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WHY QUESTION #46

How did Pythagoras come up with the Pythagorean theorem? Is ‘a’ squared plus ‘b’ squared always equal to ‘c’ squared? If so, why?

I chose this question because the Pythagorean theorem is part of the 8^{th} grade math curriculum that I teach. I figured I would definitely find something new to share with my kids.

I began by focusing on the first question and I discovered that the Pythagoreans were absolutely not the first group of people to have knowledge of the relationship of the sides of a right triangle.

The Egyptians: 2650 B.C (#1)

The Egyptians knew that a triangle with sides 3, 4, and 5 make a 90^{o} angle. As a matter of fact, they had a rope with 12 evenly spaced knots like this one:

that they used to build perfect corners in their buildings and pyramids. It is believed that they only knew about the 3, 4, 5 triangle and not the general theorem that applies to all right triangles.

The Babylonians: 1900 B.C – 1600 B.C. (#2)

4 Babylonian tablets (Plimpton 322, Yale Tablet, Susa Tablet and Tell Dhibayi Tablet) were discovered all having a connection to the Pythagorean theorem. Certainly the Babylonians were familiar with Pythagoras's theorem. A translation of a Babylonian tablet which is preserved in the British museum goes as follows:-

> 4 *is the length and* 5 *the diagonal*. *What is the breadth* ?
> *Its size is not known*.
> 4 *times* 4 *is* 16.

> 5 *times* 5 *is* 25.
> *You take* 16 *from 25 and there remains* 9.
> *What times what shall I take in order to get* 9 ?
> 3 *times* 3 *is* 9.
> 3 *is the breadth*.

All the tablets we wish to consider in detail come from roughly the same period, namely that of the Old Babylonian Empire which flourished in Mesopotamia between 1900 BC and 1600 BC.

The Chinese: 1100 B.C. (#3)

Could the origin of "Pythagoras Theorem" be in China? Swetz and Kao believe so (1977). Why? A search back in history finds a proof in an ancient Chinese mathematical text, 'Chou pei Suan Ching'. 'This book is the oldest Chinese mathematics text known. While the exact date of its origin is controversial, with estimates ranging as far back as 1100 B.C.'(Swetz and Kao, 1977, p 14). The diagram that describes the proof (Diagram 2) was so well known in China that it had a special name, 'hsuan-thu'. This diagram shows 'the square on the hypotenuse folded backwards ... and demonstrably containing three further identical triangles together with a square of the difference between the base and the altitude' (Needham, p95). This picture was described as

> The diagram giving the relations between the hypotenuse and the sum and difference of the other two sides whereby one can find the unknown from the known.
>
> -- J.Needham, 1959, p96

Diagram 2. The 'hsuan-thu'.

Problems from Chiu chang suan shu also offering evidence of the knowledge of right angled triangles

The Chiu chang suan shu is the most famous Chinese mathematical text. Its context is a summary of the mathematical knowledge possessed in China up to the middle of the 3rd century. Alexander Wylie 'was the first to translate parts of the text into English and he was marvelled at his findings' (Swetz and Kao, 1977, p17).

A few examples are as follows :

Problem 1

Given: kou = 3 ch'ih, ku = 4 ch'ih. What is the length of hsien?

Answer

hsien = 5 ch'ih

Method

Add the square of kou and ku. The square root of the sum is equal to hsien (Look at diagram 3).

Diagram 3. Pictorial representation of Method 1.

INDIANS: 800 B.C. – 200 B.C. (#4)

In India, the *Baudhayana* *Sulba Sutra*, the dates of which are given variously as between the 8th century BC and the 2nd century BC, contains a list of Pythagorean triples discovered algebraically, a statement of the Pythagorean theorem, and a geometrical proof of the Pythagorean theorem for an isosceles right triangle.[63] The *Apastamba* *Sulba Sutra* (circa 600 BC) contains a numerical proof of the general Pythagorean theorem, using an area computation. Van der Waerden believes that "it was certainly based on earlier traditions". According to Albert Bŭrk, this is the original proof of the theorem; he further theorizes that Pythagoras visited Arakonam, India, and copied it. Boyer (1991) thinks the elements found in the Śulba-sũtram may be of Mesopotamian derivation.

