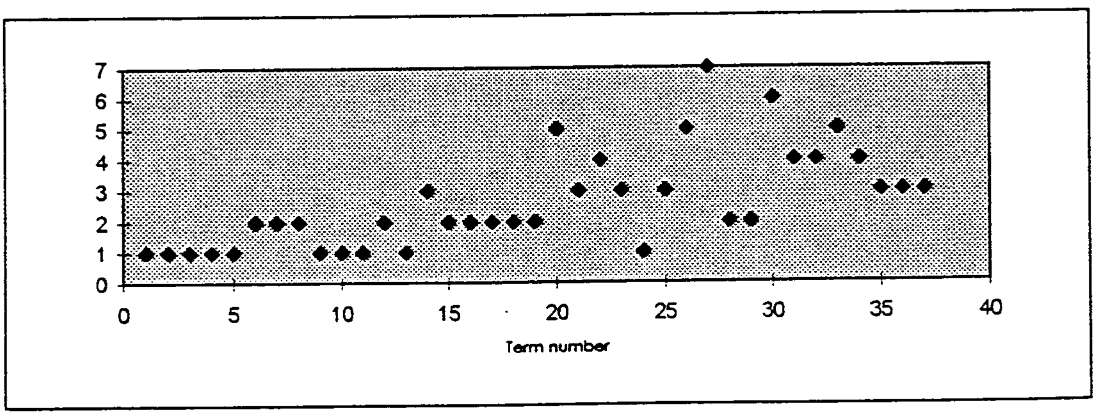

Diagram 4. The number of prime factors in the first 40 terms of the square-product sequence.

Diagram 4 is based on table 3 which shows the prime factorization of the 40 first terms in the square-product sequence. The number of factors of each term is denoted f. The factorization is not complete for terms numero 38 and 39. A +-sign in the column for f indicates that the last factor is not a prime. The terms of this sequence are in general much more time consuming to factorize than those of the prime-product sequence which accounts for the more limited results in this section. Using the same method as for the prime-product sequence the terms t_n in the interval $40<n\leq200$ which may possible be primes were identified. There are only two of them:

Term #65. $N=(65!)^2+1$. N is a 182 digit number.
68023740289078328950450781972622203792902576953271358034279380104027100652464382649659623724446578151412858996571534385340563792951822384455180747800576000000000000000000000000000001

Term #76 $N=(76!)^2+1$. N is a 223 digit number.
35550902700107478542025131357707726481943256669255416479770052502800500841772266884421391665890651643920912930369944999452531006264950776782697850719865801162529840993176478638638115061760000000000000000000000000000000000001

Table 3. Prime factorization of square-product terms.

