Trigonometry

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November 15, 2019

Table of Contents

Preface

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Special thanks to Michael Corral who generously provided the LaTeXcode for many of the images in this text.

Chapter 1

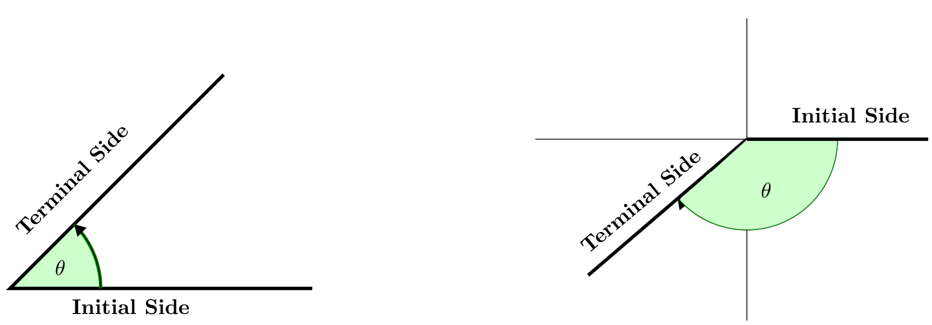
Trigonometric Functions

1.1 Angles and Their Measure

Angles

Definition 1.1. An angle is the shape formed when two rays come together. In trigonometry we think of one of the sides as being the Initial Side and the angle is formed by the other side (Terminal Side) rotating away from the initial side. See **Figure 1.1**.

We will usually draw our angles on the coordinate axes with the positive x-axis being the Initial Side. If we sweep out an angle in the counter clockwise direction we will say the angle is positive and if we sweep the angle in the clockwise direction we will say the angle is negative. An angle is in **standard position** if the initial side is the positive x-axis and the vertex is at the origin.



(a) Positive angle **(b)** Negative Angle

Figure 1.1: Positive and Negative Angles

When representing angles using variables, it is traditional to use Greek letters. Here is a list of commonly encountered Greek letters.

alpha	beta	gamma	theta	phi
α	β	γ	θ	ϕ

Measuring an Angle

When we measure angles we can think of them in terms of pieces of a circle. We have two units for measuring angles. Most people have heard of the degree but the radian is often more useful in trigonometry.

NOTE: By convention if the units are not specified they are radians.

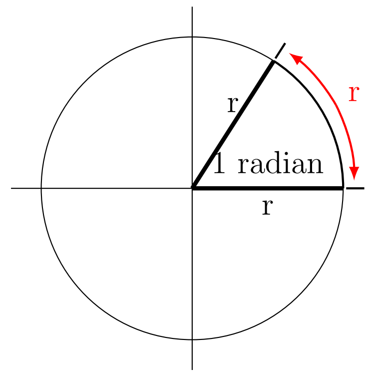


Figure 1.2: One Radian

Degrees: One degree (1°) is a rotation of 1/360 of a complete revolution about the vertex. There are 360 degrees in one full rotation which is the terminal side going all the way around the circle.

Radian: One Radian is the measure of a central angle θ that intercepts an arc equal in length to the radius r of the circle. See **Figure 1.2** at right. Since the radian is measured in terms of r on the arc of a circle and the complete circumference of the circle is $2\pi r$ then there are 2π radians in one full rotation.

Since $360^\circ = 2\pi$ `radians`, this gives us a way to convert between degrees and radians:

$$180^\circ = \pi \texttt{ radians}$$

Converting Degrees and Radians

To convert from degrees $\rightarrow$ radians we multiply degrees by $\frac{\pi}{180}$

$$\texttt{degrees} \cdot \frac{\pi}{180} = \texttt{radians}$$

To convert from radians $\rightarrow$ degrees we multiply radians by $\frac{180}{\pi}$

$$\texttt{radians} \cdot \frac{180}{\pi} = \texttt{degrees}$$

Example 1.1.1

Consider the following two angles: $240°$ and $-120°$. Sketch them and convert to radians.

(a) 240° **(b)** -120°

Figure 1.3: Example 1.1.1

Solution:

To convert to radians we need to multiply by the appropriate factor.

$$240° \cdot \frac{\pi}{180} = \boxed{\frac{4\pi}{3}} \qquad \text{and} \qquad -120° \cdot \frac{\pi}{180} = \boxed{-\frac{2\pi}{3}}$$

If we sketch these two angles from Example 1.1.1 on a single graph and in standard position (**Figure 1.4**) we will see that they look exactly the same. Since these two angle terminate at the same place we call them **Coterminal Angles**.

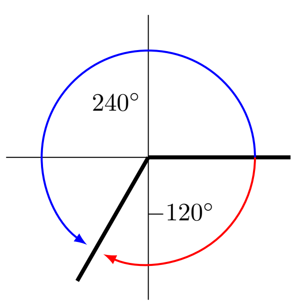


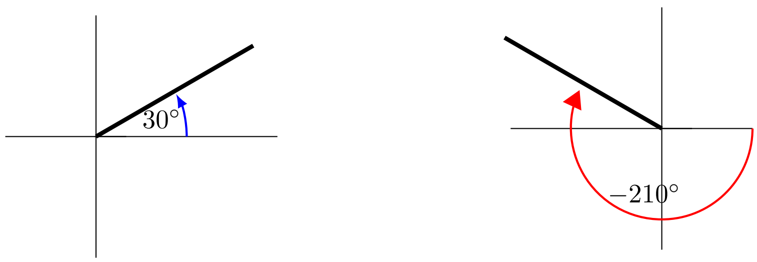
Figure 1.4: Coterminal angles end up in the same position but have different angle measures.

There are an infinite number of ways to draw an angle on the coordinate axes. By simply adding or subtracting $360°$ (or 2π rad) you will arrive at the same place. For example if you draw the angles $240° + 360° = 600°$ and $-120° - 360° = -480°$ you will end up in the same positions as the angles in **Figure 1.4**.

Example 1.1.2

Convert 30° and -210° to radians, sketch the angle and find two coterminal angles (one positive and one negative).

Solution:



(a) 30° **(b)** -210°

Figure 1.5: Example 1.1.2

a) $30° \cdot \dfrac{\pi}{180} = \boxed{\dfrac{\pi}{6}}$

Coterminal angles: $30° + 360\ ° = \boxed{390°}$ and $30° - 360° = \boxed{-330°}$

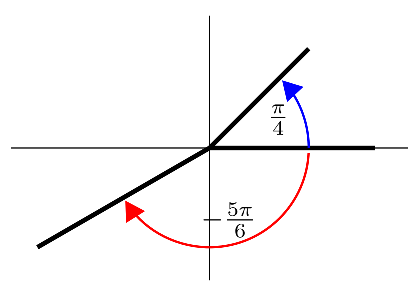
b) $-210 \cdot \dfrac{\pi}{180} = \boxed{-\dfrac{7\pi}{6}}$

Coterminal angles: $-210° + 360° = \boxed{150°}$ and $-210° - 360° = \boxed{-570°}$

Example 1.1.3

Convert $\dfrac{\pi}{4}$ and $-\dfrac{5\pi}{6}$ to degrees, sketch the angles and find two coterminal angles for each (one positive and one negative). Leave exact answers

Solution:



a) Convert to degrees: $\dfrac{\pi}{4} \cdot \dfrac{180}{\pi} = \boxed{45°}$

Coterminal angles: $\dfrac{\pi}{4} + 2\pi = \boxed{\dfrac{9\pi}{4}}$ and $\dfrac{\pi}{4} - 2\pi = \boxed{-\dfrac{7\pi}{4}}$

b) Convert to degrees: $-\dfrac{5\pi}{6} \cdot \dfrac{180}{\pi} = \boxed{-150°}$

Coterminal angles: $-\dfrac{5\pi}{6} + 2\pi = \boxed{\dfrac{7\pi}{6}}$

and $-\dfrac{5\pi}{6} - 2\pi = \boxed{-\dfrac{17\pi}{6}}$

Example 1.1.4

Convert 1 radian to degrees.

Solution:

$$1 \cdot \frac{180}{\pi} = \boxed{57.29^\circ}$$

Example 1.1.5

Find an angle θ that is coterminal with 970°, where $0 \le \theta < 360^\circ$

Solution:

Since adding or subtracting a full rotation, 360°, would result in an angle with terminal side pointing in the same direction, we can find coterminal angles by adding or subtracting multiples of 360°. An angle of 970° is coterminal with an angle of $970 - 360 = 610^\circ$. It would also be coterminal with an angle of $610 - 360 = 250^\circ$.

$\boxed{\text{The angle } \theta = 250^\circ \text{ is coterminal with } 970^\circ.}$

By finding the coterminal angle between 0 and 360°, it can be easier to sketch the angle in standard position.

Example 1.1.6

Find an angle β that is coterminal with $\dfrac{19\pi}{4}$, where $0 \le \beta < 2\pi$

Solution:

As in Example 1.1.5, adding or subtracting a full rotation (2π) will result in an angle with terminal side pointing in the same direction. In this case we need an angle $0 \le \beta < 2\pi$ so we need to subtract 2π twice. An angle of $\dfrac{19\pi}{4}$ is coterminal with an angle of

$$\frac{19\pi}{4} - (2) \cdot 2\pi = \frac{19\pi}{4} - \frac{16\pi}{4} = \frac{3\pi}{4}.$$

$$\boxed{\text{The angle } \beta = \frac{3\pi}{4} \text{ is coterminal with } \frac{19\pi}{4}.}$$

Degrees, Minutes and Seconds

The Babylonians who lived in modern day Iraq from about 5000BC to 500BC used a base 60 number system (link to Wikipedia). It is believed that this is the origin of having 60 minutes in an hour and 60 seconds in a minute. This may also explain why our degree measures are multiples of 60, once around the circle is 6 60s. Similar to the way hours are divided into minutes and seconds the degree ($^\circ$) can also be divided into 60 minutes ($'$) and each of those minutes is divided into 60 seconds ($''$). This form is often abbreviated DMS ($^\circ\ '\ ''$).

Example 1.1.7

Convert $5°37'15''$ to a decimal.

Solution:

First we need to understand that $1' = \frac{1}{60}^{\circ}$ and $1'' = \frac{1}{60}' = \frac{1}{3600}^{\circ}$. To convert to a decimal you have to write all the parts as fractions. $37' = \frac{37}{60}^{\circ}$

$$5°37'15'' = 5 + \frac{37}{60} + \frac{15}{3600} = \boxed{5.6208°}$$

Example 1.1.8

Convert $15.67°$ to DMS.

Solution:

We know our answer will look like

$$15°\ x'\ y''.$$

This direction is a bit more difficult because you have to work your way up to $0.67°$ using minutes and seconds. First we have to determine how many minutes we have. $\dfrac{x'}{60} = 0.67°$ so $x' = 0.67 \cdot 60 = 40.2'$. We can only use whole numbers so we take $x' = 40$. Now we have $15°40'y''$. y'' is the seconds and there are still $0.2'$ left. We can convert that to seconds because there are 60 seconds in a minute and we have 0.2 minutes. $(0.2')(60) = 12''$. Now our answer is

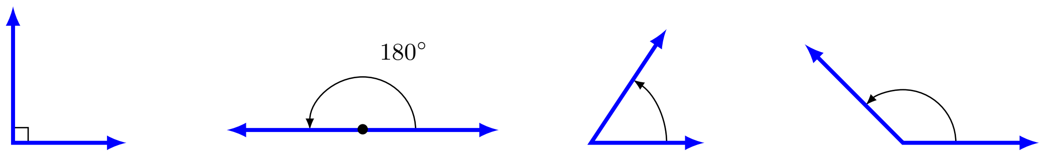
$$\boxed{15°40'12''}$$

and we can verify that this is true using the same technique we used in Example 1.1.7:

$$15°40'12'' = 15 + \frac{40}{60} + \frac{12}{3600} = 15.67°$$

Some basic angles

Name of angle	Measure in degrees	Measure in radians
Right angle	$90°$	$\frac{\pi}{2}$
Straight angle	$180°$	π
Acute angle	between 0 & $90°$	between 0 & $\frac{\pi}{2}$
Obtuse angle	between 90 & $180°$	between $\frac{\pi}{2}$ and π



(a) Right **(b)** Straight **(c)** Acute **(d)** Obtuse

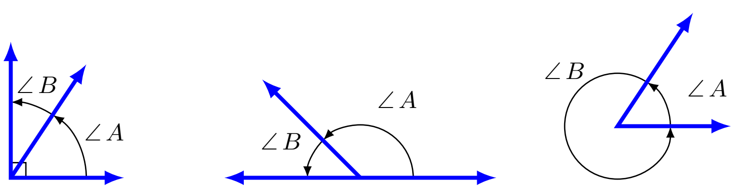
Figure 1.6: Basic Angles

Some special angles

(a) Two acute angles are **complementary** if their sum equals $90°$. In other words, if $0° \leq \angle A, \angle B \leq 90°$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90°$.

(b) Two angles between $0°$ and $180°$ are **supplementary** if their sum equals $180°$. In other words, if $0° \leq \angle A, \angle B \leq 180°$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180°$.

(c) Two angles between $0°$ and $360°$ are **conjugate** (or **explementary**) if their sum equals $360°$. In other words, if $0° \leq \angle A, \angle B \leq 360°$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360°$.



(a) complementary **(b)** supplementary **(c)** conjugate

Figure 1.7: Types of pairs of angles

Notation: Notice that we use the $\angle$ symbol here to denote angle A. Very often we will drop the $\angle$ symbol and simply refer to the angle by its letter if there is no chance for confusion. Angles are often labeled with Greek letters as seen earlier or with Latin letters as seen here. It is common to use upper case letters to denote angles but sometimes we use lowercase variable names (e.g. x , y , t).

Arc Length and Area

There is another way to define the radian. The **radian measure** of an angle is the ratio of the length of the circular arc subtended by the angle to the radius of the circle as seen in **Figure 1.8**. So the radian measure of an arc or length s on a circle of radius r is

$$\text{radian measure} = \theta = \frac{s}{r}$$

This formulation of the radian gives us a formula for the arc length s if we know the angle θ in radians:

$$\text{arc length} = s = r\theta$$

Example 1.1.9

Find the length of the arc of a circle with radius 4 cm and central angle 5.1 radians.

Solution:

$$\begin{aligned} s &= r\theta \\ &= (4)(5.1) \\ &\boxed{= 20.4\text{cm}} \end{aligned}$$

Example 1.1.10

Because Pluto orbits much further from the Sun than Earth, it takes much longer to orbit the Sun. In fact, Pluto takes 248 years to orbit the Sun. That's because Pluto orbits at an average distance of 5.9 billion km from the Sun, while Earth only orbits at 150 million km. Assuming that Pluto has a circular orbit how far does it travel in the time it takes the Earth to go around the sun once?

Solution: Since it takes 248 years to orbit the sun that means that in one year Pluto has completed $\frac{1}{248}$ of an orbit. To calculate the distance it has traveled we need to calculate the arc length so we need to convert $\frac{1}{248}$ of an orbit to radians. Since one rotation $= 2\pi$ radians then

$$\frac{1}{248} \text{ rotations} = 2\pi\left(\frac{1}{248}\right) = 0.025335425 \text{ radians}$$

$$s = r\theta = (5,900,000,000)(0.025335425) = 149,479,000\text{km}$$

$\boxed{\text{Pluto travels approximately 150 million km in a year}}$

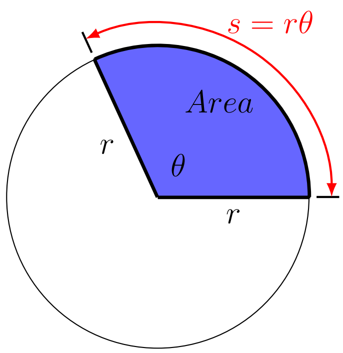


Figure 1.8: Area of sector and arc length

From geometry we know that the area of a circle of radius r is πr^2. We want to find the area of a sector of a circle. A **sector of a circle** is the region bounded by a central angle and its intercepted arc, such as the shaded region in **Figure 1.8**. The area of this sector is proportional to the angle by the following relationship:

$$\frac{\text{sector area}}{\text{circle area}} = \frac{\text{sector angle}}{\text{one revolution}} = \frac{Area}{\pi r^2} = \frac{\theta}{2\pi}$$

This gives a formula for the area of the sector of circle radius r with central angle θ:

$$Area = \frac{1}{2} r^2 \theta$$

Example 1.1.11

A farmer wants to irrigate her field with a central pivot irrigation system[1] with a radius of 400 feet. Due to water restrictions she can only water a portion of the field each day. She calculated that she could irrigate an arc of 130° each day. How much area is being irrigated each day?

Solution: To use our area formula we need to convert the angle to radians.

$$\theta = 130^\circ \left(\frac{\pi}{180}\right) = \frac{13\pi}{18}$$

$$Area = \frac{1}{2} r^2 \theta = \left(\frac{1}{2}\right) (400)^2 \left(\frac{13\pi}{18}\right) \approx 181514 \text{ft}^2$$

$\boxed{\text{The area is about } 181514\text{ft}^2}$.

1.1 Exercises

For Exercises 1-20,

a) draw the angle in standard position

b) find two coterminal angles, one positive and one negative.

Leave your answer in the same units (degrees/radians) as the original problem.

[1] https://en.wikipedia.org/wiki/Center_pivot_irrigation

1. $120°$ **2.** $-120°$ **3.** $-30°$ **4.** $217°$ **5.** $-217°$

6. $-115°$ **7.** $928°$ **8.** $1234°$ **9.** $-1234°$ **10.** $-515°$

11. $\frac{\pi}{2}$ **12.** $\frac{5\pi}{3}$ **13.** $-\frac{5\pi}{3}$ **14.** $\frac{3\pi}{7}$ **15.** $\frac{11\pi}{6}$

16. 5π **17.** -17 **18.** $-\frac{35\pi}{3}$ **19.** $-\frac{15\pi}{4}$ **20.** $\frac{122\pi}{3}$

For Exercises 21-32, convert to radians or degrees as appropriate. Leave an exact answer.

21. $120°$ **22.** $115°$ **23.** $135°$ **24.** $-425°$

25. $-270°$ **26.** $15°$ **27.** $\frac{\pi}{2}$ **28.** $\frac{\pi}{3}$

29. $\frac{\pi}{4}$ **30.** $\frac{\pi}{5}$ **31.** $-\frac{\pi}{6}$ **32.** $-\frac{11\pi}{6}$

For Exercises 33-36, write the following angles in DMS format. Round the seconds to the nearest whole number.

33. $12.5°$ **34.** $125.7°$ **35.** $539.25°$ **36.** $7352.12°$

For Exercises 37-40, write the follwing angles in decimal format. Round to two decimal places.

37. $12°12'12''$ **38.** $25°50'50''$ **39.** $0°22'17''$ **40.** $1°1'1''$

41. Saskatoon, Saskatchewan is located at 52.1332°N, 106.6700°W. Convert these map coordinates to DMS format.

42. On a circle of radius 7 miles, find the length of the arc that subtends a central angle of 5 radians.

43. On a circle of radius 6 feet, find the length of the arc that subtends a central angle of 1 radian.

44. On a circle of radius 12 cm, find the length of the arc that subtends a central angle of 120 degrees.

45. On a circle of radius 9 miles, find the length of the arc that subtends a central angle of 200 degrees.

46. A central angle in a circle of radius 5 m cuts off an arc of length 2 m. What is the measure of the angle in radians? What is the measure in degrees?

47. Mercury orbits the sun at a distance of approximately 36 million miles. In one Earth day, it completes 0.0114 rotation around the sun. If the orbit was perfectly circular, what distance through space would Mercury travel in one Earth day?

48. Find the distance along an arc on the surface of the Earth that subtends a central angle of $1°5'$. The radius of the Earth is 6,371 km.

49. Find the distance along an arc on the surface of the sun that subtends a central angle of $1''$ (1 second). The radius of the sun is 695,700 km.

50. On a circle of radius 6 feet, what angle in degrees would subtend an arc of length 3 feet?

51. On a circle of radius 5 feet, what angle in degrees would subtend an arc of length 2 feet?

52. A sector of a circle has a central angle of $\theta = 45°$. Find the area of the sector if the radius of the circle is 6 cm.

53. A sector of a circle has a central angle of $\theta = \frac{10\pi}{7}$. Find the area of the sector if the radius of the circle is 20 cm.

1.2 Right Triangle Trigonometry

Pythagorean Theorem

In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the other two sides are called its **legs**. For example, in **Figure 1.9** the right angle is C, the hypotenuse is the line segment $\overline{AB}$, which has length c, and $\overline{BC}$ and $\overline{AC}$ are the legs, with lengths a and b, respectively. The hypotenuse is always the longest side of a right triangle. When using Latin letters to label a triangle we use upper case letters $(A, B, C, \ldots)$ to denote the angles and we use the corresponding lower case letters $(a, b, c, \ldots)$ to represent the side opposite the angle. So in **Figure 1.9** side a is opposite angle A.

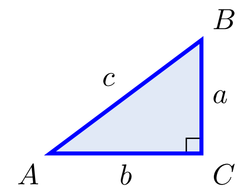


Figure 1.9: $a^2 + b^2 = c^2$

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

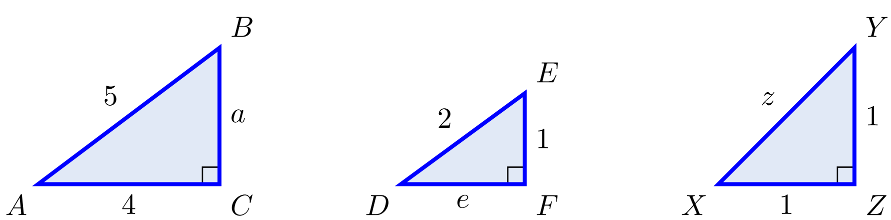
Pythagorean Theorem

Pythagorean Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.
Thus, if a right triangle has a hypotenuse of length c and legs of lengths a and b, as in **Figure 1.9**, then the Pythagorean Theorem says:

$$a^2 + b^2 = c^2 \tag{1.1}$$

Example 1.2.1

For each right triangle below, determine the length of the unknown side:



Solution: For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad\Rightarrow\quad a^2 = 25 - 16 = 9 \quad\Rightarrow\quad \boxed{a = 3}\,.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

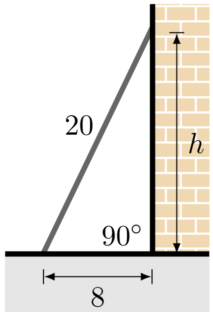
$$e^2 + 1^2 = 2^2 \quad\Rightarrow\quad e^2 = 4 - 1 = 3 \quad\Rightarrow\quad \boxed{e = \sqrt{3}}\,.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad\Rightarrow\quad z^2 = 2 \quad\Rightarrow\quad \boxed{z = \sqrt{2}}\,.$$

Example 1.2.2

A ladder 20 feet long leans against the side of a house. Find the height h from the top of the ladder to the ground if the base of the ladder is placed 8 feet from the base of the building.