Pythagoras: 569 B.C. – 475 B.C. (#1, #4, #5, #6, #7)

So why is it called the Pythagorean theorem? Even though the theorem was known long before his time, Pythagoras certainly generalized it and made it popular. It was Pythagoras who is attributed with its first geometrical demonstration. That is why it is known as the Pythagorean theorem. There are hundreds of purely geometric demonstrations as well as an unlimited (that is right -- an infinite number) of algebraic proofs.

We do not know for sure how Pythagoras himself proved the theorem that bears his name because he refused to allow his teachings to be recorded in writing. But probably, like most ancient proofs of the Pythagorean theorem, it was geometrical in nature. That is, such proofs are demonstrations that the combined areas of squares with sides of length *a* and *b* will equal the area of a square with sides of length *c*, where *a*, *b*, and *c* represent the lengths of the two sides and hypotenuse of a right triangle.

While the theorem that now bears his name was known and previously utilized by the Babylonians and Indians, he, or his students, are often said to have constructed the first proof. Because of the secretive nature of his school and the custom of its students to attribute everything to their teacher, there is no evidence that Pythagoras himself worked on or proved this theorem. The earliest known mention of Pythagoras's name in connection with the theorem occurred five centuries after his death, in the writings of Cicero and Plutarch. However, when authors such as Plutarch and Cicero attributed the theorem to Pythagoras, they did so in a way which suggests that the attribution was widely known and undoubted.[6][73] "Whether this formula is rightly attributed to Pythagoras personally, [...] one can safely assume that it belongs to the very oldest period of Pythagorean mathematics."[31]

Historians believe that Pythagoras traveled extensively throughout Egypt and Babylonia at an early age. No doubt Pythagoras picked up on their observations that the square on the longer side of a right triangle is equal in area to the sum of the squares of the other two sides. However, with the Greeks, this property became a rule applied to all right-angled triangles and was finally refined into a proof that covered all right-angled triangles. So what might be attributed to Pythagoras is the establishment of a general rule. The proof would come later. It would be left to Euclid to present a final proof in his book *Elements* nearly 200 years later. The proof comes at the end of Book I, under the form of Preposition 47. Since Book I is comprised of 48 prepositions, the proof appears as a crowning moment in the gradual progression of the Elements of Book I.

The second question asked if always equaled ? (#8, #9 &#10)

As long as 'a' and 'b' are the legs of a right triangle and 'c' is the hypotenuse, the answer is YES. I have included two proofs below. The first is geometric, while the second is more algebraic.

Proofs of the Pythagorean Theorem

Proof 1. (Gardner 154)

Begin with the figure on the left below. Arrange four duplicated right triangles in this pattern. Now by rearranging the four triangles inside the largest square, the figure on the right can be reached. The two white squares are each squares on the legs of the triangle. Since their area is that of the large square minus the four triangles, we know it must equal the area of the white square on the left. Therefore, the square on the hypotenuse (left picture) has an area equal to the area of the sum of the squares on the legs (right picture).

Choupei's Proof

The following figure is made up of a large square, a small square and four right triangles.

The steps of the proof are as follows:

As the large square consists of one small square and four right triangles, thus we can write

Therefore, the sum of the squares of the sides of a right triangle equals the square of the hypotenuse.

The Pythagorean Proposition by an early 20th century professor Elisha Scott Loomis is a collection of 367 proofs of the Pythagorean Theorem and was republished by NCTM in 1968. In the Foreword, the author rightly asserts that the number of algebraic proofs is limitless as is also the number of geometric proofs.

Sources:

#1: http://www.arcytech.org/java/pythagoras/history.html

#2: http://www-history.mcs.st-andrews.ac.uk/HistTopics/Babylonian_Pythagoras.html

#3: http://www.unisanet.unisa.edu.au/07305/pythag.htm

#4: http://en.wikipedia.org/wiki/Pythagorean_theorem#Euclid.27s_proof

#5: http://ualr.edu/lasmoller/pythag.html

#6: http://en.wikipedia.org/wiki/Pythagoras#Pythagorean_theorem

#7: http://www.completepythagoras.net/overview.html

#8: http://www.ms.uky.edu/~lee/ma502/pythag/proofs.htm

#9: http://jwilson.coe.uga.edu/EMT668/EMAT6680.2002/Rouhani/Essay1/pythagorastheorem.html

#10: http://www.cut-the-knot.org/pythagoras/index.shtml (This site contains 84 proofs)