n	L	f	$N=(n!)^2+1$ and its factors
1	1	1	2
2	1	1	5
3	2	1	37
4	3	1	577
5	5	1	14401
6	6	2	518401=13-39877
7	8	2	25401601= 101· 251501
8	10	2	1625702401= 17· 95629553
9	12	1	131681894401
10	14	1	13168189440001
11	16	1	1593350922240001
12	18	2	229442532802560001=101· 2271708245569901
13	20	1	38775788043632640001
14	22	3	7600054456551997440001= 29· 109· 2404319663572286441
15	25	2	1710012252724199424000001=134169-1272170577304043929
16	27	2	437763136697395052544000001=149· 2938007628841577533852349
17	30	2	126513546505547170185216000001= 9049·13980942259426143240713449
18	32	2	40990389067797283140009984000001= 37·1107848353183710355135404972973
19	35	2	14797530453474819213543604224000001=710341·20831587158104092560535861261
20	37	5	5919012181389927685417441689600000001=41·10657· 86816017· 348046955609·448324749841
21	40	3	2610284371992958109269091785113600000001= 61·157·2725576247251705240961774861766631513
22	43	4	1263377636044591724886240423994982400000001= 337· 8017· 51404983644027 7481· 9096748233235378 49
23	45	3	668326769467589022464821184293345689600000001= 509·15448374629· 849940026045327476874017 41723441
24	48	1	384956219213331276939737002152967117209600000001
25	51	3	24059763700833204808733562634560444825600000000001= 941· 81576983190847975 8733- 3134253313493312902433994 17
26	54	5	162644002617632464507038883409628607021056000000000001= 53- 53- 418633- 6017159668589· 22985889712876096222556462301797
27	57	7	118567477908254066625631346005619254518349824000000000001= 113- 42461· 745837· 2460281· 7566641· 15238649· 116793504008451126962009
28	59	2	929569026800711882344949752684054955423862620160000000000001= 2122590346576634509· 437940852929979393039522414749827534643 89
29	62	2	781767551539398693052102742007290217511468463554560000000000001= 171707860473207588349837· 4552893207014140637164693965317582487 73
30	65	6	703590796385458823746892467806561195760321617199104000000000000001= 61· 1733- 15661· 359525849· 1006363811265686901 10069· 117459224951820775953 7897
31	68	4	676150755326425929620763661562105309125669074128338944000000000000001= 353- 422041· 13400767181· 338676081809484090853058207938321915703246678216 77
32	71	4	692378373454260151931661989439595836544685131907419078656000000000000001= 10591621681· 6415450838021· 522303293914660001204969· 1950882388585355532025429
33	74	5	754000048691689305453579906499719865997162108647179376656384000000000000001= 37· 3121· 4421· 407333288284593625 3- 3625813512324448042738745076210857822314805 2301
34	77	4	8716240562875928371043383719136761650927193975961393594147799040000000000000001= 193- 13217· 866100731693- 3945214344314564523147689464409690129119741062428681 6576197
35	81	3	1067739468952301225452814505594253302238581262055270715283105382400000000000000001= 317 · 373 · 9030196538868084889782854556323553608634748201176163219890777161898157 15361
36	84	3	1383790351762182388186847599250152279701201315623630847006904575590400000000000000001= 73- 57986941373 · 32690174316982778045052076286249517817882480631366146754011637734149869
37	87	3	18944089915624276894277943633734584709109446010887506295524523639832576000000000000000001= 127406364297881· 491055711943381280219101 09 · 30279720114524038428292430769814272052327643069
38	90	4+	2735526583816145583533735060711274031995404003972155909073741213591823974400000000000000001= 233-757· 15509190807491428121700948859068006372542416722731790323638834191845062531678582160 21
39	93	4+	4160735933984357432554811027341847802665009490041649137701160385873164265062400000000000000001= 61·10045457577 41·67900128269327066547084588212509348973296153689729372855102292431572730336178 01
40	96	4	66571774943749718920876976437469564842640151840666386203218566173970628240998400000000000000000001= 89· 701· 187100101949· 5703061928798691519563156731422223621393496539544379492647728174834902025511 3441

IV. The Smarandache Prime-Digital Sub-Sequence

Definition: The prime-digital sub-sequence is the set $\{M=a_0+a_1\cdot 10+a_2\cdot 10^2+\ldots a_k 10^k$:M is a prime and all digits $a_0, a_1, a_2\ldots a_k$ are primes$\}$

The first terms of this sequence are {2, 3, 5, 7, 23, 37, 53, 73, ... }. Sylvester Smith [1] conjectured that this sequence is infinite. In this paper we will prove that this sequence is in fact infinite. Let's first calculate some more terms of the sequence and at the same time find how many terms there is in the sequence in a given interval, say between 10^k and 10^{k+1} .The program below is written in *Ubasic.* One version of the program has been used to produce table 4 showing the first 100 terms of the sequence. The output of the actual version has been used to produce the calculated part of table 5 which we are going to compare with the theoretically estimated part in the same table.

Ubasic program

```
10  point 2
20  dim A%(6),B%(4)
30  for I%=1 to 6:read A%(I%):next
40  data 1,4,6,8,9,0                            'Digits not allowed stored in A%()
50  for I%=1 to 4:read B%(I%):next
60  data 2,3,5,7                                'Digits allowed stored in B%()
70  for K%=1 to 7                               'Calc. for 7 separate intervals
80  M%=0:N=0
90  for E%=1 to 4                               'Only 2,3,5 and 7 allowed as first digit
100 P=B%(E%)*10^K%:P0=P:S=(B%(E%)+1)*10^K%:gosub 150
110 next
120 print K%,M%,N,M%/N
130 next
140 end
150 while P<S
160 P=nxtprm(P):P$=str(P)                       'Select prime and convert to string
170 inc N                                       'Count number of primes
180 L%=len(P$):C%=0                             'C% will be set to 1 if P not member
190 for I%=2 to L%
200 for J%=1 to 6                               'This loop examines each digit of P
210 if val(mid(P$,I%,1))=A%(J%) then C%=1
220 next:next
230 if C%=0 then inc M%                         'If criteria filled count member (m%)
240 wend
250 return
```

Table 4. The first 100 terms in the prime-digital sub sequence.