Solution: Since the house and the ground are perpendicular to each other they make right angle at the base of the wall. Then the ladder, the ground and the wall form a right triangle and we can use the Pythagorean theorem to find the height.

$$h^2 + 8^2 = 20^2 \quad \Rightarrow \quad h^2 = 400 - 64 = 336 \quad \Rightarrow \quad \boxed{h \approx 18.3 \text{ ft.}}\ .$$

Basic Trigonometric Functions

Consider a right triangle where one of the angles is labeled θ. The longest side is called the **hypotenuse**, the side opposite the angle θ is called the **opposite side** and the side adjacent to the angle is called the **adjacent side**, see **Figure 1.10**. Using the lengths of these sides you can form 6 ratios which are the trigonometric functions of the angle θ. These ratios are irrespective of the size of the triangle. If the angles in two triangles are the same then the triangles are similar which means the ratios of the sides will be the same. *When calculating the trigonometric functions of an acute angle θ, you may use any right triangle which has θ as one of the angles.*

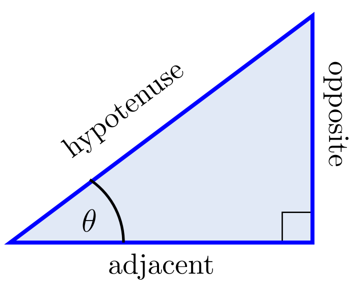


Figure 1.10: Standard right triangle

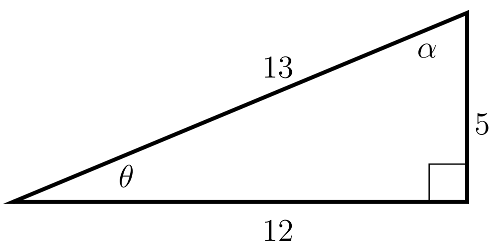
The Six Trigonometric Functions

Function	Abbreviation	Function	Abbreviation
Sine of θ:	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	**Cosecant of θ:**	$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$
Cosine of θ:	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	**Secant of θ:**	$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
Tangent of θ:	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$	**Cotangent of θ:**	$\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

We will usually use the abbreviated names of the functions.

Example 1.2.3

Given the following triangle find the six trigonometric functions of the angles θ and α.



Solution:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{5}{13}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{\frac{13}{5}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{12}{13}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\frac{13}{12}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\frac{5}{12}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\frac{12}{5}}$$

The same thing can be done for α but now the opposite and adjacent sides are switched:

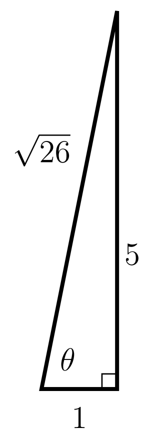
$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{12}{13}} \qquad \csc\alpha = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{\frac{13}{12}}$$

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{5}{13}} \qquad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\frac{13}{5}}$$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\frac{12}{5}} \qquad \cot\alpha = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\frac{5}{12}}$$

Example 1.2.4

Suppose θ is an angle such that $\tan\theta = 5$ and $0 \le \theta \le \frac{\pi}{2}$, solve for the other five trigonometric functions.



Solution: You know that $\tan\theta = 5 = \frac{5}{1}$ is the ratio $\frac{\text{opposite}}{\text{adjacent}}$ so if we draw a right triangle and label one of the angles θ then we know that the side opposite θ is 5 and the side adjacent to θ is 1. We can draw a triangle and solve for the hypotenuse ($\sqrt{26}$) using the Pythagorean theorem. Then we read the values of the trigonometric functions from the triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{5}{\sqrt{26}}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{\frac{\sqrt{26}}{5}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{1}{\sqrt{26}}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\frac{\sqrt{26}}{1}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\frac{5}{1}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\frac{1}{5}}$$

Two Special Triangles

For the angles 45°, 30° and 60° we have two special triangles which allow us to find the their trigonometric functions. To construct a right triangle with a 45° angle we will start with a square with sides of length 1 and cut it in half with a diagonal. Since the square is completely symmetric a diagonal will cut the angle in half creating two 45° angles. Consider the lower triangle in **Figure 1.11**. We found the length of the diagonal by the Pythagorean theorem. Then we read the values of the trigonometric functions from the triangle.

$$\sin 45^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{1}{\sqrt{2}}} \quad \cos 45^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{1}{\sqrt{2}}} \quad \tan 45^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = \boxed{1}$$

$$\csc 45^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{\frac{\sqrt{2}}{1}} \quad \sec 45^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\frac{\sqrt{2}}{1}} \quad \cot 45^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = \boxed{1}$$

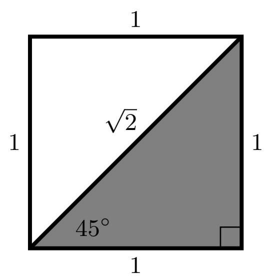


Figure 1.11

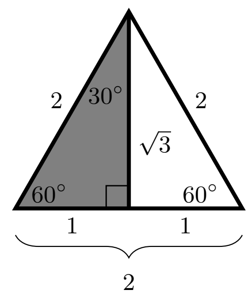


Figure 1.12

We can also construct a triangle for 30° and 60° angles. To do this we start with an equilateral triangle where each side has length 2. We then cut it in half vertically to create two right triangles with 30° and 60° angles as shown in **Figure 1.12**. To find the height of the triangle, $\sqrt{3}$, we once again used the Pythagorean theorem. With this triangle we can now find the values of the six trigonometric functions for both 30° and 60° angles.

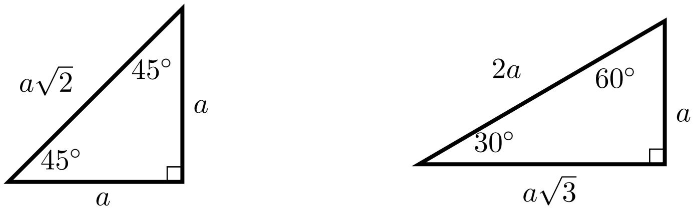
$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{1}{2}} \quad \cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{\sqrt{3}}{2}} \quad \tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\frac{1}{\sqrt{3}}}$$

$$\csc 30^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{2} \quad \sec 30^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\frac{3}{\sqrt{3}}} \quad \cot 30^\circ = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\frac{\sqrt{3}}{1}}$$

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\frac{\sqrt{3}}{2}} \quad \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{1}{2}} \quad \tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\frac{\sqrt{3}}{1}}$$

$$\csc 60^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{\frac{3}{\sqrt{3}}} \quad \sec 60^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{2} \quad \cot 60^\circ = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\frac{1}{\sqrt{3}}}$$

Note that we could have done this with a square or equilateral triangle with side length a and still have come up with the same ratios. **Figure 1.13** shows the two triangles and our trigonometric ratios are summarized in the table. The angles are presented in both degrees and radians. Here we will simplify and rationalize denominators where possible. If our ratio is $\frac{a}{a\sqrt{2}}$ we will move the $\sqrt{2}$ to the numerator by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$ to get $\frac{a\cdot\sqrt{2}}{a\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}$



(a) 45-45-90 **(b)** 30-60-90

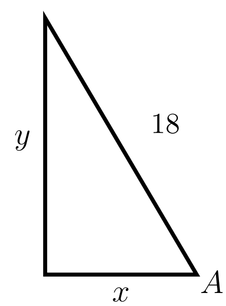
Figure 1.13: Two general special right triangles (any $a > 0$)

Trigonometric Ratios for the Special Triangles

$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\tan 45° = \tan\frac{\pi}{4} = 1$
$\csc 45° = \csc\frac{\pi}{4} = \sqrt{2}$	$\sec 45° = \sec\frac{\pi}{4} = \sqrt{2}$	$\cot 45° = \cot\frac{\pi}{4} = 1$
$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$	$\cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\csc 30° = \csc\frac{\pi}{6} = 2$	$\sec 30° = \sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$	$\cot 30° = \cot\frac{\pi}{6} = \sqrt{3}$
$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$	$\tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$
$\csc 60° = \csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$	$\sec 60° = \sec\frac{\pi}{3} = 2$	$\cot 60° = \cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$

Example 1.2.5

Use the triangle below to find the lengths of the other two sides, x and y. Angle A is 60°



Solution: Since we know the angle is 60° we can use the sine and cosine to find the lengths of the missing sides. From our 30-60-90 triangle we can see that $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$ set up equations to solve for x and y.

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{18} = \frac{1}{2}$$

$$x = 18\left(\frac{1}{2}\right) = \boxed{9}$$

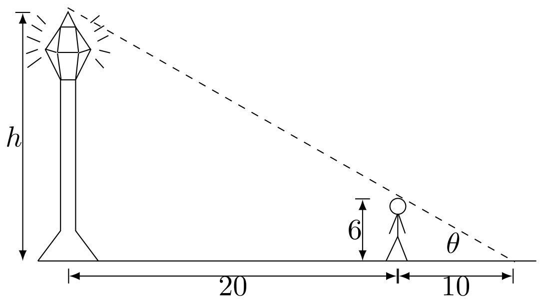
$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{18} = \frac{\sqrt{3}}{2}$$

$$y = 18\left(\frac{\sqrt{3}}{2}\right) = \boxed{9\sqrt{3}}$$

Example 1.2.6

Benjamin is 6 feet tall and casts a 10 foot shadow when he is standing 20 feet from the base of a street light. What is the height of the street light?

Solution: First we start with a labeled picture. We will call the angle of elevation from the end of the shadow to the top of the light θ. Then we will draw two right triangles from our picture.



We can find the value of $\tan\theta$ from both triangles. From the large one $\tan\theta = \frac{h}{30}$ and from the small one $\tan\theta = \frac{6}{10}$. Then set them equal and solve for h

$$\tan\theta = \frac{h}{30} = \frac{6}{10} \implies \boxed{h = 18}$$

Identities

Example 1.2.7

Show that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Solution:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{hypotenuse}} \cdot \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

We can similarly show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$

These properties in **Example 1.2.7** are true no matter what angle we use. When you have an equation that is always true it is known as an **identity**. We will see through the course of this book that there are many identities that can be formed using the 6 trigonometric functions.

Basic Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Notice that the trigonometric functions come in reciprocal pairs. The cosecant is the reciprocal of the sine, the secant is the reciprocal of the cosine and the cotangent is the reciprocal of the tangent. These reciprocal relations are presented below.

Reciprocal Trigonometric Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

There is a set of important identities known as the **Pythagorean identities** . They come from using the Pythagorean theorem on the trigonometric functions. We will state them here and then prove them.

Pythagorean Identities

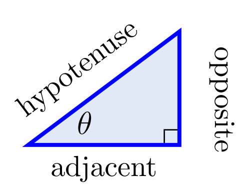
$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

We should say something about the notation here. When we write $\sin^2\theta$ what we mean is $(\sin\theta)^2$.

Example 1.2.8

Show that $\sin^2\theta + \cos^2\theta = 1$

Solution: Consider our standard right triangle:



The Pythagorean theorem states that

$$\text{opposite}^2 + \text{adjacent}^2 = \text{hypotenuse}^2$$

Lets look at $\sin^2\theta + \cos^2\theta$ and replace the trigonometric functions with the appropriate ratios.

$$\begin{aligned}\sin^2\theta+\cos^2\theta &= \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)^2+\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2\\ &= \frac{(\text{opposite})^2}{(\text{hypotenuse})^2}+\frac{(\text{adjacent})^2}{(\text{hypotenuse})^2}\\ &= \frac{(\text{opposite})^2+(\text{adjacent})^2}{(\text{hypotenuse})^2}\end{aligned}$$

Now we can use the Pythagorean theorem to replace $(\text{opposite})^2 + (\text{adjacent}^2)$ with $(\text{hypotenuse})^2$ and we see that

$$\sin^2\theta+\cos^2\theta \;=\; \frac{(\text{hypotenuse})^2}{(\text{hypotenuse})^2}=1$$

Example 1.2.9

Show that $\tan^2\theta+1=\sec^2\theta$

Solution: We will start with $\sin^2\theta+\cos^2\theta=1$ and divide by $\cos^2\theta$ on both sides.

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta} \quad\Longrightarrow\quad \frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta} \quad\Longrightarrow\quad \boxed{\tan^2\theta+1=\sec^2\theta}$$

We can similarly show that $1+\cot^2\theta=\csc^2\theta$.

Note: The relations and identities presented in this section appear frequently in our study of trigonometry and it will be useful to memorize them.

1.2 Exercises

1. Fill in the missing word(s) for the fractions.

(a) $\sin\theta=\dfrac{\rule{2cm}{0.4pt}}{\text{hypotenuse}}$

(b) $\csc\theta=\dfrac{\rule{2cm}{0.4pt}}{\text{opposite}}$

(c) $\cos\theta=\dfrac{\text{adjacent}}{\rule{2cm}{0.4pt}}$

(d) $\sec\theta=\dfrac{\rule{2cm}{0.4pt}}{\text{adjacent}}$

(e) $\tan\theta = \dfrac{\rule{2cm}{0.4pt}}{\text{adjacent}}$ (f) $\cot\theta = \rule{2cm}{0.4pt}$

For Exercises 2 - 9, find the values of all six trigonometric functions of angles A and B in the right triangle $\triangle ABC$ in **Figure 1.14**

2. $a = 5,\ b = 6$

3. $a = 5,\ c = 6$

4. $a = 6,\ b = 10$

5. $a = 6,\ c = 10$

6. $a = 7,\ b = 24$

7. $a = 1,\ c = 2$

8. $a = 5,\ b = 12$

9. $b = 24,\ c = 36$

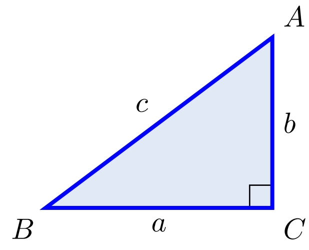


Figure 1.14

For Exercises 10 - 17, find the values of the other five trigonometric functions of the acute angle $0 \leq \theta \leq \frac{\pi}{2}$ given the indicated value of one of the functions.

10. $\sin\theta = \frac{3}{4}$ **11.** $\cos\theta = \frac{3}{4}$ **12.** $\tan\theta = \frac{3}{4}$ **13.** $\cos\theta = \frac{1}{3}$

14. $\tan\theta = \frac{12}{5}$ **15.** $\cos\theta = \frac{\sqrt{5}}{5}$ **16.** $\sin\theta = \frac{\sqrt{2}}{3}$ **17.** $\cos\theta = \frac{3}{\sqrt{17}}$

18. Suppose that for acute angle θ you know that $\sin\theta = x$. Find a simplified algebraic expression for both $\cos\theta$ and $\tan\theta$. (Hint: draw a triangle where the ratio of the opposite to the hypotenuse is $\frac{x}{1}$.)

For Exercises 19 - 24, use the special triangles to fill in the following table.
$(0 \leq \theta \leq 90°,\ 0 \leq \theta \leq \pi/2)$

Function	θ (deg)	θ (rad)	Function Value
19. $\sin\theta$	45°		
20. $\sec\theta$	60°		
21. $\tan\theta$		$\dfrac{\pi}{6}$	
22. $\csc\theta$		$\dfrac{\pi}{4}$	
23. $\cot\theta$			1
24. $\cos\theta$			$\dfrac{\sqrt{2}}{2}$

25. Using the special triangles, determine the exact value of side a and side b in **Figure 1.15**. Express your answer in simplified radical form.

26. Using the special triangles, determine the exact value of segment $\overline{DE}$ in **Figure 1.16**. Segments $\overline{BA}$ and $\overline{BC}$ have length 4. Express your answer in simplified radical form.

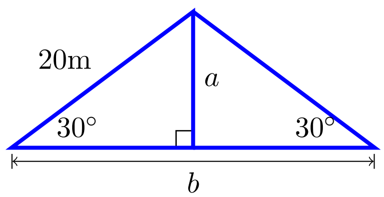


Figure 1.15: Problem 25

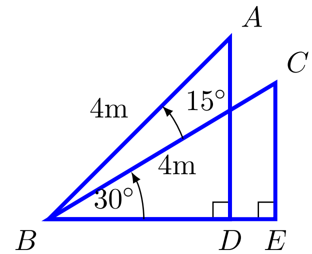


Figure 1.16: Problem 26

27. A metal plate has the form of a quarter circle with a radius of 100 cm. Two 3 cm holes are to be drilled in the plate 95 cm from the corner at 30° and 60° as shown in **Figure 1.17**. To use a computer controlled milling machine you must know the Cartesian coordinates of the holes. Assuming the origin is at the corner what are the coordinates of the holes (x_1, y_1) and (x_2, y_2)? (Round to 3 decimal places.)

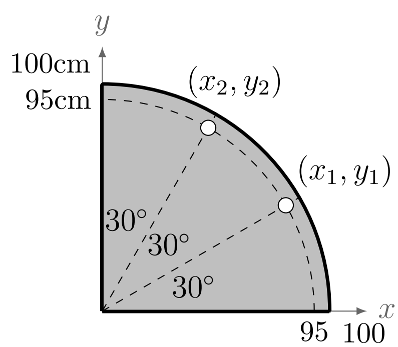


Figure 1.17: Problem 27

1.3 Trigonometric Functions of Any Angle

So far we have only looked at trigonometric functions of acute (less than 90°) angles. We would like to be able to find the trigonometric functions of any angle.

To do this follow these steps:

1. Draw the angle in standard position on the coordinate axes
2. Draw a reference triangle and find the reference angle
3. Label the reference triangle
4. Write down the answer

OR use your calculator.

Note: Your calculator will only give you decimal approximations but, where possible, the answers will be exact. For example if you ask your calculator for $\cos(30°)$ it might return an answer of 0.86602540378 whereas in this text we will present the answer as $\frac{\sqrt{3}}{2}$

Before we can talk about reference triangles and reference angles we need to review the coordinate plane. We can define the trigonometric functions of any angle in terms of **Cartesian coordinates**. You will recall that the xy - coordinate plane (Cartesian coordinates) consists of points represented as coordinate pairs (x, y) of real numbers. The plane is divided into 4 quadrants called quadrants 1 through 4 (see **Figure 1.18**). These are often abbreviated **QI, QII, QIII and QIV** or 1$^{\text{st}}$ 2$^{\text{nd}}$ 3$^{\text{rd}}$ 4$^{\text{th}}$.

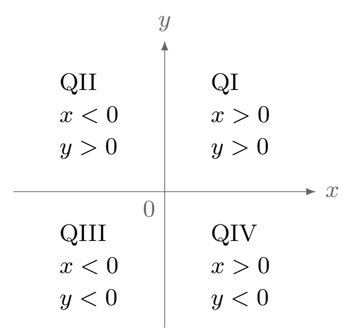


Figure 1.18: Cartesian plane divided into 4 quadrants

Reference Angles

Definition 1.2. If you draw the angle θ in the standard position (see **Definition 1.1**) its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis. The reference angle is always positive and always between 0 and 90°
(or between 0 and $\dfrac{\pi}{2}$).

Definition 1.3. The **reference triangle** is the triangle which is formed by drawing a perpendicular line from *any* point (x, y) on the terminal side of θ in standard position to the horizontal axis (x-axis).

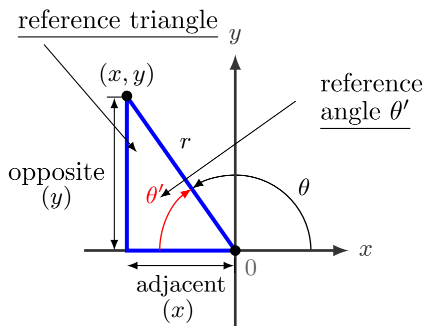


Figure 1.19: Quadrant II reference triangle

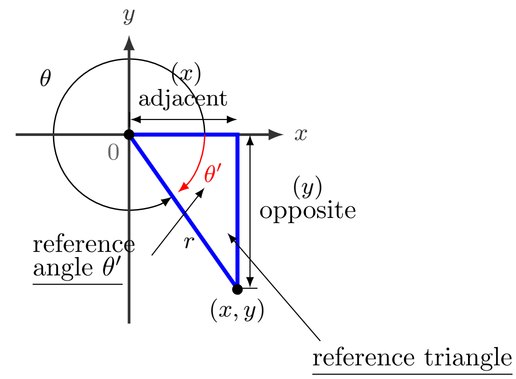


Figure 1.20: Quadrant IV reference triangle

Figure 1.19 is a reference angle and triangle in the 2^{nd} quadrant. **Figure** 1.20 is a reference angle and triangle in the 4^{th} quadrant:

The size of the reference angle in the second quadrant (QII) will be $180-\theta$ or $\pi-\theta$ depending on whether the angle is given in degrees or radians respectively.

The size of the reference angle in the fourth quadrant (QIV) will be $360-\theta$ or $2\pi-\theta$ depending on whether the angle is given in degrees or radians respectively.

What formula will give you the size of a reference angle in the third quadrant?

The six trigonometric functions can be defined in the same way as before but now the lengths are read off the reference triangle. Since the coordinates (x, y) can be negative, when we take the ratios of the sides of the triangle we often find negative results. The distance from the origin to the point (x, y) is the hypotenuse and is always a positive value $(r > 0)$. The trigonometric functions of θ are as follows.

The Six Trigonometric Functions for Any Angle θ

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

Example 1.3.1

Sketch the following angles in standard position. Draw the reference triangles and find the size of the reference angles:

(a) $\theta = 309°$

Solution: The reference angle will be $\theta' = 360 - 309 = \boxed{51°}$ **Figure** 1.21 **(a)**

(b) $\theta = -\dfrac{7\pi}{4}$

Solution: The reference angle will be $\theta' = 2\pi - \frac{7\pi}{4} = \boxed{\frac{\pi}{4}}$ **Figure 1.22 (b)**

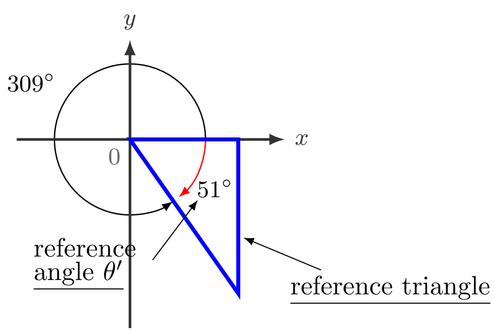


Figure 1.21: Example 1.3.1 (a)

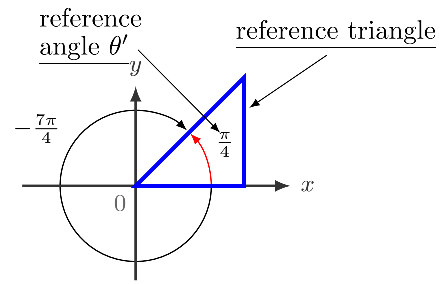


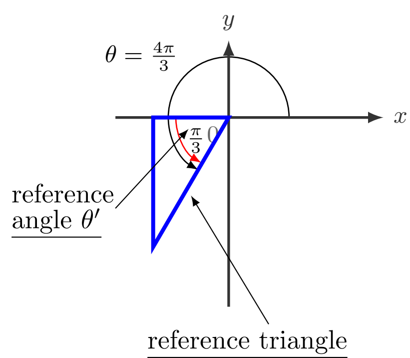
Figure 1.22: Example 1.3.1 (b)

(c) $\theta = \dfrac{10\pi}{3}$

Solution: This angle is larger than one full revolution so we need to find a coterminal angle that is between 0 and 2π (one time around the circle) to find it in standard position. To do this we subtract multiples of 2π until our angle is less than 2π.

$$\frac{10\pi}{3} - 2\pi = \frac{10\pi}{3} - \frac{6\pi}{3} = \frac{4\pi}{3}$$

Since $\dfrac{10\pi}{3}$ is coterminal with $\dfrac{4\pi}{3}$, to find the reference angle start with the coterminal angle $\dfrac{4\pi}{3}$ and subtract π to get
$\theta' = \dfrac{4\pi}{3} - \pi = \boxed{\dfrac{\pi}{3}}$.



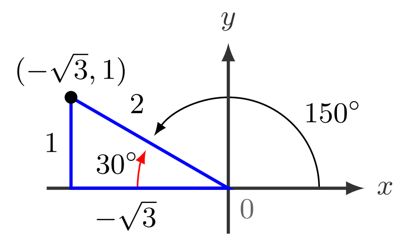
Now we will use these reference angles to find the values of some trigonometric functions. We can follow the steps outlined at the beginning of the section:

1. Draw the angle in standard position on the coordinate axes
2. Draw a reference triangle and find the reference angle
3. Label the reference triangle
4. Write down the answer

Example 1.3.2

Find the values of the six trigonometric functions for $\theta = 150°$.

Solution:



1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle. Here we will label using the standard $30 - 60 - 90$ triangle.

Note: The point we selected on the terminal side of our angle is $(-\sqrt{3}, 1)$. Since the adjacent side of the reference triangle is on the negative x-axis that side is labeled as $-\sqrt{3}$. This is VERY IMPORTANT. You will notice that this makes the cosine, secant, tangent and cotangent negative.

4. Find the 6 trigonometric functions by reading them off the reference triangle:

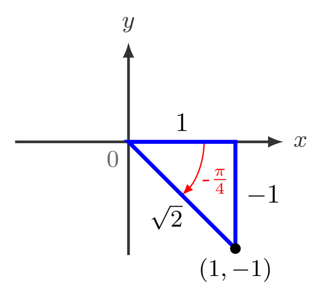
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-\sqrt{3}}{2} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = -\frac{1}{\sqrt{3}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{1} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = -\frac{2}{\sqrt{3}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = -\frac{\sqrt{3}}{1}$$

Example 1.3.3

Find the values of the six trigonometric functions for $\theta = -\frac{\pi}{4}$. Note that the angle is negative.

Solution:



1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle. Here we will label using the standard $45 - 45 - 90$ triangle.

NOTE: The point we selected on the terminal side of our angle is $(1, -1)$. Since the opposite side of the reference triangle is in the negative y direction that side is labeled as -1. This is VERY IMPORTANT. You will notice that this makes the sine, cosecant, tangent and cotangent negative.

4. Find the 6 trigonometric functions by reading them off the reference triangle:

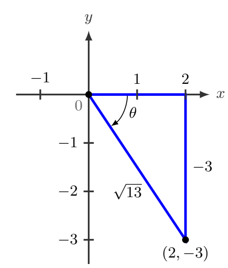
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = -\frac{1}{\sqrt{2}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = -1$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = -\frac{\sqrt{2}}{1} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{2}}{1} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = -1$$

Example 1.3.4

Suppose the terminal side of negative angle θ passes through the point $(2, -3)$. Sketch the angle in standard position, draw a reference triangle and then find the exact values for the sine, cosine and tangent of θ.

Solution:



1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle.

NOTE: The point we selected on the terminal side of our angle is $(2, -3)$. Since the opposite side of the reference triangle is in the negative y direction that side is labeled as -3. This is VERY IMPORTANT. You will notice that this makes the sine and tangent.

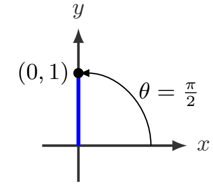
4. Now we can find the 3 trigonometric functions by reading them off the reference triangle:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = -\frac{3\sqrt{13}}{13} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{13}}{13} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = -\frac{3}{2}$$

Example 1.3.5

Find the values of the six trigonometric functions for $\theta = \frac{\pi}{2}$.

Solution:



1. Draw the angle in standard position
2. Draw the reference triangle and angle
3. Label the triangle. The triangle is just a vertical line.

NOTE: We can select any point on the terminal side so the easiest point is probably $(x, y) = (0, 1)$. Here $r = 1$ because the length of the adjacent side is zero and the opposite side is the same length as the hypotenuse. You could also use the Pythagorean theorem $x^2 + y^2 = r^2$.

4. Find the 6 trigonometric functions by using the x, y, r version of the definitions:

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = 1 \qquad \cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \tan\theta = \frac{y}{x} = \frac{1}{0} = undefined$$

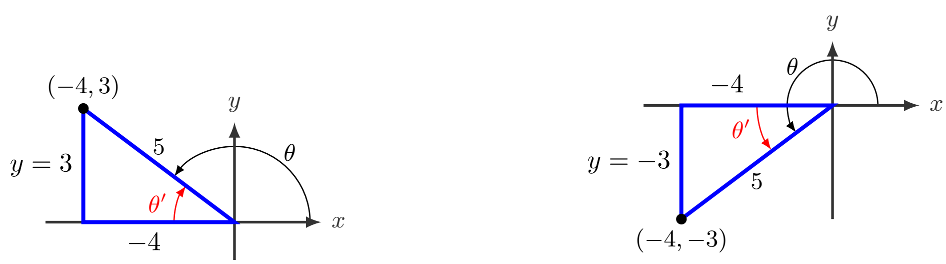
$$\csc\theta = \frac{r}{y} = \frac{1}{1} = 1 \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} = undefined \qquad \cot\theta = \frac{x}{y} = \frac{0}{1} = 0$$

It is important to notice that the tangent and the secant are undefined because division by zero is not permitted. You can never divide by zero. This division by zero will show up at each of the angles that terminate at one of the axes: 0°, 90°, 180°, 270°, 360° or in radians: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π.

Example 1.3.6

Suppose $\cos\theta = -\frac{4}{5}$. Find the exact values of $\sin\theta$ and $\tan\theta$.

Solution: The first thing we need to do is to draw a reference triangle. Since the cosine is negative there are two choices for our terminal side of θ. One in the second quadrant and one in the third quadrant. See **Figure 1.23**. We will need two reference triangles to find the values of the missing trigonometric functions because the signs (+/-) will depend on the quadrant. $\cos\theta = -\frac{4}{5} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ so two of the three sides of the triangles are known. Use the Pythagorean theorem to find the last side $(-4)^2 + y^2 = 5^2$ so $y = 3$ for the triangle in QII or $y = -3$ for the triangle in QIII.



(a) Solution in QII **(b)** Solution in QIII

Figure 1.23: $\cos\theta = -\frac{4}{5}$

Since there are two different triangles there are two different solutions to the problem. For the triangle in QII $\boxed{\sin\theta = \frac{3}{5} \text{ and } \tan\theta = -\frac{3}{4}}$. For the triangle in QIII $\boxed{\sin\theta = -\frac{3}{5} \text{ and } \tan\theta = \frac{3}{4}}$.

What this has shown us is that we can determine the sign of the trigonometric functions by the quadrant of the terminal side. When constructing the reference triangle, the hypotenuse is always positive but the two legs can be either positive or negative depending on where the triangle is drawn. In the first quadrant both legs are positive, in the second quadrant the adjacent side (x) is negative (**Figure 1.23(a)**), in the third quadrant both legs (x and y) are negative (**Figure 1.23(b)**) and in QIV the opposite side (y) is negative. Since the trigonometric functions are ratios of the sides of the reference triangle then **A**ll the functions are positive in the first quadrant, the **S**ine is positive in the second, the **T**angent is positive in the third and the **C**osine is positive in the fourth. This information is summarized in **Figure 1.24**. The mnemonic **A**ll **S**tudents **T**ake **C**alculus tells you which function is positive in which quadrant.

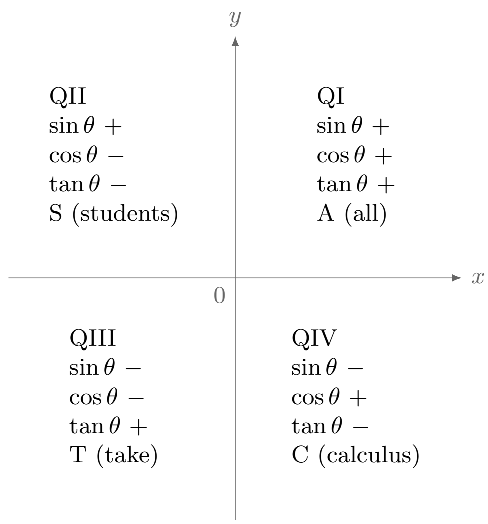


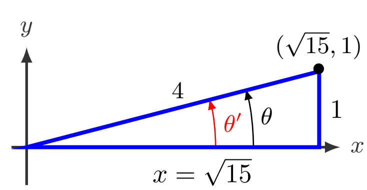
Figure 1.24: The signs of the trigonometric functions

Since $\csc\theta = \dfrac{1}{\sin\theta}$ then the cosecant has the same sign as the sine function. Similarly $\sec\theta$ has the same sign as $\cos\theta$ and $\cot\theta$ has the same sign as $\tan\theta$.

Example 1.3.7

Suppose $\csc\theta = 4$ and $\cot\theta > 0$. Find the values of the six trigonometric functions for θ.

Solution:



Since the $\csc\theta = 4$ the sine is positive so θ is in quadrant I or II. Since the $\cot\theta > 0$ the tangent is positive so θ is in quadrants I or III .

The overlap of these two regions is quadrant I so we can draw our triangle knowing that $\csc\theta = \frac{4}{1} = \frac{\text{hypotenuse}}{\text{opposite}}$. To solve for x we use the Pythagorean theorem: $1^2 + x^2 = 4^2$ so $x = \sqrt{15}$. Since we are in the first quadrant all sides of the triangle will be positive.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \boxed{\tfrac{1}{4}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\tfrac{\sqrt{15}}{4}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \boxed{\tfrac{1}{\sqrt{15}}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \boxed{4} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \boxed{\tfrac{4}{\sqrt{15}}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \boxed{\sqrt{15}}$$

1.3 Exercises

1. In which quadrant(s) do sine and cosine have the same sign?

2. In which quadrant(s) do sine and cosine have the opposite sign?

3. In which quadrant(s) do sine and tangent have the same sign?

4. In which quadrant(s) do sine and tangent have the opposite sign?

5. In which quadrant(s) do cosine and tangent have the same sign?

6. In which quadrant(s) do cosine and tangent have the opposite sign?

For Exercises 7 - 11, find the reference angle for the given angle.

7. $127°$ **8.** $250°$ **9.** $-250°$ **10.** $882°$ **11.** $-323°$

12. Let $(-3, 4)$ be a point on the terminal side of θ. Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ without a calculator.

13. Let $(-12, -5)$ be a point on the terminal side of θ. Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ without a calculator.

14. Let $(8, -15)$ be a point on the terminal side of θ. Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ without a calculator.

For Exercises 15 - 24,

a) Find the reference angle for the given angle.

b) Draw the reference triangle and label the sides

c) Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ without a calculator.

15. $30°$ **16.** $135°$ **17.** $-150°$ **18.** $-45°$ **19.** $945°$

20. $\frac{\pi}{4}$ **21.** $-\frac{2\pi}{3}$ **22.** $\frac{7\pi}{6}$ **23.** $-\frac{29\pi}{3}$ **24.** $\frac{29\pi}{4}$

For Exercises 25 - 29, find the values of $\sin\theta$ and $\tan\theta$ given the following $\cos\theta$ values.

25. $\cos\theta = \frac{3}{4}$ **26.** $\cos\theta = -\frac{3}{4}$ **27.** $\cos\theta = \frac{1}{4}$ **28.** $\cos\theta = 0$ **29.** $\cos\theta = 1$

For Exercises 30 - 34, find the values of $\cos\theta$ and $\tan\theta$ given the following $\sin\theta$ values.

30. $\sin\theta = \frac{3}{4}$ **31.** $\sin\theta = -\frac{3}{4}$ **32.** $\sin\theta = \frac{1}{4}$ **33.** $\sin\theta = 0$ **34.** $\sin\theta = 1$

For Exercises 35 - 39, find the values of $\sin\theta$ and $\cos\theta$ given the following $\tan\theta$ values.

35. $\tan\theta = \frac{3}{4}$ **36.** $\tan\theta = -\frac{3}{4}$ **37.** $\tan\theta = \frac{1}{4}$ **38.** $\tan\theta = 0$ **39.** $\tan\theta = 1$

For Exercises 40 - 44, find the values of the six trigonometric fucntions of θ with the given restriction.

Function Value	Restriction
40. $\sin\theta = \dfrac{15}{17}$	$\tan\theta < 0$
41. $\sec\theta = -\dfrac{15}{12}$	$\sin\theta < 0$
42. $\tan\theta = \dfrac{20}{21}$	$\csc\theta > 0$
43. $\cos\theta = -\dfrac{20}{21}$	$\csc\theta > 0$
44. $\sec\theta$ is undefined	$\pi \leq \theta \leq \frac{3\pi}{2}$

For Exercises 45 - 54, use a calculator to evaluate the following trigonometric functions. Round your answer to 4 decimal places.

45. $\sin 127°$ **46.** $\cos 250°$ **47.** $\csc(-250°)$ **48.** $\cot 882°$ **49.** $\sec(-323°)$

50. $\tan\left(\frac{\pi}{5}\right)$ **51.** $\cot\left(-\frac{\pi}{5}\right)$ **52.** $\csc\left(\frac{\pi}{5}\right)$ **53.** $\cot\pi$ **54.** $\sec\left(-\frac{14}{5}\right)$

55. In engineering the motion of the spring - mass - damper system shown in **Figure 1.25** can be modeled by the the equation

$$x = \sqrt{221}e^{-0.2t}\cos(14t - 0.343)$$

where x is the position of the mass relative to equilibirum (no motion), t is the time measured in seconds after the system is set into motion and the angles are in radians. Find the positions x of the mass when the time is

$t = 1$ sec, $t = 5$ sec, $t = 10$ sec, and $t = 20$ sec.

What does a negative position mean?

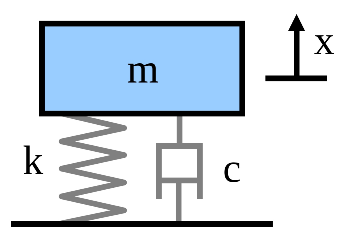


Figure 1.25

1.4 The Unit Circle

Definition 1.4. The Unit Circle is a circle with radius 1. $x^2 + y^2 = 1$

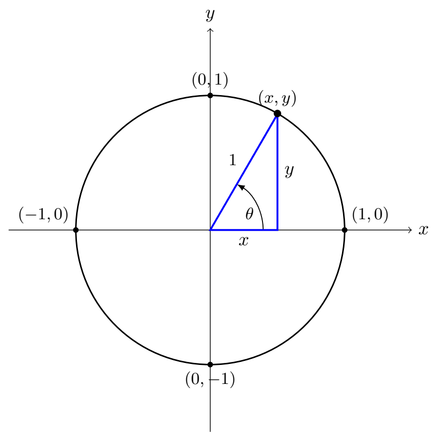


Figure 1.26: A circle of radius 1 with a reference triangle drawn in the first quadrant.

Every point (x, y) on the unit circle corresponds to some angle θ. For example:

Point (x, y)	Angle θ
$(1, 0)$	$0°$ or 0
$(0, 1)$	$90°$ or $\frac{\pi}{2}$
$(-1, 0)$	$180°$ or π
$(0, -1)$	$270°$ or $\frac{3\pi}{2}$

We can define trigonometric functions based on the coordinates of the point on the unit circle which corresponds to the angle. Notice that since the circle has radius 1 the reference triangle in **Figure 1.26** above has hypotenuse 1, height length y and base length x. We can now use the techniques from Section 1.3 to define the six trigonometric functions:

The Six Trigonometric Functions on the Unit Circle

$$\sin\theta = y \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

$$\cos\theta = x \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Then every point on the unit circle is $(x, y) = (\cos\theta, \sin\theta)$ for some angle θ.

We can use the two special triangles we looked at in Section 1.2 to fill in the unit circle for many "standard" angles. In the following diagram, each point on the unit circle is labeled with its coordinates $(x, y) = (\cos\theta, \sin\theta)$ (exact values) and, with the angle in degrees and radians.

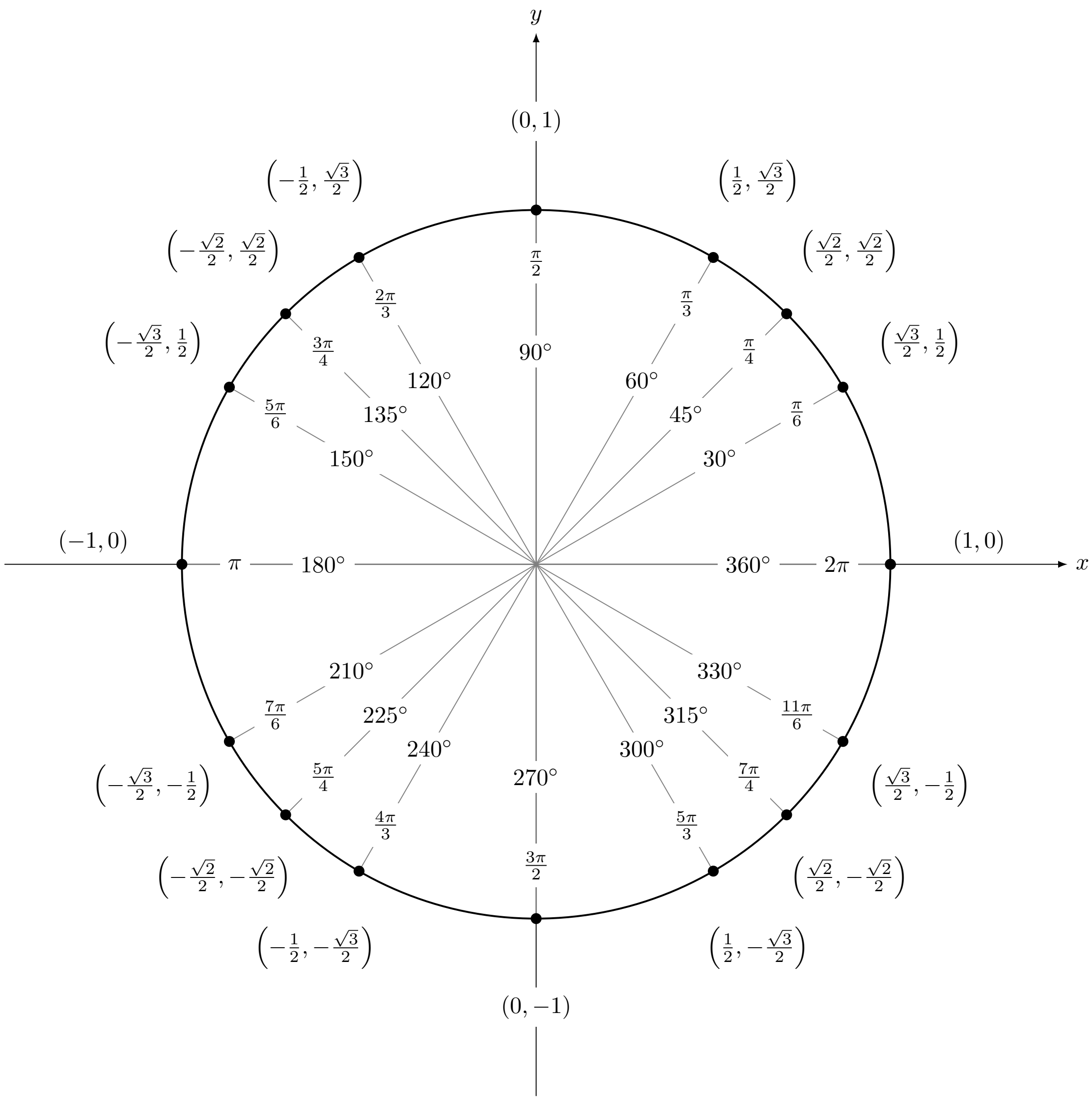


Figure 1.27: The Unit Circle has radius 1. The coordinates on the circle give you the values of the cosine and the sine of the angle θ. $(x, y) = (\cos\theta, \sin\theta)$

For any trigonometry problem involving one of the nice angles (multiples of 30°, 45°, or 60°) you can either use the unit circle or the triangle techniques in Section 1.3.

Example 1.4.1

Find the six trigonometric functions for the following angles:

1. $\theta = -\dfrac{2\pi}{3}$

Solution: $\theta = -\dfrac{2\pi}{3}$ is coterminal with the angle $\frac{4\pi}{3}$ which corresponds to the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = (\cos\theta, \sin\theta)$ on the unit circle. Now the other trigonometric functions can be found from the identities.

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = -\frac{1}{2} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{2}{\sqrt{3}} \qquad \sec\theta = \frac{1}{\cos\theta} = -2 \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$$

2. $\theta = \dfrac{3\pi}{4}$

   Solution: $\theta = \frac{3\pi}{4}$ corresponds to the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = (\cos\theta, \sin\theta)$ on the unit circle.

$$\sin\theta = \frac{\sqrt{2}}{2} \qquad \cos\theta = -\frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = -1$$

$$\csc\theta = \frac{1}{\sin\theta} = \sqrt{2} \qquad \sec\theta = \frac{1}{\cos\theta} = -\sqrt{2} \qquad \cot\theta = \frac{1}{\tan\theta} = -1$$

3. $\theta = 180°$

   Solution: $\theta = 180°$ corresponds to the point $(-1, 0) = (\cos\theta, \sin\theta)$ on the unit circle.

   Note: We can not divide by zero so cosecant and cotangent are both undefined.

$$\sin\theta = 0 \qquad \csc\theta = \frac{1}{\sin\theta} = \text{undefined}$$

$$\cos\theta = -1 \qquad \sec\theta = \frac{1}{\cos\theta} = -1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = 0 \qquad \cot\theta = \frac{1}{\tan\theta} = \text{undefined}$$

4. $\theta = \dfrac{3\pi}{2}$

   Solution: $\theta = \frac{3\pi}{2}$ corresponds to the point $(0, -1) = (\cos\theta, \sin\theta)$ on the unit circle. Since the tangent is undefined it would be difficult to find the reciprocal so instead use the identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\sin\theta = -1 \qquad \csc\theta = \frac{1}{\sin\theta} = -1$$

$$\cos\theta = 0 \qquad \sec\theta = \frac{1}{\cos\theta} = \text{undefined}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \text{undefined} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = 0$$

Domain and Period of sine, cosine and tangent

Recall that the **domain** of a function $f(x)$ is the set of all numbers x for which the function is defined. For example, the domain of $f(x) = \sin\ x$ and $f(x) = \cos x$ is the set of all real numbers, whereas the domain of $f(x) = \tan\ x$ is the set of all real numbers except $x = \pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, ... The **range** of a function $f(x)$ is the set of all values that $f(x)$ can take over its domain. For example, the range of $f(x) = \sin\ x$ and $f(x) = \cos x$ is the set of all real numbers between -1 and 1 (i.e. the interval $[-1, 1]$), whereas the range of $f(x) = \tan\ x$ is the set of all real numbers. (Why?)

Recall that by adding or subtracting 360° or 2π to any angle you get back to the same angle on the graph (coterminal). So the following relationships are true:

$$\sin(x) = \sin(x + 2\pi) \qquad \text{and} \qquad \cos(x) = \cos(x + 2\pi) \tag{1.2}$$

In fact any integer multiple of 2π can be added to the angle to arrive at a coterminal angle. Multiples of 2π are represented as

$$2n\pi, \qquad \text{where } n \in \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\,.$$

The integers are represented by $\mathbb{Z}$: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We can abbreviate the above multiples of 2π as:

$$2n\pi,\ \text{where } n \in \mathbb{Z}.$$

The relationships in equation (1.2) are said to be **periodic** with **period** 2π.