2	3	5	7	23	37	53	73	223	227
233	257	277	337	353	373	523	557	577	727
733	757	773	2237	2273	2333	2357	2377	2557	2753
2777	3253	3257	3323	3373	3527	3533	3557	3727	3733
5227	5233	5237	5273	5323	5333	5527	5557	5573	5737
7237	7253	7333	7523	7537	7573	7577	7723	7727	7753
7757	22273	22277	22573	22727	22777	23227	23327	23333	23357
23537	23557	23753	23773	25237	25253	25357	25373	25523	25537
25577	25733	27253	27277	27337	27527	27733	27737	27773	32233
32237	32257	32323	32327	32353	32377	32533	32537	32573	33223

Table 5. Comparison of results.

k	1	2	3	4	5	6	7
Computer count:							
m	4	15	38	128	389	1325	4643
log(m)	0.6021	1.1761	1.5798	2.1072	2.5899	3.1222	3.6668
n	13	64	472	3771	30848	261682	2275350
m/n	0.30769	0.23438	0.08051	0.03394	0.01261	0.00506	0.00204
Theoretical estimates:							
m	4	11	34	109	364	1253	4395
log(m)	0.5922	1.0430	1.5278	2.0365	2.5615	3.0980	3.6430
n	7	55	421	3399	28464	244745	2146190
m/n	0.50000	0.20000	0.08000	0.03200	0.01280	0.00512	0.00205

Theorem:
The Smarandache prime-digital sub sequence is infinite.

Proof:
We recall the prime counting function $\pi(x)$. The number of primes $p \leq x$ is denoted $\pi(x)$. For sufficiently large values of x the order of magnitude of $\pi(x)$ is given by $\pi(x) \approx \frac{x}{\log x}$. Let a and b be digits such that $a > b \neq 0$ and n(a,b,k) be the approximate number of primes in the interval $(b \cdot 10^k, a \cdot 10^k)$. Applying the prime number counting theorem we then have:

$$n(a,b,k) \approx \frac{10^k}{k}\left(\frac{a}{\log 10 + \frac{\log a}{k}} - \frac{b}{\log 10 + \frac{\log b}{k}}\right) \qquad (1)$$

Potential candidates for members of the prime-digital sub sequence will have first digits 2,3,5 or 7, i.e. for a given k they will be found in the intervals $(2 \cdot 10^k, 4 \cdot 10^k)$, $(5 \cdot 10^k, 6 \cdot 10^k)$ and $(7 \cdot 10^k, 8 \cdot 10^k)$. The approximate number of primes n(k) in the interval $(10^k, 10^{k+1})$ which might be members of the sequence is therefore:

$$n(k)=n(4,2,k)+n(6,5,k)+n(8,7,k) \qquad (2)$$

The theoretical estimates of n in table 5 are calculated using (2) ignoring the fact that results may not be all that good for small values of k.

We will now find an estimate for the number of candidates m(k) which qualify as members of the sequence. The final digit of a prime number >5 can only be 1,3,7 or 9. Assuming that these will occur with equal probability only half of the candidates will qualify. The first digit is already fixed by our selection of intervals. For the remaining k-1 digits we have ten possibilities, namely 0,1,2,3,4,5,6,7,8 and 9 of which only 2,3,5 and 7 are good. The probability that all k-1 digits are good is therefore $(4/10)^{k-1}$. The probability q that a candidate qualifies as a member of the sequence is

$$q = \frac{1}{2} \cdot \left(\frac{4}{10}\right)^{k-1} \qquad (3)$$

The estimated number of members of the sequence in the interval $(10^k, 10^{k+1})$ is therefore given by $m(k)=q \cdot n(k)$. The estimated values are given in table 5. A comparison between the computer count and the theoretically estimated values shows a very close fit as can be seen from diagram 5 where $\log_{10} m$ is plotted against k.