Definition 1.5. Functions that repeat values at a regular interval are called **periodic**.

Formally: A function $f(x)$ is **periodic** if there exists a number $C > 0$ such that

$$f(x) = f(x + C).$$

There can be many numbers C that satisfy the above requirement.

$f(x) = \sin x$ and $f(x) = \cos x$ are periodic with period 2π and $f(x) = \tan x$ is periodic with period π.

Recall from algebra that even and odd functions have special properties when the sign of the variable is changed. An **even function** satisfies the property $f(x) = f(-x)$ so it returns the same result with both positive and negative x values. An **odd function** is one that has the property $-f(x) = f(-x)$ so the function returns the negative result for $-x$. The cosine and sine satisfy the same properties where:

Negative Angle Identities

cosine is even	$\cos(\theta)$	$=$	$\cos(-\theta)$
sine is odd	$-\sin(\theta)$	$=$	$\sin(-\theta)$
tangent is odd	$-\tan(\theta)$	$=$	$\tan(-\theta)$

You can see this by examining the corresponding values on the unit circle.

We can also construct what are known as **cofunction identities** which relate two different functions.

Cofunction Identities Radians

$$\sin\left(\theta+\tfrac{\pi}{2}\right)=\cos\theta \qquad \sin\left(\theta-\tfrac{\pi}{2}\right)=-\cos\theta$$

$$\cos\left(\theta+\tfrac{\pi}{2}\right)=-\sin\theta \qquad \cos\left(\theta-\tfrac{\pi}{2}\right)=\sin\theta$$

$$\tan\left(\theta+\tfrac{\pi}{2}\right)=-\cot\theta \qquad \tan\left(\theta-\tfrac{\pi}{2}\right)=-\cot\theta$$

Cofunction Identities Degrees

$$\sin\left(\theta+90^\circ\right)=\cos\theta \qquad \sin\left(\theta-90^\circ\right)=-\cos\theta$$

$$\cos\left(\theta+90^\circ\right)=-\sin\theta \qquad \cos\left(\theta-90^\circ\right)=\sin\theta$$

$$\tan\left(\theta+90^\circ\right)=-\cot\theta \qquad \tan\left(\theta-90^\circ\right)=-\cot\theta$$

Example 1.4.2

Suppose $\cos(t)=-\dfrac{3}{4}$. Find (a) $\cos(-t)$, (b) $\sec(-t)$, (c) $\csc(90^\circ-t)$, (d) $\sin\left(t+\frac{\pi}{2}\right)$

Solution:

(a) $\cos(-t)=\cos(t)=\boxed{-\dfrac{3}{4}}$

(b) $\sec(-t) = \dfrac{1}{\cos(-t)} = \boxed{-\dfrac{4}{3}}$

(c) $\csc(90° - t) = \dfrac{1}{\sin(90° - t)} = \dfrac{1}{\sin[-(t - 90°)]} = \dfrac{1}{-\sin(t - 90°)} = \dfrac{1}{\cos(t)} = \boxed{-\dfrac{4}{3}}$

(d) $\sin\left(t + \dfrac{\pi}{2}\right) = \cos(t) = \boxed{-\dfrac{3}{4}}$

Example 1.4.3

Find $\cos(5\pi)$

Solution: 5π is larger than 2π (one time around the circle) so we need to find a coterminal angle θ between 0 and 2π. To do this subtract 2π until $0 \leq \theta < 2\pi$.

$$\theta = 5\pi - 2\pi - 2\pi = \pi$$

so

$$\cos(5\pi) = \cos(\pi) = \boxed{-1}$$

Example 1.4.4

Find $\sin\left(-\dfrac{9\pi}{4}\right)$

Solution: $-\frac{9\pi}{4}$ is not between 0 and 2π (one time around the circle) so we need to find a coterminal angle between 0 and 2π. To do this add 2π to find an angle θ such that $0 \leq \theta < 2\pi$.

$$\theta = -\frac{9\pi}{4} + 2\pi + 2\pi = \frac{7\pi}{4}$$

So

$$\sin\left(-\frac{9\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

1.4 Exercises

Fill in the blanks for problems 1 - 8.

1. Every point on the unit circle is $(x, y) =$ __________________ for some angle θ.

2. The equation for the unit circle is __________________.

3. The unit circle is a circle of radius __________________.

4. Functions that repeat values at a regular interval are called __________________.

5. An even function satisfies the property __________________.

6. The range of $y = \cos x$ is __________________.

7. The range of $y = \tan x$ is __________________.

8. An odd function satisfies the property __________________.

For Exercises 9 - 18, find the corresponding point (x, y) on the unit circle and then find the the six trigonometric functions for the given angle.

9. $\alpha = 150°$ **10.** $\theta = 135°$ **11.** $\gamma = -135°$ **12.** $\beta = 720°$ **13.** $\alpha = -540°$

14. $\alpha = \dfrac{3\pi}{4}$ **15.** $\theta = \dfrac{5\pi}{3}$ **16.** $\gamma = -\dfrac{5\pi}{3}$ **17.** $\beta = 17\pi$ **18.** $\alpha = -\dfrac{11\pi}{2}$

19. Suppose $\sin(t) = -\dfrac{3}{4}$. Find

a) $\sin(-t)$

b) $\csc(-t)$

c) $\sec(90° - t)$

d) $\cos\left(t + \frac{\pi}{2}\right)$

20. Suppose $\tan(t) = -\dfrac{3}{4}$. Find

a) $\tan(-t)$

b) $\cot(-t)$

c) $\tan(t - 90°)$

d) $\tan\left(t + \frac{\pi}{2}\right)$

1.5 Applications and Models

In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is 90°. Later we will see how to do this when we do not have a right triangle. We also know that the angles of a triangle add up to 180°.

Example 1.5.1

Use the triangle in **Figure 1.28** to solve the triangles for the missing parts.

(a) $c = 10,\ A = 22°$

Solution: The unknown parts are a, b, and B. Solving yields:

$$\begin{aligned} a &= c\ \sin\ A &&= 10\ \sin\ 22° &&= \boxed{3.75} \\ b &= c\ \cos\ A &&= 10\ \cos\ 22° &&= \boxed{9.27} \\ B &= 90° - A &&= 90° - 22° &&= \boxed{68°} \end{aligned}$$

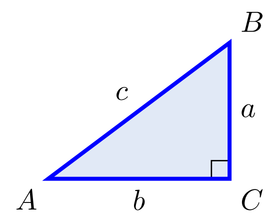


Figure 1.28

(b) $b = 8,\ A = 40°$

Solution: The unknown parts are a, c, and B. Solving yields:

$$\frac{a}{b} = \tan\ A \quad\Rightarrow\quad a = b\ \tan\ A = 8\ \tan\ 40° = \boxed{6.71 = a}$$

$$\frac{b}{c} = \cos\ A \quad\Rightarrow\quad c = \frac{b}{\cos\ A} = \frac{8}{\cos\ 40°} = \boxed{10.44 = c}$$

$$B = 90° - A = 90° - 40° = \boxed{50° = B}$$

(c) $a = 3,\ b = 4$

Solution: The unknown parts are c, A, and B. By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = \boxed{5}\ .$$

Now, $\tan\ A = \frac{a}{b} = \frac{3}{4} = 0.75$. So how do we find A? There should be a key labeled $\boxed{\tan^{-1}}$ on your calculator, which works like this: give it a number x and it will tell you the angle θ such that $\tan\ \theta = x$. In our case, we want the angle A such that $\tan\ A = 0.75$:

Press: $\boxed{\tan^{-1}}$ Enter: 0.75 Answer: 36.86989765

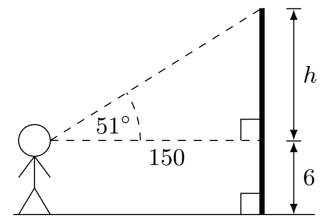
This tells us that $\boxed{A = 36.87°}$. Thus $B = 90° - A = 90° - 36.87° = \boxed{53.13°}$.

Note: The $\boxed{\sin^{-1}}$ and $\boxed{\cos^{-1}}$ keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*. The inverse trigonometric functions will be discussed in detail in Section 2.3.

Example 1.5.2

Sandra is standing 150 feet from the base of a platform from which people are bungee jumping. The *angle of elevation*[2] from her horizontal line of sight to the top of the platform from which they jump is 51°. Assume her eyes are a vertical distance of 6 feet from the ground. From what height are the people jumping?

Solution: The picture on the right describes the situation. We see that the height of the platform is $h + 6$ ft, where



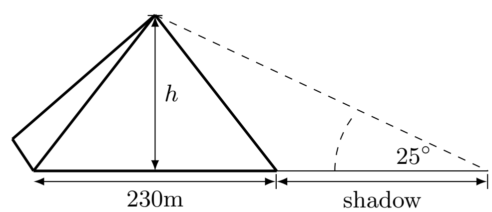
$$\frac{h}{150} = \tan\ 51^\circ \quad \Rightarrow \quad h = 150\ \tan\ 51^\circ = 185\text{ft} .$$

We can calculate tan 51° by using a calculator. Be careful that your calculator is in degree mode. Since none of the numbers we were given had decimal places, we rounded off the answer for h to the nearest integer. Thus, the height of the platform is $h + 6 = 185 + 6 = \boxed{191\text{ ft}}$.

Example 1.5.3

While visiting Cairo an ancient Greek mathematician wanted to measure the height of the Great Pyramid of Giza. He was able to measure the length of one side of the pyramid to be 230 meters. At that time the sun was about 25° above the horizon and the shadow cast by the pyramid extended 200 meters from its base. Using trigonometry what height did the mathematician calculate for the pyramid?

Solution: The picture on the right describes the situation. We need to measure the distance from the middle of one edge of the pyramid to the end of the shadow. Thus the length of the adjacent side of the triangle is $115 + 200$ and we can use the tangent function to write an equation relating the height and the adjacent side:



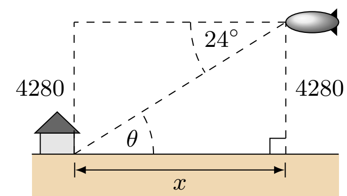
$$\frac{h}{315} = \tan\ 25^\circ \quad \Rightarrow \quad h = 315\ \tan\ 25^\circ = 146.9\text{m} .$$

We can calculate tan 25° by using a calculator. Again, be careful that your calculator is in degree mode. Since none of the numbers we were given had decimal places, we round off the answer for h to the nearest integer. Thus, the height of the pyramid is about $h = \boxed{147\text{ m}}$.

Example 1.5.4

A blimp 4280 ft above the ground measures an *angle of depression* of 24° from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?

[2]The **angle of elevation** is the angle made from the horizontal looking up to some object. Similarly the **angle of depression** is the angle from the horizontal looking down to some object.



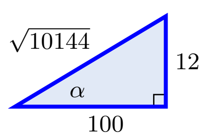
Solution: Let x be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we can construct the rectangle shown. Using 4280 ft as the opposite side and x as the adjacent we can use the tangent to calculate the desired distance. (Note: Alternatively, we know from elementary geometry that the angle of elevation θ from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house and this gives us the lower triangle i.e. $\theta = 24°$.) Hence,

$$\frac{4280}{x} = \tan\ 24° \quad \Rightarrow \quad x = \frac{4280}{\tan\ 24°} = \boxed{9613 \text{ ft}}\ .$$

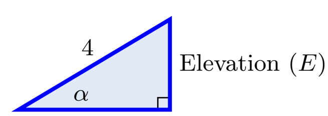
Example 1.5.5

A roadway sign at the top of a mountain indicates that for the next 4 km the grade is 12%[3]. Find the change in elevation for a car descending the mountain.

Solution: Even though the road probably winds around the mountain and the slope is not exactly 12% everywhere we can assume that if we straighten out the road it is 4 km long and descends at a constant rate of $\frac{12}{100}$. If we draw a triangle for the grade the opposite side would be 12 and the adjacent side would be 100. Using the Pythagorean theorem we can find that the hypotenuse is $h = \sqrt{12^2 + 100^2} = \sqrt{10144}$. If we call the angle of elevation α then we can find the value of any trigonometric function for α from our triangle. The second triangle represents the the mountain where the hypotenuse is the length of the road, 4 km.



(a) The grade is 12%



(b) The car travels 4km along the road

Figure 1.29: Figures for Example 1.5.5

The sine function relates the opposite side to the hypotenuse so we can set up two equations for the $\sin\alpha$ using both triangles. To make the calculations easier we convert km to m by multiplying by 1000.

$$\sin\alpha = \frac{12}{\sqrt{10144}} = \frac{E}{4000m}$$

$$E = 4000\left(\frac{12}{\sqrt{10144}}\right)$$

$$\boxed{E = 477 \text{ m}}$$

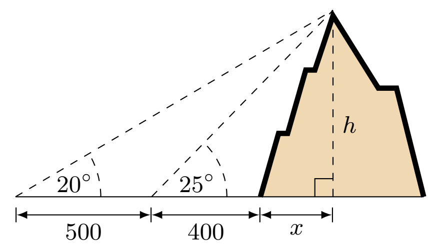
[3] The grade is the slope (rise over run) of the road. When expressed as a percentage: grade $= 100\left(\frac{\text{rise}}{\text{run}}\right)$

We round to the nearest meter because the length is probably not exactly 4.000 km. Also note that we never found the value of α. We were able to find the value of $\sin\alpha$ from the triangle.

Example 1.5.6

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. The person then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?

Solution: We will assume that the ground is flat and not inclined relative to the base of the mountain. Let h be the height of the mountain, and let x be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that



$$\frac{h}{x+400} \;=\; \tan\ 25^\circ \quad\Rightarrow\quad h \;=\; (x+400)\ \tan\ 25^\circ\ ,\ \text{and}$$

$$\frac{h}{x+400+500} \;=\; \tan\ 20^\circ \quad\Rightarrow\quad h \;=\; (x+900)\ \tan\ 20^\circ\ ,\ \text{so}$$

$(x+400)\ \tan\ 25^\circ \;=\; (x+900)\ \tan\ 20^\circ$, since they both equal h. Use that equation to solve for x:

$$x\ \tan\ 25^\circ \;-\; x\ \tan\ 20^\circ \;=\; 900\ \tan\ 20^\circ \;-\; 400\ \tan\ 25^\circ$$

$$\Rightarrow\quad x \;=\; \frac{900\ \tan\ 20^\circ \;-\; 400\ \tan\ 25^\circ}{\tan\ 25^\circ \;-\; \tan\ 20^\circ} \;=\; 1378\ \text{ft}$$

Finally, substitute x into the first formula for h to get the height of the mountain:

$$h \;=\; (1378+400)\ \tan\ 25^\circ \;=\; 1778\,(0.4663) \;=\; \boxed{829\ \text{ft}}$$

1.5 Exercises

For Exercises 1 - 8, solve the right triangle $\triangle ABC$ in **Figure 1.30** using the given information.

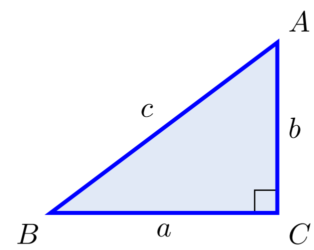


Figure 1.30

1. $A = 35°, \;\; b = 6$

2. $a = 5, \;\; B = 6°$

3. $a = 1, \;\; B = 36°$

4. $A = 6°, \;\; c = 10$

5. $c = 7, \;\; B = 24°$

6. $A = 1°, \;\; a = 2$

7. $A = \frac{\pi}{4}, \;\; b = 12$

8. $B = \frac{\pi}{3}, \;\; c = 36$

For Exercises 9 - 11 find the length of x in **Figure 1.31**

9. $\alpha = 55°\ 30', \;\; \beta = 62°\ 30'', \;\; h = 15$

10. $\alpha = 25°, \;\; \beta = 30°, \;\; h = 15$

11. $\alpha = \pi/5, \;\; \beta = \pi/3, \;\; h = 15$

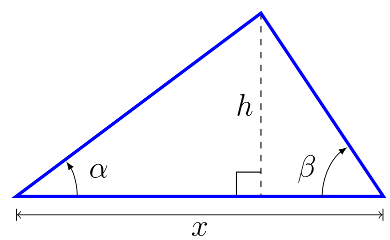


Figure 1.31: Problems 9 - 11

12. To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 29°. Find the height of the tree.

13. The angle of elevation to the top of a building is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

14. The angle of elevation to the top of the Space Needle in Seattle is found to be 31 degrees from the ground at a distance of 1000 feet from its base. Using this information, find the height of the Space Needle.

15. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 60°. How high does the ladder reach up the side of the building?

16. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building?

17. As the angle of elevation from the top of a tower to the sun decreases from 64° to 49° during the day, the length of the shadow of the tower increases by 92 ft along the ground. Assuming the ground is level, find the height of the tower.

18. Find the length c in **Figure 1.32**

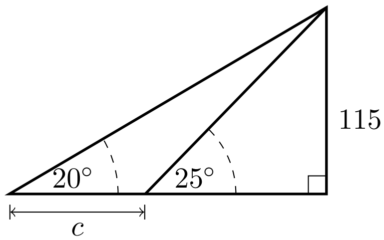


Figure 1.32

19. Find the length c in **Figure 1.33**

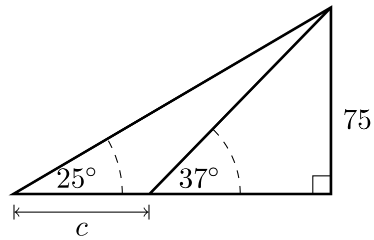


Figure 1.33

20. Two banks of a river are parallel, and the distance between two points A and B along one bank is 500 ft. For a point C on the opposite bank, $\angle BAC = 56°$ and $\angle ABC = 41°$, as in **Figure 1.34**. What is the width w of the river? (*Hint: Divide* $\overline{AB}$ *into two pieces.*)

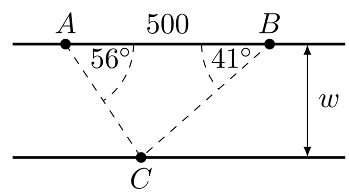


Figure 1.34

21. A person standing on the roof of a 100 m building is looking towards a skyscraper a few blocks away, wondering how tall it is. She measures the angle of depression from the roof of the building to the base of the skyscraper to be 20° and the angle of elevation to the top of the skyscraper to be 40°. Calculate the distance between the buildings x and the height of skyscraper h. See **Figure 1.35**.

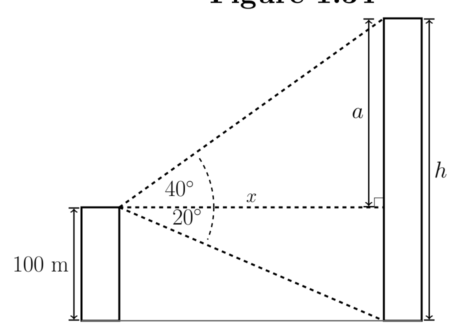


Figure 1.35

22. 2200 years ago the greek Aristarchus realized that using trigonometry it is possible to calculate the distance to the sun.[4] Let O be the center of the earth and let A be the center of the moon. Aristarchus began with the premise that, during a half moon, the moon forms a right triangle with the Sun and Earth. By observing the angle between the Sun and Moon, $\phi = 89.83°$ and knowing the distance to the moon, about 239,000 miles[5] it is possible to estimate the distance from the center of the earth to the sun. Estimate the distance to the sun using these values. See **Figure 1.36**.

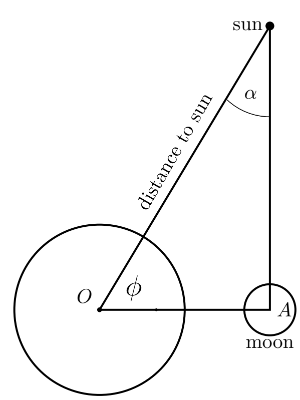


Figure 1.36

[4] https://en.wikipedia.org/wiki/On_the_Sizes_and_Distances_(Aristarchus)
[5] https://en.wikipedia.org/wiki/Lunar_distance_(astronomy)

23. A plane is flying 2000 feet above sea level toward a mountain as shown in **Figure 1.37**. The pilot observes the top of the mountain to be $\alpha = 18°$ above the horizontal, then immediately flies the plane at an angle of $\beta = 20°$ above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain?

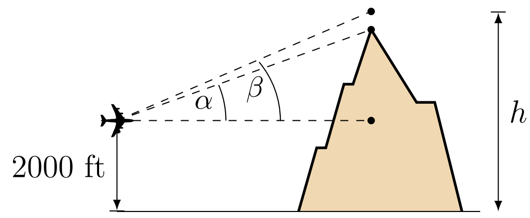


Figure 1.37

24. Parallax is a displacement or difference in the apparent position of an object viewed along two different lines of sight.[6] (A simple everyday example of parallax can be seen in the dashboard of motor vehicles that use a needle-style speedometer gauge. When viewed from directly in front, the speed may show exactly 60; but when viewed from the passenger seat the needle may appear to show a slightly different speed, due to the angle of viewing.) Parallax can be used to calculate the distance to near stars. By measuring the distance a star moves when taking two observations when the earth is on opposite sides of the sun we can calculate the parallax angle. **Figure 1.38** shows the parallax angle labeled p. Knowing that the distance from the earth to the sun is about 92,960,000 miles how far is it from the sun to a star that creates a parallax angle $p = 1''$ (one second)? This is a distance known as 1 parsec.[7]

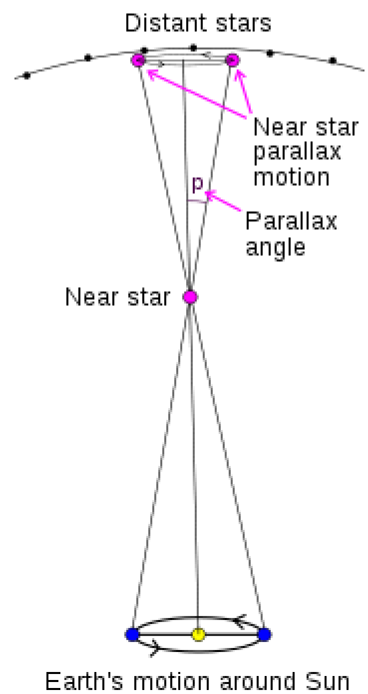


Figure 1.38

[6] https://en.wikipedia.org/wiki/Parallax
[7] https://en.wikipedia.org/wiki/Parsec

Chapter 2

Graphs and Inverse Functions

2.1 Graphs of Sine and Cosine

Basic Sine and Cosine Graphs

We can graph trigonometric functions the same as we can graph any other function. We will graph the trigonometric functions on the xy-plane and the x coordinate will always be in radians. We will demonstrate two ways to look at the graph of $y = \sin x$. First we will plot points by selecting angle values for x and calculating the y values. Second we will use the unit circle.

The following table (Table 2.1) is a list of common angles and their trigonometric function values.

Table 2.1: **Table of Common Trigonometric Function Values**

θ radians	$y = \sin\theta$	$y = \cos\theta$	$y = \tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
π	0	-1	0

continued on next page

Table 2.1: *Common trigonometric function values continued*

θ radians	$y = \sin\theta$	$y = \cos\theta$	$y = \tan\theta$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	-1	0	undefined
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$

Using the numbers in Table 2.1 we can plot the sine function from $0 \le x \le 2\pi$. In **Figure 2.1** the points are indicated on the graph and some have been labeled. We saw in Section 1.4 that the trigonometric functions are periodic. This means that the values repeat at regular intervals. The sine repeats every 2π radians so this graph repeats forever in both directions as seen in in **Figure 2.3**.

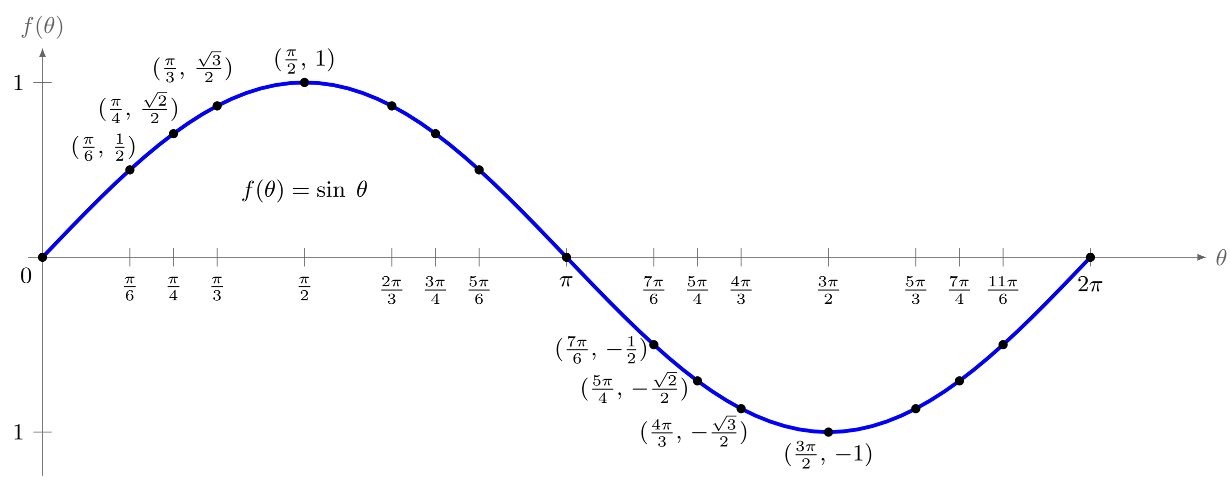


Figure 2.1: Graph of $y = \sin x$ for $0 \le x \le 2\pi$

Another way to consider the graph of the sine is to remember that every point on the unit circle (circle of radius 1) is $(x, y) = (\cos\theta, \sin\theta)$ on the terminal side of θ. Here you can see how for each angle, we use the y value of the point on the circle to determine the output value of the sine function. The correspondence is shown in **Figure 2.2**.

It is most common to use the variable x and y to represent the horizontal and vertical axes so we will relabel the axes when we draw the graphs of the trigonometric functions from now

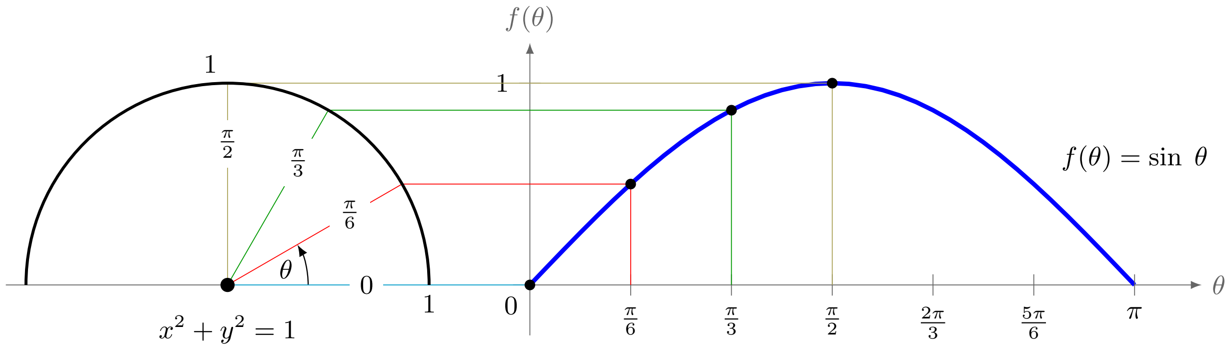


Figure 2.2: Graph of sine function based on y-coordinate of points on unit circle

on. In our graph in **Figure 2.3** we have plotted both positive an negative angles. You will notice that if you pick any starting x value and move 2π units in either direction the values of the function are the same because the period of the sine function in 2π.

WARNING: Be careful because we reuse variables. x and y are used to represent the cosine and sine on the unit circle but here x is the angle and y is the trigonometric value of that angle.

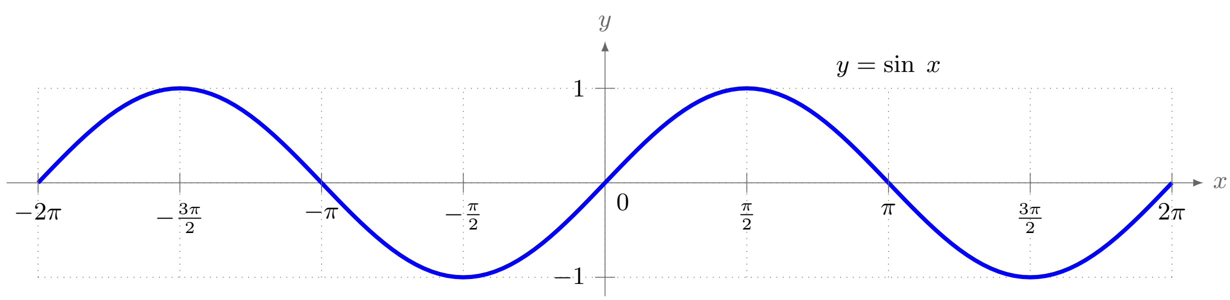


Figure 2.3: Graph of sine function where x is the angle and $y = \sin x$

Similarly we can construct a graph for the cosine function. Note that the cosine function has the same shape as the sine function but it is shifted $\dfrac{\pi}{2}$ units to the left. From algebra you may recall that a $\dfrac{\pi}{2}$ shift to the left can be represented $f\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right) = \cos(x)$. This is the same cofunction identity presented in Section 1.4.

Both the sine and cosine functions alternate between $+1$ and -1 passing through zero at regular points. When we label the axes of the graphs we want to make sure we label the angles where the functions are 0, 1 or -1 on the x-axis and the values for the maximum, minimum and center line for the y-axis. You can certainly include more labels but this would generally be the minimum amount of information for a graph. Notice that all the multiples of $\frac{\pi}{2}$ have been labeled on the graphs in **Figures 2.3** and **2.4**.

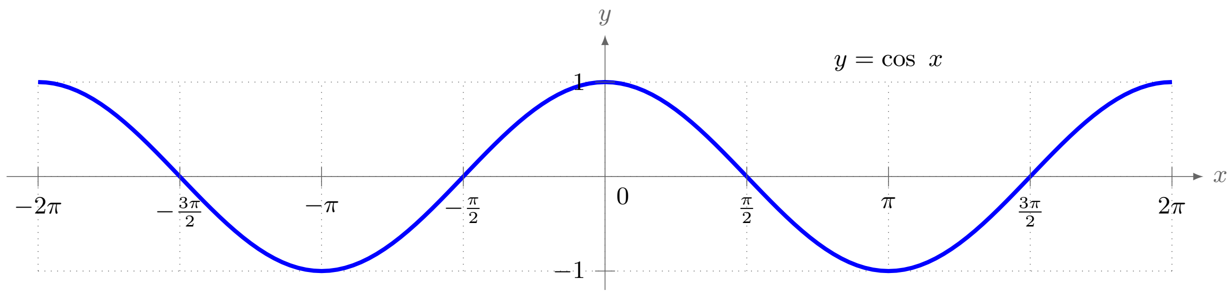


Figure 2.4: Graph of cosine function where x is the angle and $y = \cos x$

Algebraic Transformations

The graphs can be altered by standard algebraic transformations. A function may be stretched or compressed vertically by multiplying it by a number.

Stretching the function $f(x)$ vertically

$h(x) = A \cdot f(x)$ stretches $f(x)$ vertically by a factor of A.

In the case of the sine and cosine this has the effect of making the **amplitude** of the function larger or smaller. The amplitude of the function is the distance from the center line to the maximum height. It can be calculated using the formula:

$$\text{amplitude of } f(x) = \frac{(\text{maximum of } f(x)) - (\text{minimum of } f(x))}{2}$$

Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ then for any $A > 0$

$$-A \leq A \sin x \leq A \quad \text{and} \quad -A \leq A \cos x \leq A$$

Notice that the x-axis is labeled at the maximums, minimums and zeros of the function in **Figure 2.5**.

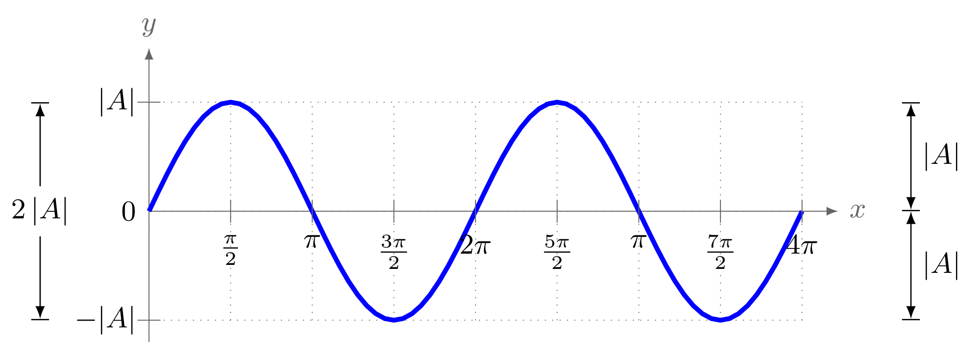


Figure 2.5: The amplitude of a graph $\frac{\max - \min}{2} = |A|$

Example 2.1.1

Sketch the graph of $y = 2\cos x$ for two complete cycles.

Solution: Since the period of the cosine is 2π two complete cycles can be $0 \leq x \leq 4\pi$. We could have also done negative angles and graphed $-2\pi \leq x \leq 2\pi$.

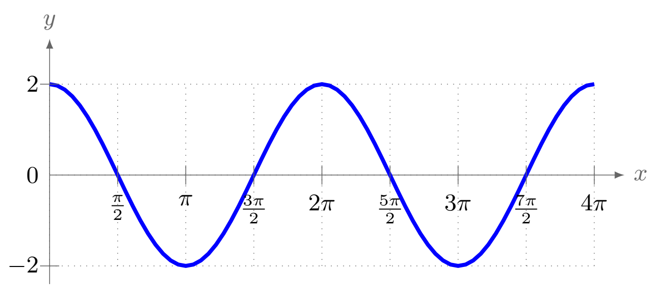


Figure 2.6: $y = 2\cos x$

A function may be shifted up or down by adding or subtracting a number on the outside.

Moving the function $f(x)$ up and down

$$h(x) = f(x) + D \quad \text{moves } f(x) \text{ up "}D\text{" units.}$$

$$h(x) = f(x) - D \quad \text{moves } f(x) \text{ down "}D\text{" units.}$$

Example 2.1.2

Sketch the graph of $y = 2\cos x + 3$

Solution: This is the same graph as **Example** 2.1.1 but moved up 3 units. It has the same amplitude $A = 2$.

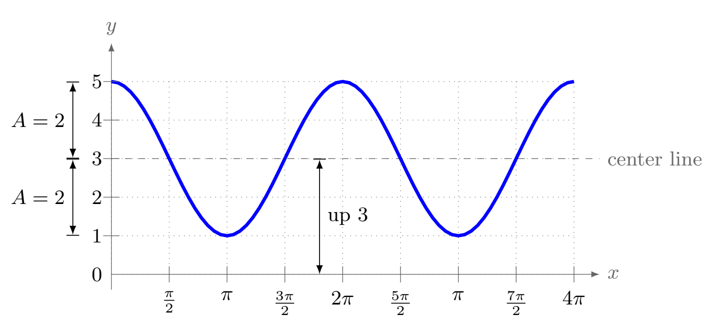


Figure 2.7: $y = 2\cos x + 3$

A function may be stretched or compressed horizontally by multiplying the variable by a number.

Stretching the function $f(x)$ horizontally

$h(x) = f(B \cdot x)$ stretches or compresses $f(x)$ horizontally by a factor of $\frac{1}{B}$.

If $B > 1$ the function is compressed horizontally and if $0 < B < 1$ the function is stretched horizontally.

In the case of the sine and cosine multiplying the variable by a number B changes the period. The period of $y = \sin(Bx)$ and of $y = \cos(Bx)$ is

$$\text{period of } y = \sin(Bx) \text{ is } \frac{2\pi}{B}$$

$$\text{period of } y = \cos(Bx) \text{ is } \frac{2\pi}{B}$$

Example 2.1.3

Sketch the graph of $y = \cos(2x)$ and $y = \cos x$ on the same set of axes.

Solution: Since we have a $2x$ inside the cosine it goes around the circle twice as fast which is why in the space of 2π the graph will repeat twice. We will graph both $y = \cos x$ and $y = \cos(2x)$ on the same set of axes.

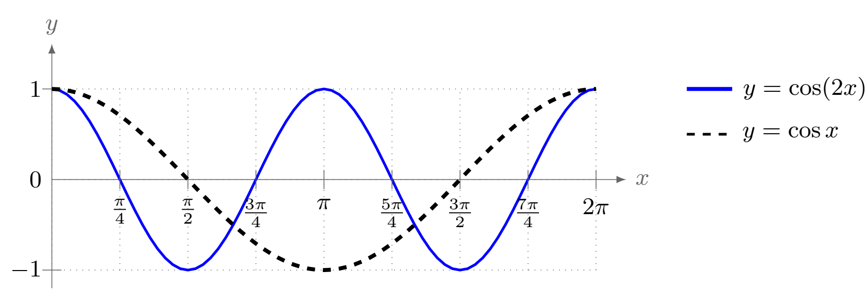


Figure 2.8: $y = \cos(2x)$ and $y = \cos x$

A function may be reflected across the x-axis by multiplying by (-1). (Making it negative.)

Reflecting the function $f(x)$ over the x-axis

$h(x) = -f(x)$ reflects $f(x)$ across the x-axis.

Example 2.1.4

Sketch the graphs of $y = -\cos\left(\frac{x}{2}\right)$ and $y = -\cos\left(\frac{x}{2}\right) + 3$ on the same set of axes. Draw two complete periods for each function.

Solution: Here we will have to adjust the period using the period formula period $= \frac{2\pi}{B}$. Since we have $\cos\left(\frac{x}{2}\right) = \cos\left(\frac{1}{2}x\right)$ we can see that $B = \frac{1}{2}$ and the period is $= \frac{2\pi}{1/2} = 4\pi$. The function will repeat every 4π units. Since the horizontal axis is divided into 4 pieces for each period those divisions are all of size π. See **Figure 2.9**.

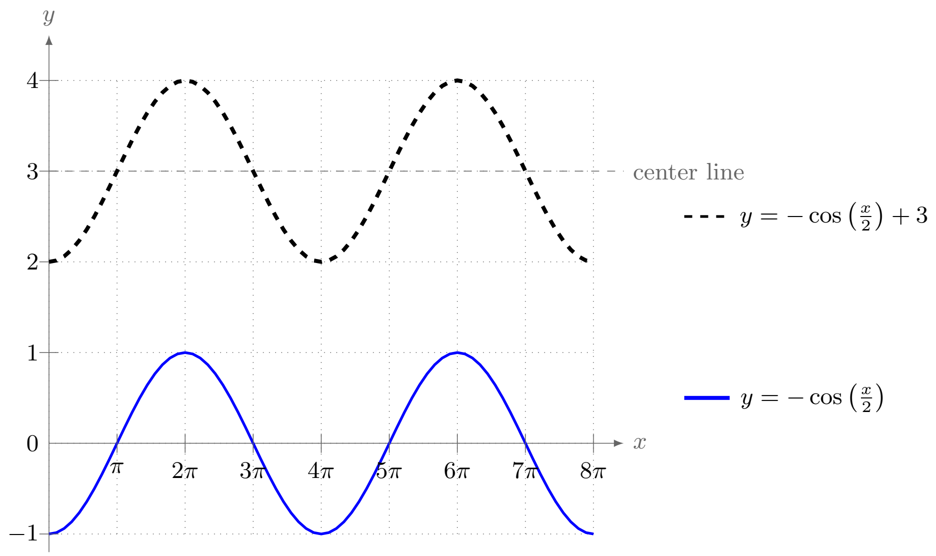


Figure 2.9: $y = -\cos\left(\frac{x}{2}\right)$ and $y = -\cos\left(\frac{x}{2}\right) + 3$

A function may be shifted left or right by adding or subtracting a number on the inside. This shift is called the **phase shift** .

Shifting the function $f(x)$ left and right

$$h(x) = f(x + C) \quad \text{moves } f(x) \text{ to the left "}C\text{" units.}$$
$$h(x) = f(x - C) \quad \text{moves } f(x) \text{ to the right "}C\text{" units.}$$

Example 2.1.5

Graph $y = \sin\left(x + \frac{\pi}{4}\right)$.

Solution: Since we have added $\frac{\pi}{4}$ inside the function the graph will be the same as the graph of $y = \sin x$ but shifted to the left $\frac{\pi}{4}$ units. Rather than having zeros at 0, $\pm\pi$ and $\pm 2\pi$ the zeros are now at $-\frac{5\pi}{4}$, $-\frac{\pi}{4}$, $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. The graphs of both $y = \sin\left(x + \frac{\pi}{4}\right)$ and $y = \sin x$ are presented in **Figure 2.10**.

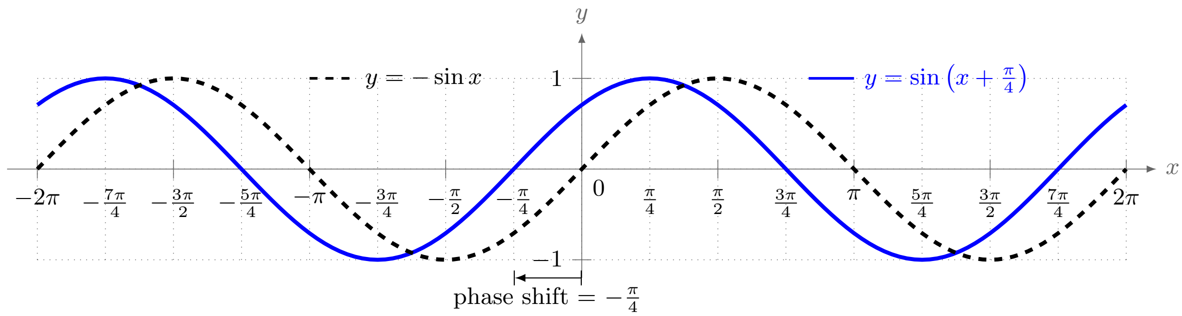


Figure 2.10: $y = \sin\left(x + \frac{\pi}{4}\right)$ and $y = \sin x$

Example 2.1.6

Graph $y = 3\cos(2x - \pi)$ for two complete cycles.

Solution: Here we have to be careful because there are three of our transformations in the same problem. First we need to identify the amplitude. That is given to us by the number multiplied in front of the function so $A = 3$. The period is determined by the number multiplied by the x, in this case $B = 2$. The period of the function is $\frac{2\pi}{B} = \pi$.

The phase shift is a bit more difficult because our original definition of phase shift was written as $f(x+C)$ but we don't have that, we have $f(2x+\phi)$. That 2 multiplied by the x is going to influence our shift. We have to write the function as $f(2(x+C))$ to find the correct value of the phase shift. To see why this is true let's consider that the cosine function goes through an entire cycle when its angle goes from 0 to 2π. In this case our angle is represented by $2x - \pi$ so that cycle starts when

$$2x - \pi = 0 \quad \Longrightarrow \quad x = \frac{\pi}{2}$$

and ends when

$$2x - \pi = 2\pi \quad \Longrightarrow \quad x = \frac{2\pi}{2} + \frac{\pi}{2} = \pi + \frac{\pi}{2}$$

Our phase shift is $\frac{\pi}{2}$ and the period is π which is exactly what we see when we write the function as $y = 3\cos\left[2\left(x - \frac{\pi}{2}\right)\right]$.

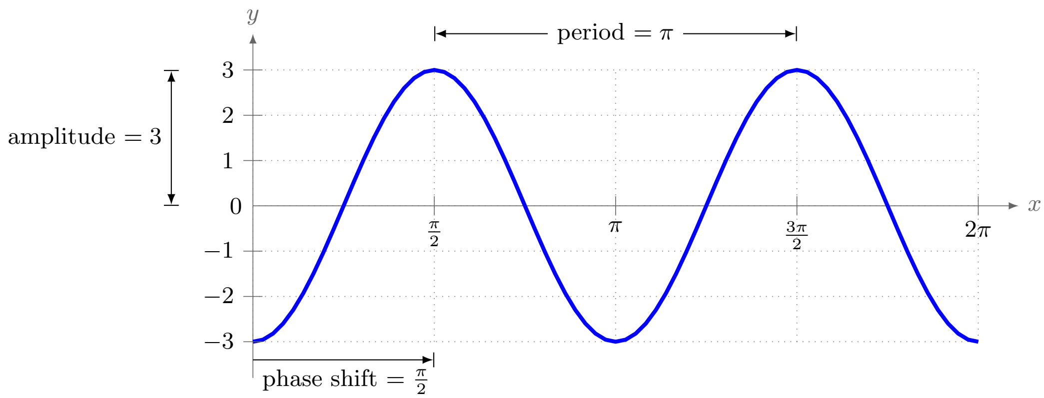


Figure 2.11: $y = 3\cos(2x - \pi)$

Summary of trigonometric transformations for sine and cosine.

Given the functions

$$y = A\sin(Bx + C) + D = A\sin\left(B\left(x + \frac{C}{B}\right)\right) + D$$

or

$$y = A\cos(Bx + C) + D = A\cos\left(B\left(x + \frac{C}{B}\right)\right) + D$$

the following transformations occur:

1. The amplitude of the function is $|A|$.
2. The period of the function is $\dfrac{2\pi}{B}$
3. The phase shift of the function is $-\dfrac{C}{B}$.

   The shift is to the left for $\left(x + \frac{C}{B}\right)$ and to the right for $\left(x - \frac{C}{B}\right)$
4. The vertical shift is D

A negative sign in front of the function will reflect it over the x-axis.

Example 2.1.7

Find the amplitude, period and phase shift of $y = -2\sin\left(3x + \frac{\pi}{2}\right)$

Solution: The amplitude is 2, the period is $\frac{2\pi}{3}$, and the phase shift is $-\frac{\frac{\pi}{2}}{3} = -\frac{\pi}{6}$. Since the phase shift is negative we move the graph to the left. Or if you write the function as

$$y = -2\sin\left(3\left(x + \frac{\pi}{6}\right)\right)$$

we are adding $\frac{\pi}{6}$ inside the sine function which is a shift to the left. Also note the negative in front of the sine, this reflects the graph over the x-axis.