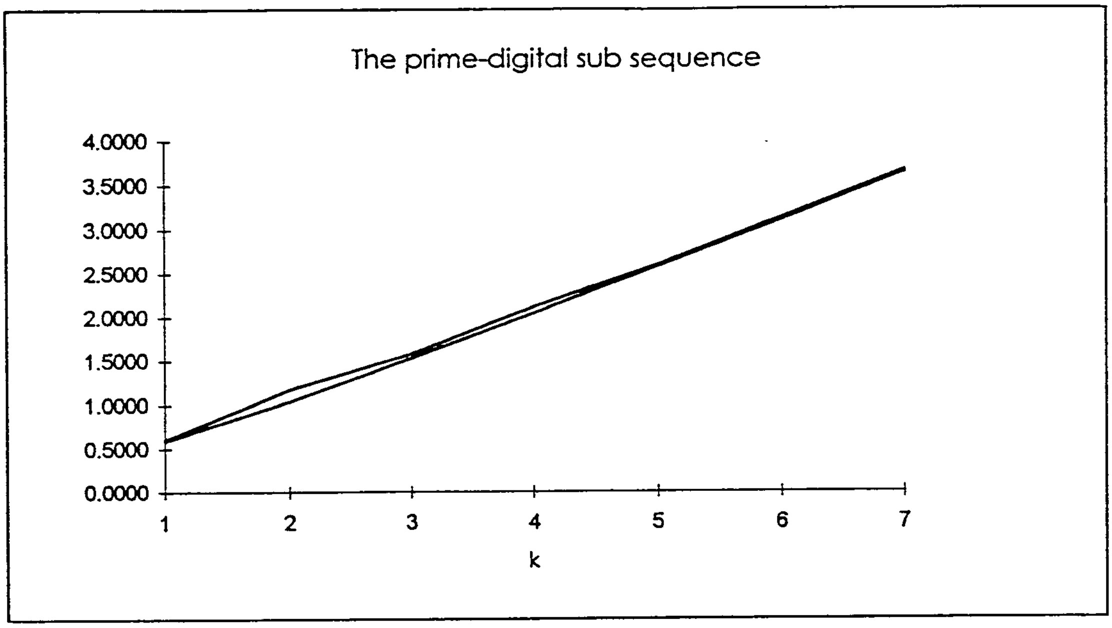

Diagram 5. $\log_{10} m$ as a function of k. The upper curve corresponds to the computer count.

For large values of k we can ignore the terms $\frac{\log a}{k}$ and $\frac{\log b}{k}$ in comparison with log 10 in (1). For large k we therefore have

$$n(a,b,k) \approx \frac{(a-b)10^k}{k\log 10} \tag{1'}$$

and (2) becomes

$$n(k) \approx \frac{4\cdot 10^k}{k\log 10} \tag{2'}$$

Combining this with (3) we get

$$m(k) \approx \frac{5\cdot 2^{2k}}{k\log 10} \tag{4}$$

From which we see (apply for instance l'Hospital's rule) that m(k)→∞ as k→∞. A fortiori the prime-digital sub sequence is infinite.

V. Smarandache Concatenated Sequences

Smarandache formulated a series of very artificially conceived sequences through concatenation. The sequences studied below are special cases of the Smarandache Concatenated S-sequence.

Definition: Let G={g_1, g_2, ... g_k, ... } be an ordered set of positive integers with a given property G. The corresponding concatenated S.G sequence is defined through

$$S.G = \{a_i : a_1 = g_1, a_k = a_{k-1} \cdot 10^{1+\log_{10} g_k} + g_k, k \geq 1\}$$

In table 6 the first 20 terms are listed for three cases, which we will deal with in some detail below.