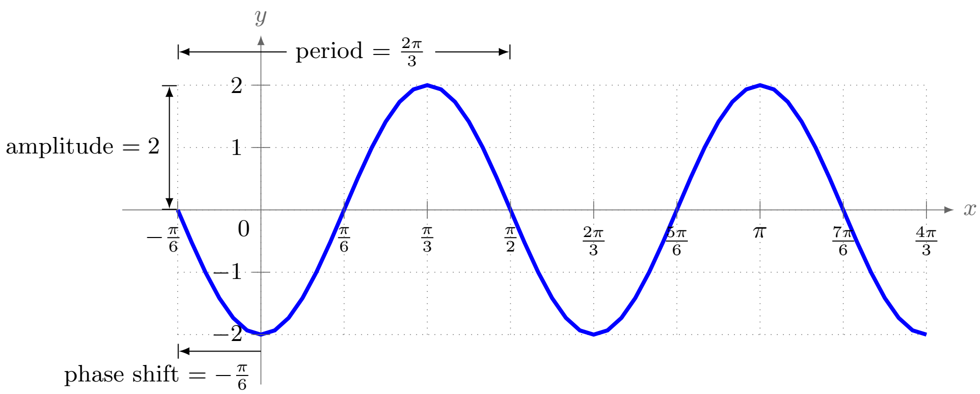


Figure 2.12: $y = -2\sin\left(3x + \frac{\pi}{2}\right) = -2\sin\left[3\left(x + \frac{\pi}{6}\right)\right]$

2.1 Exercises

For Exercises 1-12, determine the amplitude, period, vertical shift, horizontal shift, and draw the graph of the given function for two complete periods.

1. $y = 3\sin x$

2. $f(x) = -3\sin x$

3. $y = -3\sin(2x)$

4. $f(x) = -3\sin(2x) + 4$

5. $y = \dfrac{\cos x}{4}$

6. $y = \cos\left(\frac{x}{4}\right)$

7. $f(x) = \frac{1}{2}\cos x - 4$

8. $y = 2\cos\left(x - \frac{\pi}{4}\right)$

9. $g(x) = -3 + 2\cos\left(x - \frac{\pi}{4}\right)$

10. $y = 2\sin\left(2x + \frac{\pi}{2}\right)$

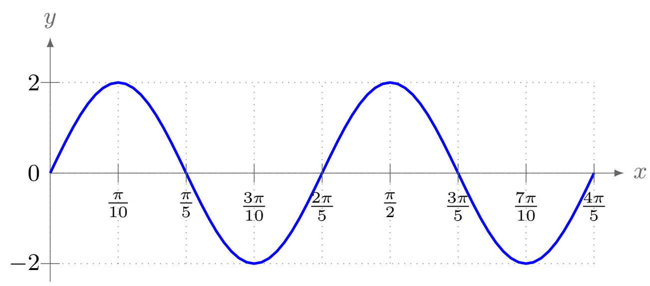
11. $y = \frac{1}{2}\sin\left(2x + \frac{\pi}{2}\right) + 1$

12. $y = 3\sin\frac{\pi t}{3}$

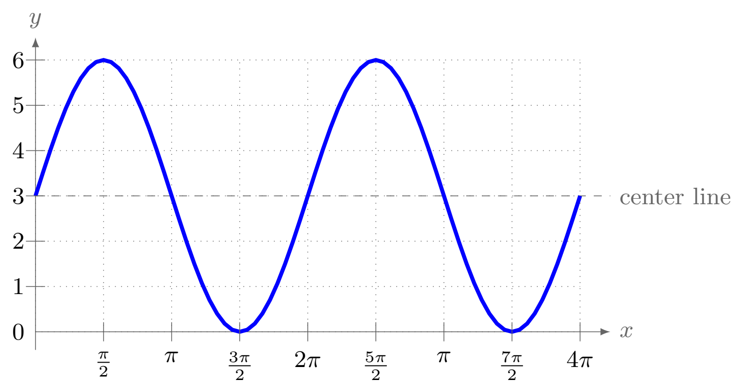
For Exercises 13-14, sketch $f(x)$ and $g(x)$ on the same set of axes for $0 \le x \le 2\pi$.

13. $f(x) = 2\sin x, \qquad g(x) = \sin(2x)$ **14.** $f(x) = 3\cos(2x), \qquad g(x) = 3\cos(2x) - 2$

For Exercises 15-19, determine the amplitude, period and vertical shift, then find a formula for the function.



15.



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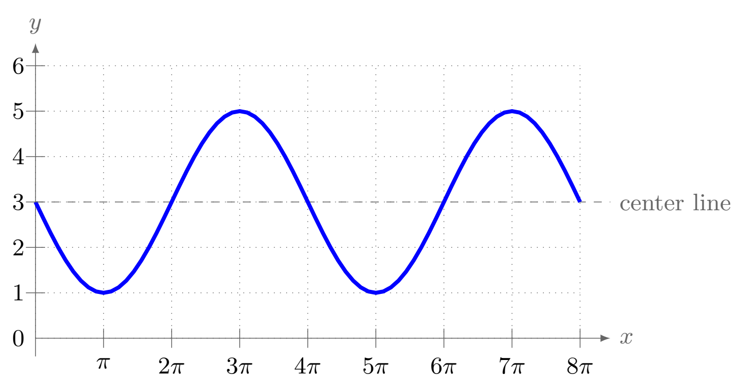

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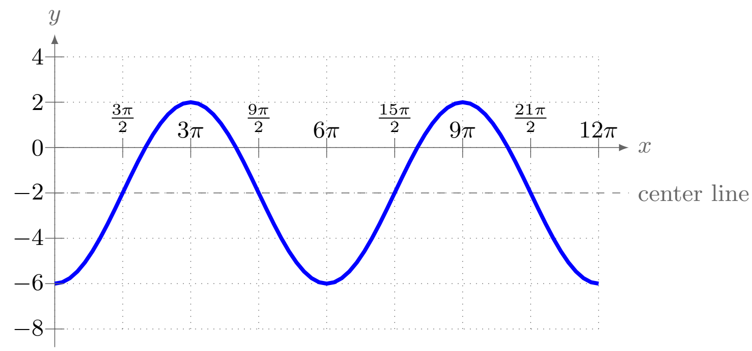
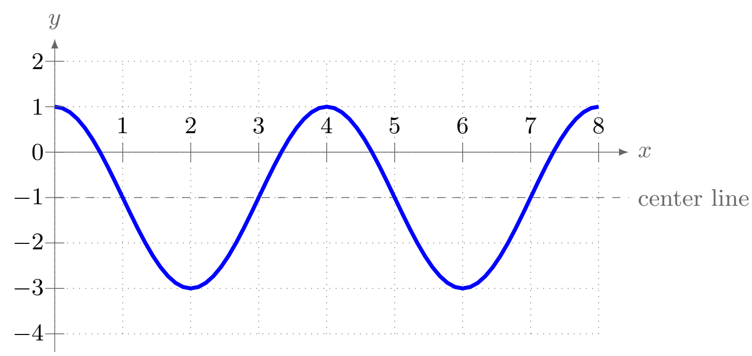


18.



19.

20. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 50 degrees at midnight and the high and low temperature during the day are 57 and 43 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

21. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68 degrees at midnight and the high and low temperature during the day are 80 and 56 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

22. Consider the device shown in **Figure 2.13** for converting rotary motion to linear motion (and vice versa). A nail on the edge of the wheel moves the arm back and forth. Relative to the coordinates shown, derive an expression for the position of point P as a function of the wheel radius R, the bar length L, and the angle θ.

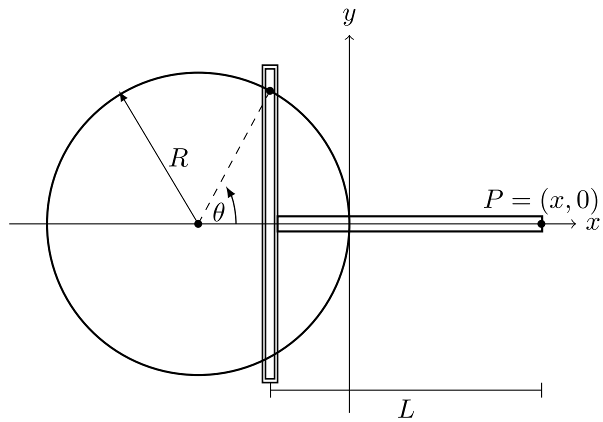


Figure 2.13: Linear motion device

2.2 Graphs of tan(x), cot(x), csc(x) and sec(x)

Tangent and Cotangent Graphs

The graph of the tangent can be constructed by plotting points from Table 2.1 or by using the identity $\tan x = \dfrac{\sin x}{\cos x}$. On the graph of the tangent notice that there are vertical asymptotes at multiples of $\dfrac{\pi}{2}$. This is because $\tan x = \dfrac{\sin x}{\cos x}$ and everywhere cosine is zero tangent is undefined. You can see from the cosine graph that it has zeros at $x = \dfrac{\pi}{2} + n\pi$ where $n \in \mathbb{Z}$. Also note that the period of the tangent function is π. The graph repeats every π units, it is identical between any two asymptotes.

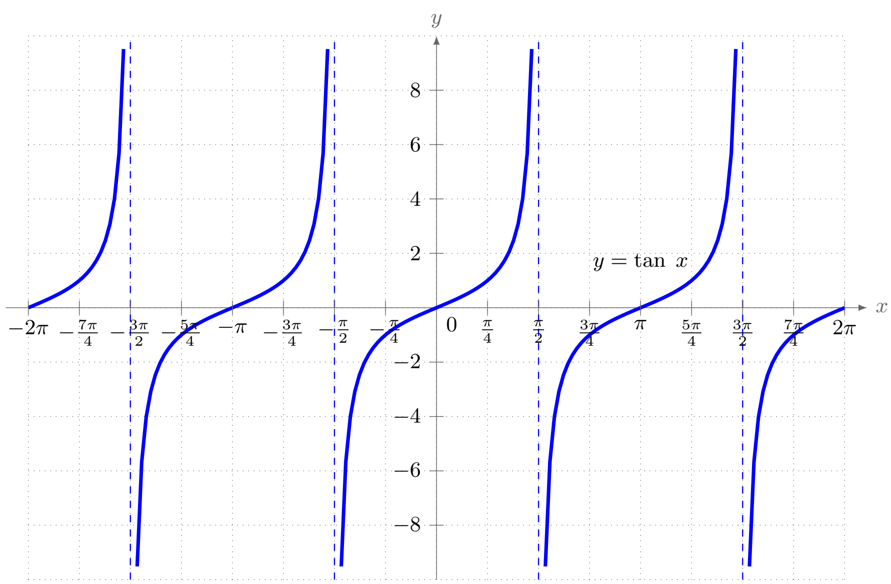


Figure 2.14: Graph of $y = \tan x$

We can perform similar transformations to what was done for the sine and cosine graphs. Those transformations are summarized here:

Summary of trigonometric transformations for tangent.

Given the function

$$y = A\tan(Bx + C) + D$$

the following transformations occur:

1. The amplitude of the function is undefined.
2. The period of the function is $\dfrac{\pi}{B}$
3. The phase shift of the function is $\dfrac{C}{B}$.
4. The vertical shift is D

A negative sign in front of the function will reflect it over the x-axis.

Example 2.2.1

Find the amplitude, period, phase shift, and vertical shift for the function $y = \frac{1}{2}\tan(2x) - 3$

Solution: The amplitude is undefined, the period is $\frac{\pi}{2}$, there is no phase shift, and the vertical shift is down 3 units.

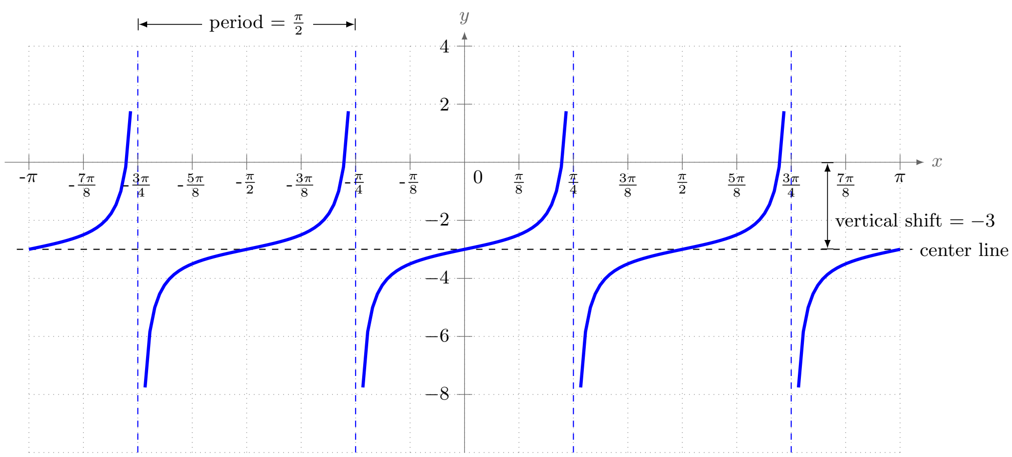


Figure 2.15: $y = \frac{1}{2}\tan(2x) - 3$

The graph of the cotangent **Figure 2.16** can be constructed by using the identity $\cot x = \dfrac{\cos x}{\sin x}$ or by using the relation $\cot x = -\tan\left(x + \frac{\pi}{2}\right)$. On the graph of the cotangent notice that there are vertical asymptotes at multiples of π. This is because $\cot x = \dfrac{\cos x}{\sin x}$ and everywhere sine is zero the cotangent is undefined. $y = \sin x$ has zeros at $x = \pi + n\pi$ where $n \in \mathbb{Z}$ so $y = \cot x$ has vertical asymptotes at $x = \pi + n\pi$. Also note that the period of the cotangent function is π. The graph repeats every π units, it is identical between any two asymptotes.

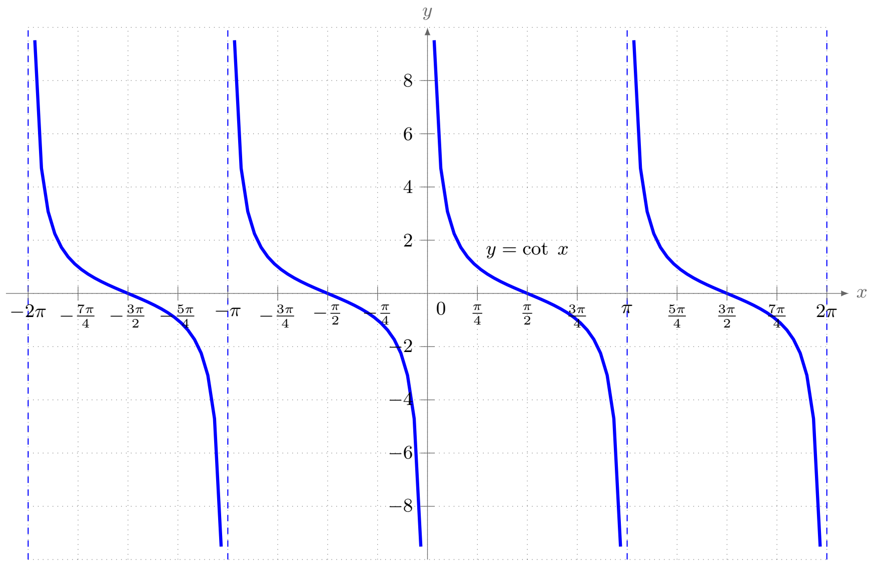


Figure 2.16: Graph of $y = \cot x$

Cosecant and Secant Graphs

The graph of the cosecant can be constructed by using the identity $\csc x = \dfrac{1}{\sin x}$. On the graph of the cosecant notice that there are vertical asymptotes at multiples of π. This is because $\csc x = \dfrac{1}{\sin x}$ and everywhere sine is zero the cosecant is undefined. The period of the cosecant function is 2π which is the same as the sine function. The graph repeats every 2π units. **Figure 2.17** shows the graph of $y = \csc x$, with the graph of $y = \sin x$ (the dashed curve) for reference.

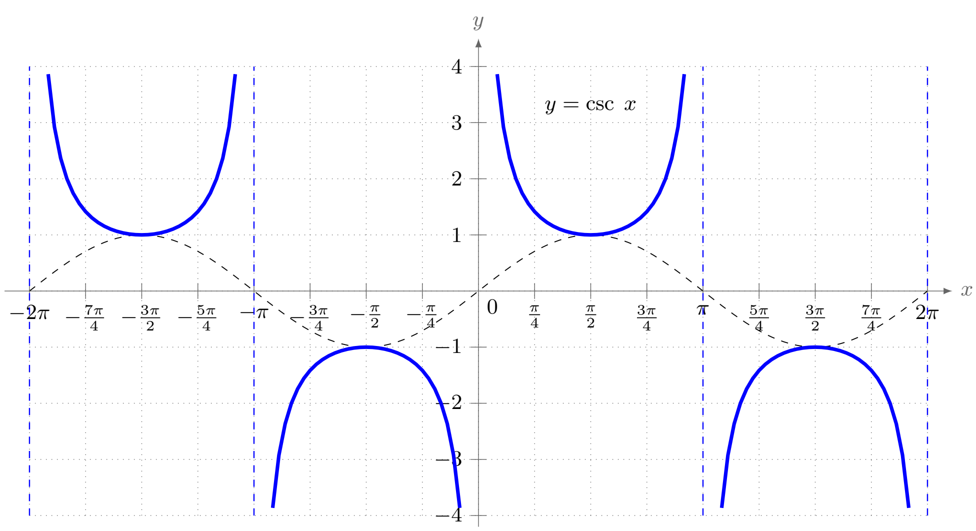


Figure 2.17: Graph of $y = \csc x$ in blue and $y = \sin x$ (dashed line)

The graph of the secant can be constructed by using the identity $\sec x = \dfrac{1}{\cos x}$. On the graph of the secant notice that there are vertical asymptotes at multiples of $\dfrac{\pi}{2}$ because the graph of $y = cosx$ has zeros at $x = \dfrac{\pi}{2} + n\pi$ where $n \in \mathbb{Z}$. The period of the secant function is 2π which is the same as the cosine function. The graph repeats every 2π units. **Figure 2.18** shows the graph of $y = \sec x$, with the graph of $y = \cos x$ (the dashed curve) for reference.

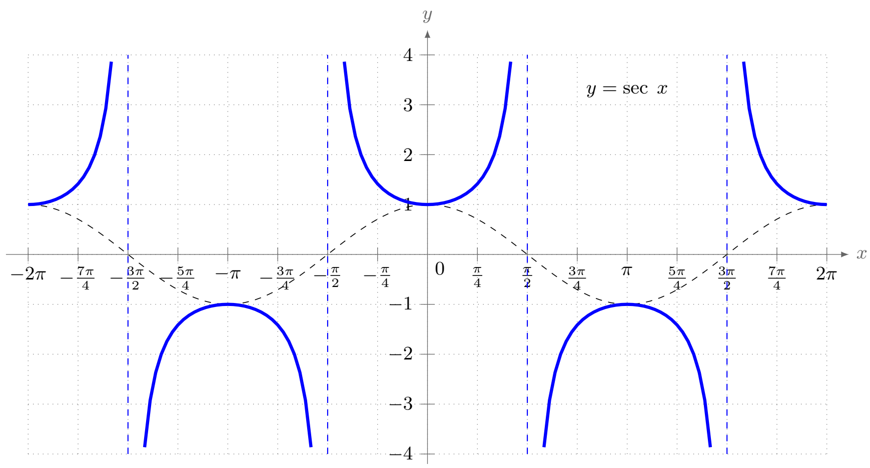


Figure 2.18: Graph of $y = \sec x$

All the same transformations that were done to the sine, cosine and tangent can be done to the other functions.

Summary of trigonometric transformations for cosecant, secant and cotangent

$y = A\csc(Bx + C)$ has undefined amplitude, period $\dfrac{2\pi}{B}$ and phase shift $\dfrac{C}{B}$

$y = A\sec(Bx + C)$ has undefined amplitude, period $\dfrac{2\pi}{B}$ and phase shift $\dfrac{C}{B}$

$y = A\cot(Bx + C)$ has undefined amplitude, period $\dfrac{\pi}{B}$ and phase shift $\dfrac{C}{B}$

A negative sign in front of the function will reflect it over the x-axis.

2.2 Exercises

For Exercises 1-9, determine the amplitude, period, vertical shift, horizontal shift, and draw the graph of the given function for two complete periods.

1. $y = 3\tan x$

2. $f(x) = -3\csc x$

3. $y = -3\sec(2x)$

4. $f(x) = -3\sec(\pi x)$

5. $y = \dfrac{\cot x}{4}$

6. $y = \cot\left(\frac{x}{4}\right)$

7. $y = \tan\left(x + \frac{\pi}{4}\right)$

8. $y = \frac{1}{2}\cot\left(x - \frac{\pi}{4}\right)$

9. $y = \sec(t) + 2$

2.3 Inverse Trigonometric Functions

Review of Functions and Inverse Functions

Definition 2.1. A **function** is a rule that establishes a correspondence between two sets of elements (called the **domain** and **range**) so that for every element in the domain there corresponds EXACTLY ONE element in the range.

Often the domain is x and the range is y but any symbols can be used. With trigonometric functions frequently θ or another Greek letter is used for the domain. For a function we can have repeated range elements but all the domain elements are unique. For example with $f(x) = x^2$ both $x = 2$ and $x = -2$ are mapped to $y = 4$ when put into the function.

There is a special type of function known as a one-to-one (sometimes written $1-1$) function where all the range values are unique as well. In other words if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. The example above of $f(x) = x^2$ is not a $1-1$ function because two different x values give the same y value. Much like there was a vertical line test for functions we have a horizontal line test for $1-1$ functions.

The vertical line test says that $f(x)$ is a function if and only if every vertical line intersects the graph of $f(x)$ at most once. Similarly the horizontal line tests says that a function $f(x)$ is $1-1$ if every horizontal line intersects the graph at most once.

This idea of a $1-1$ function is important when discussing inverse functions. An inverse function is a function $f^{-1}(x)$ such that

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

In other words if f is a function that takes x to y then the inverse function f^{-1} takes y back to x. We need the original function to be $1-1$ because when we reverse the operation we want to make sure we get a unique answer. In the $f(x) = x^2$ example we can't have an inverse function because reversing the operation results in two x-values because $f(2) = f(-2) = 4$.

None of the trigonometric functions are $1-1$. Consider the sine function $y = \sin x$. There are an infinite number of x-values that will produce every y value since the sine repeats every 2π radians. If we want to reverse the operation of the sine function with an inverse sine function we will have to restrict the domain so that the original sine produces one set of range values. We will make sure that this restriction includes the angle zero. In **Figure 2.19** the extended dotted line is to show that the sine function would fail the horizontal line test and sine is $1-1$ on the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. It also shows that we have one complete set of range values ($-1 \leq \sin x \leq 1$) for the sine function.

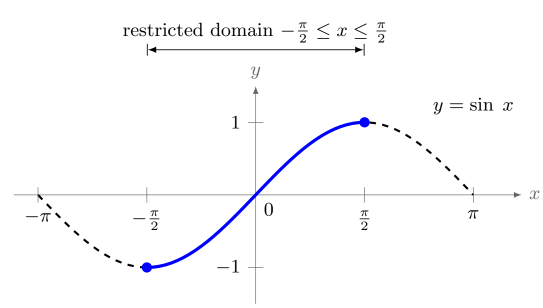
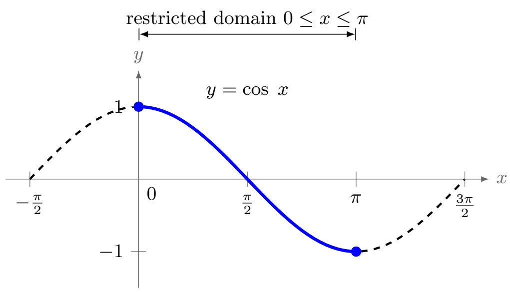
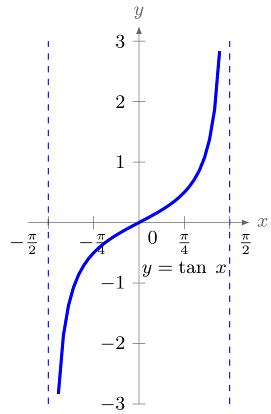


Figure 2.19: Restricted domain for sine

We will do the same for the cosine and tangent. **Figure 2.20** shows the domain restrictions.



(a) $y = \cos x$



(b) $y = \tan x$

Figure 2.20: Restricted domains for cosine and tangent

Recall that there are two ways to find the inverse of a function. The graphical way to find the inverse is to look at the graph and reflect it across the line $y = x$. The algebraic way to solve for the inverse of a function is to switch the x and y coordinates and solve for y.

We can find inverse functions of the sine, cosine and tangent using the graphing method. The graph of $y = \sin^{-1} x$ (sometimes called the **arcsine** and denoted $y = \arcsin x$) is shown in **Figure 2.21**. Notice the symmetry about the line $y = x$ with the graph of $y = \sin x$.

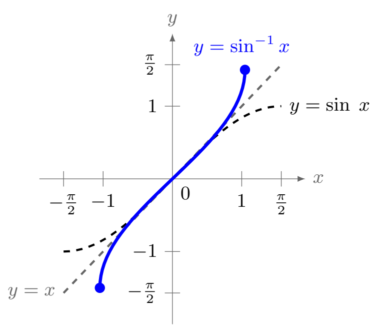


Figure 2.21: Graph of $y = \sin^{-1} x$

The sine function gives you the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ for some angle θ. The inverse sine function give you the angle θ if you know the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$. It is the reverse of the sine. It is often good to think of $y = \sin^{-1} x$ as "the inverse sine of x is the angle whose sine is x."

On your calculator these functions are not displayed as arc functions. Your calculator probably has keys that look like: $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$ and $\boxed{\tan^{-1}}$. These features are often found just above the regular trigonometric function, but different models of calculator have it in different places.

The inverse sine function $y = \sin^{-1} x = \arcsin x$

The sine has restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and range $-1 \leq \sin x \leq 1$. The inverse sine is the function whose domain is $-1 \leq x \leq 1$ and whose range is $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$ such that

$$\sin\left(\sin^{-1} x\right) = x \quad \text{for} \quad -1 \leq x \leq 1$$

and

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

There are a couple of important things to remember here.

Note 1: With the restriction we have put on the inverse sine, it is ONLY defined in quadrants I and IV so all your answers for arcsine must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Note 2: The notation for inverse functions is to have an exponent of -1 on the function. This should not be confused with the reciprocal of the function. If we want the reciprocal of the sine we would write it one of the following ways:

$$\frac{1}{\sin x} = (\sin x)^{-1} = \csc x.$$

For this reason some prefer to write $y = \arcsin x$ and both are often used interchangeably without warning.

The cosine is similar but in this case we restrict the domain to $0 \leq x \leq \pi$ because this also gives us all the y values between 1 and 1. **Figure:** 2.22 It is often good to think of $y = \cos^{-1} x$ as "the inverse cosine of x is the angle whose cosine is x."

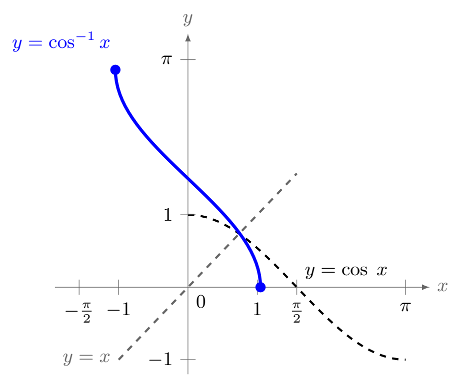


Figure 2.22: Graph of $y = \cos^{-1} x = \arccos x$

Again we reflect this dotted curve across the line $y = x$ to get the inverse cosine function.

Note: With the restriction we have put on the arccosine, it is ONLY defined in quadrants I and II so all your answers for arccosine must lie between 0 and π.

The tangent has the same restrictions as the sine but in this case we have vertical asymptotes. When we reflect across the line $y = x$ the vertical asymptotes become horizontal asymptotes.

It is often good to think of $y = \tan^{-1} x$ as "the inverse tangent of x is the angle whose tangent is x." See **Figure: 2.23**.

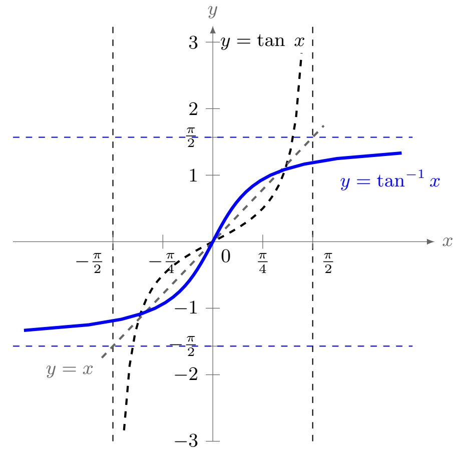


Figure 2.23: Graph of $y = \tan^{-1} x = \arctan x$

Note: With the restriction we have put on the arctangent, it is ONLY defined in quadrants I and IV so all your answers for arccosine must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

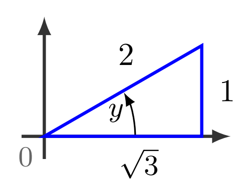
Summary of inverse trigonometric functions

Function	Definition	In Words	Range
$\sin^{-1} x = y$	$x = \sin y$	y is the angle whose sine is x	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1} x = y$	$x = \cos y$	y is the angle whose cosine is x	$0 \leq y \leq \pi$
$\tan^{-1} x = y$	$x = \tan y$	y is the angle whose tangent is x	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Example 2.3.1

Find y when $y = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

Solution: Step 1: Draw a triangle in the appropriate quadrant and label the sides. Since $\cos^{-1}$ is defined in quadrant I and II and $\dfrac{\sqrt{3}}{2}$ is positive, we draw the triangle in quadrant I (See Figure at right). Since we have the arccosine here we know that $\dfrac{\sqrt{3}}{2} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ and we can use the Pythagorean theorem to find the missing side.

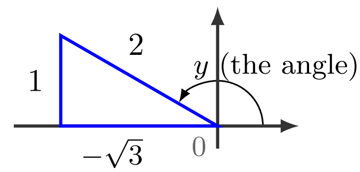


Step 2: Identify the angle in the triangle. Very often it will be one of the special triangles. In this case we have a $30 - 60 - 90$ triangle so our angle is

$$\boxed{y = 30^\circ} \text{ or } \boxed{y = \tfrac{\pi}{6}}.$$

Example 2.3.2

Find y when $y = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$



Solution: Step 1: Draw a triangle in the appropriate quadrant and label the sides. Since $\cos^{-1}$ is defined in quadrant I and II and $-\dfrac{\sqrt{3}}{2}$ is negative, we draw the triangle in quadrant II (See Figure at right). Since we have the arccosine here we know that $-\dfrac{\sqrt{3}}{2} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ and we can use the Pythagorean theorem to find the missing side.

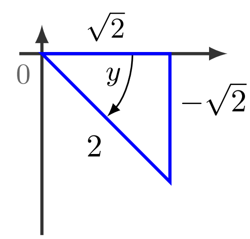
Step 2: Identify the angle in the triangle. In this case we have a $30-60-90$ triangle again so our reference angle is $\frac{\pi}{6}$ and the answer is

$$\boxed{y = \tfrac{5\pi}{6}}.$$

Notice that this falls in the range we want for answers to arccosine problems: $0 \le y \le \pi$.

Example 2.3.3

Find y when $y = \sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$



Solution: Step 1: Draw a triangle in the appropriate quadrant and label the sides. Since $\sin^{-1}$ is defined in quadrant I and IV and $-\dfrac{\sqrt{2}}{2}$ is negative, we draw the triangle in quadrant IV (See Figure at right). Since we have the arcsine here we know that $-\dfrac{\sqrt{2}}{2} = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and we can use the Pythagorean theorem to find the missing side.

Step 2: Identify the angle in the triangle. In this case we have a $45-45-90$ triangle so our reference angle is $\frac{\pi}{4}$ and the answer is

$$\boxed{y = -\frac{\pi}{4}}.$$

Notice that this falls in the range we want for answers to arcsine problems: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Example 2.3.4

Evaluate $\sin^{-1}(0.97)$ using your calculator.

Solution: Since the output of the inverse function is an angle, your calculator will give you a degree value if in degree mode, and a radian value if in radian mode.

In radian mode, $\boxed{\sin^{-1}(0.97) \approx 1.3252}$ In degree mode, $\boxed{\sin^{-1}(0.97) \approx 75.93^\circ}$

Example 2.3.5

Evaluate $\cos^{-1}\left(\cos\left(\dfrac{13\pi}{6}\right)\right)$

Solution: Here we want to be careful. The cosine and arccosine are direct inverses of each other only between 0 and π so our answer can't be $\dfrac{13\pi}{6}$. What we need to do is to first find the value of $\cos\left(\dfrac{13\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$. Once we know this we are now looking for $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ which we found in **Example 2.3.1**. So

$$\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{6}}$$

Example 2.3.6

Find $\tan^{-1}(\tan\pi)$.

Solution: Since $\pi > \frac{\pi}{2}$, tangent and arctangent are not direct inverses. But we know that $\tan\pi = 0$. Thus, $\tan^{-1}(\tan\pi) = \tan^{-1} 0$ is, by definition, the angle y such that $\tan y = 0$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. That angle is $y = 0$. Thus, $\boxed{\tan^{-1}(\tan\pi) = \tan^{-1}(0) = 0}$.

Example 2.3.7

Evaluate $\sin^{-1} 0$

Solution: We need to find an angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ such that $\sin\theta = 0$. The only angle that satisfies this is $\boxed{\theta = 0}$.

Example 2.3.8

Evaluate $\tan^{-1}(-1)$

Solution: We need to find an angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ such that $\tan\theta = -1$. The answer will be in QIV: $\boxed{\theta = -\frac{\pi}{4}}$.

Example 2.3.9

Evaluate $\cos\left(\sin^{-1}\left(-\dfrac{3}{4}\right)\right)$

Solution: We could do this problem the way we did the earlier examples where we drew a triangle but another solution is to use one of our pythagorean identities from Section 1.2.

$$\cos^2\theta + \sin^2\theta = 1$$

Let $\theta = \sin^{-1}\left(-\frac{3}{4}\right)$. Since $\sin(\theta) = -\frac{3}{4}$ we know that θ is in QIV so $\cos\theta > 0$ (positive). Using our identity we can now calculate

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(-\frac{3}{4}\right)^2 = \frac{7}{16} \quad\Longrightarrow\quad \cos\theta = \frac{\sqrt{7}}{4}.$$

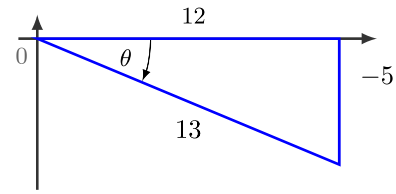
Note that we took the positive square root since $\cos\theta > 0$. Thus our answer is

$$\boxed{\cos\left(\sin^{-1}\left(-\frac{3}{4}\right)\right) = \frac{\sqrt{7}}{4}}$$

Example 2.3.10

Evaluate $\csc\left(\tan^{-1}\left(-\frac{5}{12}\right)\right)$

Solution: This problem is similar to **Example 2.3.9** but for this one we will construct a triangle to show a different way to arrive at the solution.



Let $\theta = \tan^{-1}\left(-\frac{5}{12}\right)$. Then we can draw a triangle for θ in QIV since

$$\tan\theta = -\frac{5}{12} = \frac{\text{opposite}}{\text{adjacent}}.$$

Using the Pythagorean theorem we can find the hypotenuse length of 13. Now we can read the cosecant off the triangle. $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \boxed{\dfrac{13}{-5}}$

Example 2.3.11

Find a simplified expression for $\tan\left(\sin^{-1} x\right)$ for $-1 < x < 1$

Solution: Let $\theta = \sin^{-1} x$. Then we can draw a triangle for θ since we know that

$$\sin\theta = \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

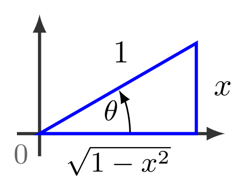


Figure 2.24

There are two triangles we can draw, one in QI for $0 < x < 1$ and one in QIV for $-1 < x < 0$ but the adjacent side length is the same for both so only the one in QI is presented in **Figure 2.24**.

The adjacent side length is calculated using the Pythagorean theorem and is $\sqrt{1-x^2}$. Notice that since the x is squared it is always positive no matter the sign of x. Then we can read the tangent right off the graph and

$$\boxed{\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}} \text{ for } -1 < x < 1}$$

Example 2.3.12

A cellular telephone tower that is 50 meters tall is placed on top of a mountain that is 1200 meters above sea level. What is the angle of depression to two decimal places from the top of the tower to a cell phone user who is 5 horizontal kilometers away and 400 meters above sea level?

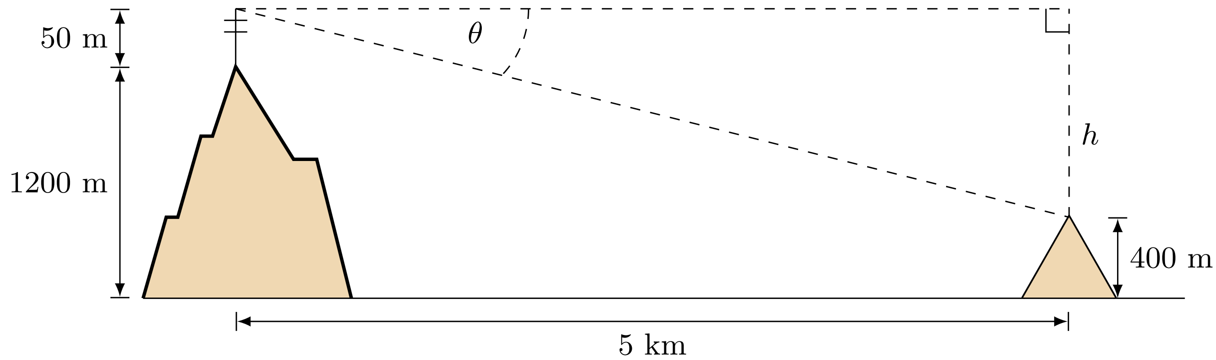


Figure 2.25

Solution: Figure 2.25 above describes the situation. We need to measure the distance from the top of the hill to the top of the cellular tower marked h. Thus $h = 1200+50-400 = 850$m. We also need to convert the horizontal distance to meters, 5 km = 5000 m and we can use the tangent function to write an equation relating the height and the adjacent side:

$$\frac{850}{5000} = \tan\,\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{850}{5000}\right) = \boxed{9.65^\circ}.$$

We can calculate the inverse function by using a calculator, the inverse button looks something like: $\boxed{\tan^{-1}}$. Again, be careful that your calculator is in degree mode.

2.3 Exercises

For Exercises 1-28, find the exact value of the given expression. If an answer is an angle answer in radians.

1. $\tan^{-1} 1$

2. $\tan^{-1}(-1)$

3. $\tan^{-1} 0$

4. $\cos^{-1} 1$

5. $\cos^{-1}(-1)$

6. $\cos^{-1} 0$

7. $\sin^{-1} 1$

8. $\sin^{-1}(-1)$

9. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

10. $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

11. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

12. $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$

13. $\sin^{-1} 0$

14. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

15. $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$

16. $\sin^{-1}\left(\sin\left(-\frac{4\pi}{3}\right)\right)$

17. $\cos^{-1}\left(\cos\frac{\pi}{5}\right)$

18. $\cos^{-1}\left(\cos\frac{6\pi}{5}\right)$

19. $\cos^{-1}\left(\cos\left(-\frac{\pi}{5}\right)\right)$

20. $\tan^{-1}\left(\tan\left(-\frac{5\pi}{6}\right)\right)$

21. $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$

22. $\cos^{-1}\left(\sin\frac{13\pi}{6}\right)$

23. $\sin^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$

24. $\csc^{-1}\left(\sec\left(-\frac{5\pi}{6}\right)\right)$

25. $\tan\left(\sin^{-1}\frac{4}{3}\right)$

26. $\sin\left(\tan^{-1}\frac{4}{3}\right)$

27. $\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

28. $\cos\left(\sin^{-1}\left(-\frac{4}{5}\right)\right)$

29. Find a simplified expression for $\cos\left(\sin^{-1} x\right)$ for $-1 \le x \le 1$.

30. Find a simplified expression for $\cot\left(\sin^{-1}\left(\dfrac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

31. Find a simplified expression for $\sin\left(\cos^{-1}\frac{x}{3}\right)$ for $-3 < x < 3$.

32. Find a simplified expression for $\csc\left(\tan^{-1}\left(\dfrac{x}{2}\right)\right)$.

33. The height of a playground basketball backboard is 12 feet 6 inches high. At 4:00 pm it casts a shadow 15 feet long. What is the angle of elevation of the sun at that time?

2.4 Solving Trigonometric Equations

To solve a trigonometric equation we use standard algebraic techniques such as combining like terms and factoring. The first goal for any trigonometric equation is to **isolate** the trigonometric function in the equation. We can't algebraically solve for the variable from inside a trigonometric function.

Example 2.4.1

Solve the equation $2\sin x + 1 = 0$

Solution: We have to have the trigonometric function by itself on one side of the equation and the numbers on the other side. In this case we solve for

$$\sin x = -\frac{1}{2}.$$

When we have it in this form we can then decide what values of x will work here. In the section on inverse functions (Section 2.3) we saw that we could ask our calculator for the value

$$x = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

While it is true that this value of x satisfies the equation, it is not the complete solution. If we plot the graph of $y = \sin x$ and $y = -\frac{1}{2}$ on the same set of axes we can find where they intersect. Four of the solutions (x, y) are labeled $(\frac{7\pi}{6}, -\frac{1}{2})$, $(\frac{11\pi}{6}, -\frac{1}{2})$, $(-\frac{\pi}{6}, -\frac{1}{2})$, $(-\frac{5\pi}{6}, -\frac{1}{2})$

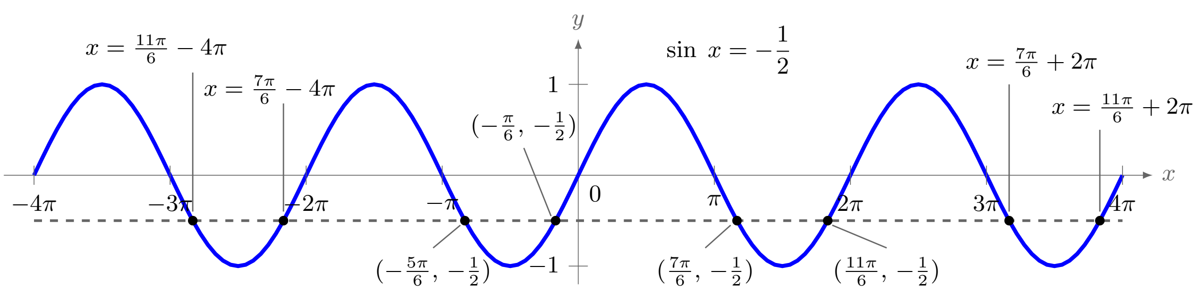


Figure 2.26: Intersections of $y = \sin x$ and $y = -\frac{1}{2}$

and four more are indicated as multiples of 2π. There are an infinite number of solutions because the graph of sine continues indefinitely in both directions and the line $y = -\frac{1}{2}$ will intersect it an infinite number of times.

We need to have a way to describe all the solutions. Since the sine is periodic we know that it repeats every 2π so our solutions will repeat every 2π. We find all the positive solutions on one time around the circle $x = \dfrac{7\pi}{6}$ and $x = \dfrac{11\pi}{6}$ and then add multiples of 2π to it. Our two solutions can be written:

$$\boxed{x = \frac{7\pi}{6} + 2n\pi \qquad \text{and} \qquad x = \frac{11\pi}{6} + 2n\pi, \qquad \text{where} \;\; n \in \mathbb{Z}}$$

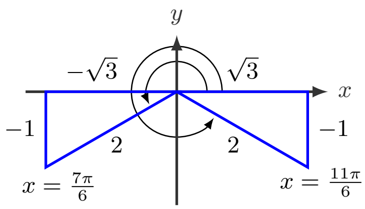


Figure 2.27: Reference triangles for Example 2.4.1

The graph of sine and cosine are not convenient for finding the x-values that satisfy the equation. Most often reference triangles or the unit circle are used. In the previous example we wanted the solutions to $\sin x = -\dfrac{1}{2} = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and we can draw two reference triangles that satisfy this angle. We need two because the sine is negative in both QIII and QIV. See **Figure 2.27**.

These two triangles are recognizable as our 30-60-90 triangle and as such we can find the reference angle $\dfrac{\pi}{6}$ and the two basic solutions $x = \dfrac{7\pi}{6}$ and $x = \dfrac{11\pi}{6}$. From there the complete solution can be written as above.

$$\boxed{x = \frac{7\pi}{6} + 2n\pi \qquad \text{and} \qquad x = \frac{11\pi}{6} + 2n\pi, \qquad \text{where} \quad n \in \mathbb{Z}}$$

Recall that the integers are represented by the symbol $\mathbb{Z}$ =0, ±1, ±2, ±3, ...

Example 2.4.2

Solve the equation $3\cot^2 x - 1 = 0$

Solution: Here we need to isolate the $\cot x$ on one side of the equation.

$$\begin{aligned} 3\cot^2 x - 1 &= 0 \\ 3\cot^2 x &= 1 \\ \cot^2 x &= \frac{1}{3} \\ \cot x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

There are two solutions because you always need to take into account both the positive and negative answers when taking a square root. We can draw reference triangles for these two solutions. There are 4 we could draw for $0 \leq x < 2\pi$, one in each quadrant. **Figure 2.28** shows the solutions in QI and QIII for $\cot x = \frac{1}{\sqrt{3}}$ and solutions in QII and QIV for the negative.

Cotangent has a period of π so we can start with two basic solutions $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$. Then add multiples of π to each of these to get the general form:

$$\boxed{x = \frac{\pi}{3} + n\pi \qquad \text{and} \qquad x = \frac{2\pi}{3} + n\pi, \qquad \text{where} \quad n \in \mathbb{Z}}$$

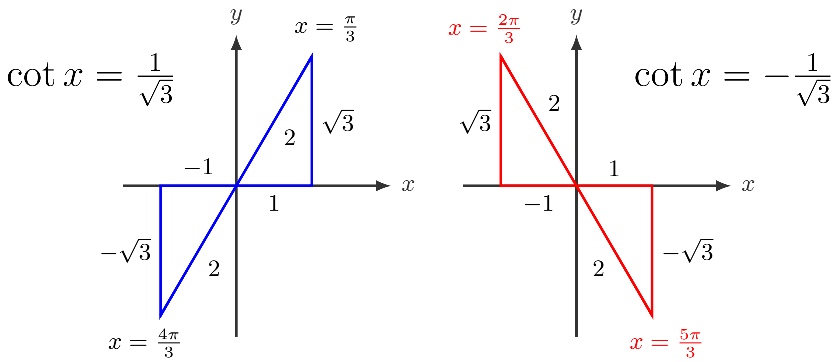


Figure 2.28: Blue reference triangles on left for $\cot x = \frac{1}{\sqrt{3}}$, red for $\cot x = -\frac{1}{\sqrt{3}}$

Example 2.4.3

Solve the equation $2\cos^2\theta - 1 \ = \ 0$.