Table 6. The first 20 terms of three concatenated sequences

The S.odd sequence	The S.even sequence	The S.prime sequence
1	2	2
13	24	23
135	246	235
1357	2468	2357
13579	246810	235711
1357911	24681012	23571113
135791113	2468101214	2357111317
13579111315	246810121416	235711131719
1357911131517	24681012141618	23571113171923
135791113151719	2468101214161820	2357111317192329
13579111315171921	246810121416182022	235711131719232931
1357911131517192123	24681012141618202224	23571113171923293137
135791113151719212325	2468101214161820222426	2357111317192329313741
13579111315171921232527	246810121416182022242628	235711131719232931374143
1357911131517192123252729	24681012141618202224262830	23571113171923293137414347
135791113151719212325272931	2468101214161820222426283032	2357111317192329313741434753
13579111315171921232527293133	246810121416182022242628303234	235711131719232931374143475359
1357911131517192123252729313335	24681012141618202224262830323436	23571113171923293137414347535961
135791113151719212325272931333537	2468101214161820222426283032343638	2357111317192329313741434753596167
13579111315171921232527293133353739	246810121416182022242628303234363840	235711131719232931374143475359616771
1357911131517192123252729313335373941	24681012141618202224262830323436384042	23571113171923293137414347535961677173

Case I. The **S.odd sequence** is generated by choosing G={1,3,5,7,9,11,.....}. Smarandache asks how many terms in this sequence are primes and as is often the case we have no answer. But for this and the other concatenated sequences we can take a look at a fairly large number of terms and see how frequently we find primes or potential primes. As in the case of prime-product sequence we will resort to Fermat's little theorem to find all primes/pseudo-primes among the first 200 terms. If they are not too big wee can then proceed to test if they are primes. For the S.odd sequence there are only five cases which all were confirmed to be primes using the elliptic curve prime factorization program. In table 7 # is the term number, L is the number of digits of N and N is a prime number member of the S.odd sequence.:

Table 7. Prime numbers in the S.odd sequence

#	L	N
2	2	13
10	15	135791113151719
16	27	135791113151719212325272931
34	63	135791113151719212325272931333537394143454749515355575961636567
49	93	135791113151719212325272931333537394143454749515355575961636567697173757779818385878991939597

Term #201 is a 548 digit number.

Case 2. The **S.even sequence** is generated by choosing G={2,4,6,8,10, ... }. The question here is : How many terms are *nth* powers of a positive integer?

A term which is a *nth* power must be of the form $2^n \cdot a$ where a is an odd *nth* power. The first step is therefore to find the highest power of 2 which divides a given member of the sequence, i.e. to determine n and at the same time we will find a. We then have to test if a is a *nth* power. The *Ubasic* program below has been implemented for the first 200 terms of the sequence. No *nth* powers were fond.

Ubasic program: (only the essential part of the program is listed)

```
60  N=2
70  for U%=4 to 400 step 2
80  D%=int(log(U%)/log(10))+1         'Determine length of U%
90  N=N*10^D%+U%                      'Concatenate U%
100  A=N:E%=0
110  repeat
120  A1=A:A=A\2:inc E%                'Determine E% (=n)
130  until res<>0
```

```
132  dec E%:A=A1                      'Determine A (=a)
140  B=round(A^(1/E%))
150  if B^E%=A then print E%,N        'Check if a is a nth power
160  next
170  end
```

So there is not even a perfect square among the first 200 terms of the S.even sequence. Are there terms in this sequence which are 2·p where p is a prime (or pseudo prime). With a small change in the program used for the S.odd sequence we can easily find out. Strangely enough not a single term was found to be of the form 2·p.

Case 3. The **S.prime sequence** is generated by {2,3,5,7,11, ...}. Again we ask: - How many are primes? - and again we apply the method of finding the number of primes/pseudo primes among the first 200 terms.

There are only 4 cases to consider: Terms #2 and #4 are primes, namely 23 and 2357. The other two cases are: term #128 which is a 355 digit number and term #174 which is a 499 digit number.

#128
235711131719232931374143475359616771737983899710110310710911312713113713914915115716316717317918
1191193197199211223227229233239241251257263269271277281283293307311313317331337347349353359367 37
3379383389397401409419421431433439443449457461463467479487491499503509521523541547557563569571 57
7587593599601607613617619631641643647653659661673677683691701709719

#174
235711131719232931374143475359616771737983899710110310710911312713113713914915115716316717317918
1191193197199211223227229233239241251257263269271277281283293307311313317331337347349353359367 37
3379383389397401409419421431433439443449457461463467479487491499503509521523541547557563569571 57
758759359960160761361761963164164364765365966167367768369170170971972773373974375175776176977378
77978098118218238278298398538578598638778818838879079119199299379419479539679719779839919971009 1
0131019102110311033

Are these two numbers prime numbers?