Solution: Isolating $\cos^2\theta$ gives us

$$\cos^2\ \theta \ = \ \frac{1}{2} \quad \Rightarrow \quad \cos\ \theta \ = \ \pm\ \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta \ = \ \frac{\pi}{4}\,,\ \frac{3\pi}{4}\,,\ \frac{5\pi}{4}\,,\ \frac{7\pi}{4}\,,$$

and since the period of cosine is 2π, we would add $2n\pi$ to each of those angles to get the general solution. But notice that the above angles differ by multiples of $\frac{\pi}{2}$. Since every multiple of 2π is also a multiple of $\frac{\pi}{2}$, we can combine those four separate answers into one:

$$\boxed{\theta \ = \ \frac{\pi}{4} \ + \ \frac{\pi}{2}\,n \qquad \text{for } n \in \mathbb{Z}}$$

Example 2.4.4

Solve the equation $4\cos^3 x - 3\cos x = 0$

Solution: This equation will require some factoring. In our previous examples we were able to isolate a squared term and then take a square root. In this case that won't be possible.

$$\begin{aligned} 4\cos^3 x - 3\cos x \ &= \ 0 \\ \cos x\left(4\cos^2 x - 3\right) \ &= \ 0 \end{aligned}$$

Now we have two things multiplied together that equal zero so one of them must be zero. Set each factor equal to zero and find all the solutions between $0 \le x < 2\pi$.

$$\cos x = 0 \qquad \text{and} \qquad \begin{aligned} 4\cos^2 x - 3 &= 0 \\ \cos^2 x &= \frac{3}{4} \\ \cos x &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

$$\cos x = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \text{and} \qquad x = \frac{\pi}{6}, \frac{11\pi}{6} \qquad \text{and} \qquad x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

Note that here $\frac{\pi}{6}$ and $\frac{7\pi}{6}$, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, as well as $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ are different by π so we can write our solutions as:

$$\boxed{x = \frac{\pi}{2} + n\pi, \qquad x = \frac{\pi}{6} + n\pi, \qquad \text{and} \qquad x = \frac{5\pi}{6} + n\pi, \qquad \text{where} \quad n \in \mathbb{Z}}$$

Example 2.4.5

Solve the equation $2\sin(5x) + 1 = 0$

Solution: This problem is similar to **Example 2.4.1** but now we have $(5x)$ in the sine. We need to isolate the $\sin(5x)$ and solve for the values of $5x$.

$$\begin{aligned} 2\sin(5x) + 1 &= 0 \\ 2\sin(5x) &= -1 \\ \sin(5x) &= -\frac{1}{2} \end{aligned}$$

In the interval $[0, 2\pi)$ we know that $5x = \frac{7\pi}{6} + 2n\pi$ and $5x = \frac{11\pi}{6} + 2n\pi$. We need to divide both sides by 5 to obtain the general solution:

$$\boxed{x = \frac{7\pi}{30} + \frac{2n\pi}{5}, \qquad x = \frac{11\pi}{30} + \frac{2n\pi}{5}, \qquad \text{where} \quad n \in \mathbb{Z}}$$

Example 2.4.6

Find all solutions on $[0, 2\pi)$.

$$2\sin^2 x + 5\sin x + 3 = 0$$

Solution: Here we need to factor the equation because it is a quadratic. Also, since we only want solutions on $[0, 2\pi)$ and the angle is just x then we don't need to write the solution with the $+2n\pi$.

$$\begin{aligned} 2\sin^2 x + 5\sin x + 3 &= 0 \\ (2\sin x + 3)(\sin x + 1) &= 0 \end{aligned}$$

Now we have two things multiplied together that equal zero so one of them must be zero. Set each factor equal to zero and find all the solutions between $0 \leq x < 2\pi$.

$$\begin{array}{rcrcl} 2\sin x + 3 = 0 & \quad\text{and}\quad & \sin x + 1 & = & 0 \\ \sin x = -\frac{3}{2} & & \sin x & = & -1 \\ \text{No solution because } -1 \leq \sin x \leq 1 & \quad\text{and}\quad & x & = & \dfrac{3\pi}{2} \end{array}$$

The only solution here is $\boxed{x = \dfrac{3\pi}{2}}$.

Example 2.4.7

Find all solutions on $[0, 2\pi)$.

$$2\sin^2(2x) = 1$$

Solution: As in **Example 2.4.5** we need to first solve for the value of $2x$ and then divide by two. We don't want the general solution but we do need to start with it to find all values of x on $[0, 2\pi)$.

We need to isolate the $\sin(2x)$ and solve for the values of $2x$.

$$\begin{aligned} 2\sin^2(2x) &= 1 \\ \sin^2(2x) &= \frac{1}{2} \\ \sin(2x) &= \pm\sqrt{\frac{1}{2}} \end{aligned}$$

There are two equations to solve: $\sin(2x) = \frac{1}{\sqrt{2}}$ and $\sin(2x) = -\frac{1}{\sqrt{2}}$ so we have 4 general solutions:

$$\begin{aligned} 2x = \frac{\pi}{6} + 2n\pi \quad &\Longrightarrow \quad x = \frac{\pi}{12} + n\pi \\ 2x = \frac{5\pi}{6} + 2n\pi \quad &\Longrightarrow \quad x = \frac{5\pi}{12} + n\pi \\ 2x = \frac{7\pi}{6} + 2n\pi \quad &\Longrightarrow \quad x = \frac{7\pi}{12} + n\pi \\ 2x = \frac{11\pi}{6} + 2n\pi \quad &\Longrightarrow \quad x = \frac{11\pi}{12} + n\pi \end{aligned}$$

To find all solutions on $[0, 2\pi)$ we will substitute values for n until we find all the solutions starting with $n = 0$:

$$n = 0: \quad x = \frac{\pi}{12} + (0)\pi = \frac{\pi}{12} \qquad n = 1: \quad x = \frac{\pi}{12} + (1)\pi = \frac{13\pi}{12}$$

$$x = \frac{5\pi}{12} + (0)\pi = \frac{5\pi}{12} \qquad x = \frac{5\pi}{12} + (1)\pi = \frac{17\pi}{12}$$

$$x = \frac{7\pi}{12} + (0)\pi = \frac{7\pi}{12} \qquad x = \frac{7\pi}{12} + (1)\pi = \frac{19\pi}{12}$$

$$x = \frac{11\pi}{12} + (0)\pi = \frac{11\pi}{12} \qquad x = \frac{11\pi}{12} + (1)\pi = \frac{23\pi}{12}$$

We don't need to go any further because any other answers will be larger than 2π. There are 8 possible solutions.

$$\boxed{x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}}$$

Example 2.4.8

There has been a murder at the Toronto docks. The coroner places the time of death around 8 AM. The main suspect claims she was on her boat fishing in Lake Ontario at the time and that she was waiting for the tide in order to tie up her boat. Detective Murdoch knows that the depth of water at the docks rises and falls with the tide, following the equation

$$f(t) = 4\sin\left(\frac{\pi}{12}t\right) + 7,$$

where t is measured in hours after midnight. The suspect's boat requires a depth of 9 feet to tie up at the dock. Between what times will the depth be 9 feet? Is the suspect lying?

Solution: To find when the depth is 9 feet, we need to solve $f(t) = 9 = 4\sin\left(\frac{\pi}{12}t\right) + 7$. We start by isolating the sine.

$$\begin{aligned} 4\sin\left(\frac{\pi}{12}t\right) + 7 &= 9 \\ 4\sin\left(\frac{\pi}{12}t\right) &= 2 \\ \sin\left(\frac{\pi}{12}t\right) &= \frac{1}{2} \end{aligned}$$

We know that $\sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ so the solutions to the equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$ are

$$\frac{\pi}{12}t = \frac{\pi}{6} + 2n\pi \qquad \text{and} \qquad \frac{\pi}{12}t = \frac{5\pi}{6} + 2n\pi \qquad n \in \mathbb{Z}$$

Multiply by $\frac{12}{\pi}$ to find the solutions $t = 2 + 24n$ and $t = 10 + 24n$. The boat will be able to approach the dock between 2AM and 10AM. Notice that because we have $+24n$ in each answer the cycle will repeat every day (24 hours). The suspect is lying about waiting for the tide at 8AM.

Example 2.4.9

Find all solutions to $\sin\theta = 0.8$.

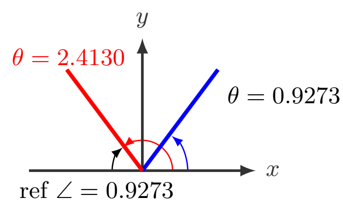


Figure 2.29: $\sin\theta = 0.8$

Solution: To find the solutions we will draw two reference angles. Since the sine is not one of the results for our special triangles we will use the inverse sine function here. When you ask your calculator for the inverse sine it will only give you one answer:

$$\theta = \sin^{-1}(0.8) \approx 0.9273.$$

Recall that the inverse sine is answering the question: "What angle has sine 0.8?" We know that on the interval $0 \le \theta < 2\pi$ there are two answers. The second answer is in QII as shown in **Figure 2.29**. The second answer can be found with the reference angle and the size of the second angle is

$$\theta \approx \pi - 0.9273 \approx 2.4130$$

To find all the solutions we add multiples of 2π.

$$\boxed{\theta = \sin^{-1}(0.8) + 2n\pi, \qquad \theta = \pi - \sin^{-1}(0.8) + 2n\pi, \qquad \text{where } \ n \in \mathbb{Z}}$$

$$\boxed{\theta \approx 0.9273 + 2n\pi, \qquad \theta = 2.4130 + 2n\pi, \qquad \text{where } \ n \in \mathbb{Z}}$$

2.4 Exercises

For Exercises 1-6, find all solutions on the interval $[0, 2\pi)$. Leave exact answers in radians.

1. $2\sin x = \sqrt{2}$

2. $2\sin x + \sqrt{3} = 0$

3. $\csc x = -2$

4. $\cos\theta = 0$

5. $2\cos\theta + 1 = 0$

6. $\tan(\theta) - \sqrt{3} = 0$

For Exercises 7-12, find the general solution for each equation. Leave exact answers in radians.

7. $\tan\theta + 1 = 0$

8. $2\sin x - 1 = 0$

9. $2\cos x = \sqrt{3}$

10. $\sqrt{3}\sec x = 2$

11. $\sin\theta = 0$

12. $\sqrt{3}\cot(x) - 1 = 0$

For Exercises 13-18, find all solutions on the interval $[0, 2\pi)$. Leave exact answers in radians.

13. $\sin(2\theta) - 1 = 0$

14. $\tan(2x) = -1$

15. $\sqrt{3}\csc\left(\frac{x}{2}\right) = -2$

16. $2\sin(2\theta) + 2 = 1$

17. $2\cos^2(2\theta) = 1$

18. $\cos(3x) = \dfrac{\sqrt{2}}{2}$

For Exercises 19-28, find all solutions on the interval $[0, 2\pi)$. Leave exact answers in radians.

19. $\tan\theta\,(\tan\theta + 1) = 0$

20. $\cot^2 x = 3$

21. $\tan x \sin x - \sin x = 0$

22. $2\cos^2 x + 3\cos x + 1 = 0$

23. $(4\sin^2 x - 3)(\sqrt{2}\cos x + 1) = 0$

24. $\sin x\,(\sec x + 2) = 0$

25. $2\sin^2 x + \sin x - 1 = 0$

26. $2\sin^3 x = \sin x$

27. $\tan^5 x = \tan x$

28. $2\cos^2 x - \sin x = 1$

For Exercises 29-34, use a calculator to find all solutions on the interval $[0, 2\pi)$. Round answers to 4 decimal places.

29. $7\sin\theta = 2$

30. $\cos x = -0.27$

31. $\tan x = 9.27$

32. $7\sin(2\theta) = 2$

33. $\sec^2 x = 7$

34. $\tan(\pi x) = 9.27$

35. An observer views a rocket take off from a distance of 7 km from the launch pad, and tracks the angle of elevation. Express the height of the rocket as a function of the angle of elevation, θ. Express the angle of elevation θ as a function of the height, h, of the rocket. When the height of the rocket is 22 km what is the angle of elevation?

36. The height of a rider on the London Eye Ferris wheel can be determined by the equation $h(t) = -67.5\cos\left(\dfrac{\pi}{15}t\right) + 69.5$. How long is the rider more than 100 meters above ground?

Chapter 3

Trigonometric Identities

3.1 Fundamental Identities

Recall in Sections 1.2 and 1.4 we saw some fundamental identities. There were the reciprocal identities, the pythagorean identities and the negative angle identities which are summarized here.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad (3.1)$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad (3.3)$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad (3.2)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad (3.4)$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad (3.5)$$

$$1 + \tan^2\theta = \sec^2 \qquad (3.6)$$

$$1 + \cot^2\theta = \csc^2\theta \qquad (3.7)$$

Negative Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

We also proved the pythagorean identities. This meant that we showed they the were true for all angles θ. We can use these identities to simplify more complicated trigonometric equations.

Simplifying Expressions

Example 3.1.1

Simplify $\cos^2\theta\tan^2\theta$

Solution: We can use identity (3.4) to simplify

$$\begin{aligned}\cos^2\theta\tan^2\theta &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\\ &= \sin^2\theta\end{aligned}$$

Example 3.1.2

Simplify $\cot^2\theta - \csc^2\theta$

Solution: In this example we have squared terms with addition or subtraction so it is going to be easiest to try to use one of the Pythagorean identities. In this case we will use identity (3.7).

$$\begin{aligned}\cot^2\theta - \csc^2\theta &= \cot^2\theta - \left(1+\cot^2\theta\right)\\ &= \cot^2\theta - 1 - \cot^2\theta\\ &= -1\end{aligned}$$

Example 3.1.3

Simplify $\dfrac{\sec^2 x - 1}{\sin^2 x}$

Solution: To simplify we will use identities (3.6), (3.4), and (3.3).

$$\begin{aligned}\frac{\sec^2 x - 1}{\sin^2 x} &= \frac{\tan^2 x}{\sin^2 x} && \text{identity (3.6)}\\ &= \tan^2 x\left(\frac{1}{\sin^2 x}\right)\\ &= \left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\frac{1}{\sin^2 x}\right) && \text{identity (3.4)}\\ &= \left(\frac{1}{\cos^2 x}\right)\\ &= \sec^2 x && \text{identity (3.3)}\end{aligned}$$

So $\boxed{\dfrac{\sec^2 x - 1}{\sin^2 x} = \sec^2 x}$

Sometimes a problem requires factorization as well:

Example 3.1.4

Factor and simplify $\tan^4 x + 2\tan^2 x + 1$

Solution: The trick to simplifying this problem to see that it is a quadratic equation in $\tan^2 x$. To see this more clearly we will do a 'u- substitution'. In this case we will let $u = \tan^2 x$ then $u^2 = \tan^4 x$. Then we can substitute into our original equation to get a quadratic equation in u:

$$\begin{aligned}\tan^4 x + 2\tan^2 x + 1 &= u^2 + 2u + 1\\ &= (u+1)(u+1)\\ &= (u+1)^2\end{aligned}$$

But we do not want a solution in u so we have to substitute for $u = \tan^2 x$ to get

$$\begin{aligned}\tan^4 x + 2\tan^2 x + 1 &= \left(\tan^2 x + 1\right)^2\\ &= \left(\sec^2 x\right)^2\\ &= \sec^4 x\end{aligned}$$

Example 3.1.5

Factor and simplify $\sin^2 x \sec^2 x - \sin^2 x$

Solution: Here we will factor the common factor $\sin^2 x$ and then apply the identity (3.6).

$$\begin{aligned}\sin^2 x \sec^2 x - \sin^2 x &= \sin^2 x\left(\sec^2 x - 1\right)\\ &= \sin^2 x \tan^2 x\end{aligned}$$

There is no more simplification that can be done to this equation. Nothing we do here will make the equation simpler in terms of only one trigonometric function.

Example 3.1.6

Simplify $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$

Solution: To combine the fractions we need to find a common denominator. In this case the common denominator is the product of the two denominators: $(\sec x + 1)(\sec x - 1)$. Once we have simplified the expression we apply identity (3.6).

$$\begin{aligned}
\frac{1}{\sec x + 1} - \frac{1}{\sec x - 1} &= \left(\frac{1}{\sec x + 1}\right)\left(\frac{\sec x - 1}{\sec x - 1}\right) - \left(\frac{1}{\sec x - 1}\right)\left(\frac{\sec x + 1}{\sec x + 1}\right) \\
&= \frac{(\sec x - 1) - (\sec x + 1)}{(\sec x + 1)(\sec x - 1)} \\
&= \frac{\sec x - 1 - \sec x - 1}{(\sec^2 x - 1)} \\
&= \frac{-2}{\tan^2 x}
\end{aligned}$$

If you would prefer to have that written without any fractions you can write:

$$\frac{1}{\sec x + 1} - \frac{1}{\sec x - 1} = -2\cot^2 x$$

Proving Identities

If we want to prove an identity we want to show that it is true for all values. If we have an equation and we want to know if it is an identity we work with one side and try to make it look like the other.

Example 3.1.7

Use trigonometric identites to transform the left side of the equation into the right side.

$$\cos x \sec x = 1$$

Solution: We will work with the left side. Convert everything to cos(x).

$$\cos x \sec x = \cos x \frac{1}{\cos x} = 1$$

So the identity is true.

Example 3.1.8

Use trigonometric identites to transform the left side of the equation into the right side.

$$\sin^2 x - \cos^2 x = 2\sin^2 x - 1$$

Solution: For this problem we need to use one of our Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$$

Now we take this expression for $\cos^2(x)$ and substitute into the original equation:

$$\begin{aligned}\sin^2 x - \cos^2 x &= \sin^2 x - (1 - \sin^2 x)\\ &= \sin^2 x - 1 + \sin^2 x\\ &= 2\sin^2 x - 1\end{aligned}$$

So the statement is true.

3.1 Exercises

For Exercises 1 -12 simplify each expression to an expression involving a single trigonometric function with no fractions.

1. $\dfrac{\tan x}{\sec x \sin x}$

2. $\csc x \tan x$

3. $\dfrac{\sec t}{\csc t}$

4. $\dfrac{1+\tan x}{1+\cot x}$

5. $\dfrac{1+\csc t}{1+\sin t}$

6. $\dfrac{1-\sin^2 x}{1+\sin x}$

7. $\dfrac{\cos\theta}{\sin^2\theta}$

8. $\dfrac{\sin\theta}{\cos^2\theta}$

9. $\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}$

10. $\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\cos\theta}$

11. $\dfrac{\sec\theta}{\tan\theta}$

12. $\dfrac{\tan x}{\cot x}$

For Exercises 13 - 18, use trigonometric identites to transform the left side of the equation into the right side.

13. $\cot\theta\tan\theta = 1$

14. $\cot\theta\sin\theta = \cos\theta$

15. $(1+\sin\alpha)(1-\sin\alpha) = \cos^2\alpha$

16. $(\sec\alpha+\tan\alpha)(\sec\alpha-\tan\alpha) = 1$

17. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2$

18. $\cos^2\theta - \sin^2\theta = 2\cos^2 -1$

3.2 Proving Identities

In this section we will be studying techniques for verifying trigonometric identities. We need to show that each of these equations is true for all values of our variable. There is no well defined set of rules for how to verify an identity but we do have some guidelines we can use.

Guidelines for Verifying Trigonometric Identities

1. Only work with one side of the equation at a time. It is usually better to work with the more complicated side first.
2. Use algebraic techniques: Factor an expression, add fractions, expand an expression, or multiply by a conjugate to create a simpler expression.
3. Look for ways to use the fundamental identities from section 3.1. Pay attention to what is in the expression you want. Sines and cosines work well together, as do secants and tangents, as do cosecants and cotangents.
4. Convert everything to sines and cosines and then use the fundamental identities.
5. Always try something. Even paths that don't end up where you want may provide insight.

NOTE: When you verify an identity you cannot assume that both sides of the equation are equal because you are trying to verify that they are equal. This means that you cannot use operations that do the same thing to both sides of the equation such as multiplying the same quantity to both sides or cross multiplication.

Example 3.2.1

Verify the identity $\cos x + \sin x \tan x = \sec x$.

Solution: We will work with the left side of the equation, because it is more complicated, and make it look like the right side.

$$\begin{aligned}
\cos x + \sin x \tan x &= \cos x + \sin x \left(\frac{\sin x}{\cos x}\right) && \text{identity (3.4)} \\
&= \cos x \left(\frac{\cos x}{\cos x}\right) + \sin x \left(\frac{\sin x}{\cos x}\right) && \text{common denominator} \\
&= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos x} \\
&= \frac{1}{\cos x} && \text{identity (3.5)} \\
&= \sec x
\end{aligned}$$

Example 3.2.2

Verify the identity $\dfrac{\sec x - 1}{1-\cos x} = \sec x$.

Solution 1: The left side is certainly more complicated so we will start there. The fraction doesn't have any squared terms so we can't use the Pythagorean identities and there isn't any algebraic simplification that can be done. We will convert the secant to cosine and then simplify.

$$\begin{aligned}
\frac{\sec x - 1}{1-\cos x} &= \frac{\frac{1}{\cos x} - 1}{1-\cos x} && \text{convert to cosine}\\
&= \frac{\left(\frac{1}{\cos x} - 1\right)(\cos x)}{(1-\cos x)(\cos x)} && \text{multiply by } 1 = \frac{\cos x}{\cos x}\\
&= \frac{1-\cos x}{(1-\cos x)(\cos x)} && \text{simplify}\\
&= \frac{1}{\cos x}\\
&= \sec x
\end{aligned}$$

Solution 2: We will show a different way to verify the identity. This method is longer but it illustrates that there is often more than one way to solve the problems. The fraction doesn't have any squared terms so we can't use the Pythagorean identities, however, we can multiply by the conjugate of the denominator to make it look like a Pythagorean identity. Remember that $(a+b)(a-b) = a^2 - b^2$ so here if we multiply $(1-\cos x)(1+\cos x) = 1-\cos^2 x$. This technique is known as "multiplying by the conjugate." A conjugate is an expression where the sign has been changed. The conjugate of $a+b$ is $a-b$ and vice versa. We can't just multiply the denominator by something because that changes the problem. What we need to do is multiply by a clever form of 1. We will multiply by $1 = \dfrac{1+\cos x}{1+\cos x}$.

$$\begin{aligned}
\frac{\sec x - 1}{1-\cos x} &= \left(\frac{\sec x - 1}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right) && \text{multiply by 1}\\
&= \frac{\sec x + (\sec x)(\cos x) - \cos x - 1}{1-\cos^2 x}\\
&= \frac{\frac{1}{\cos x} + \left(\frac{1}{\cos x}\right)(\cos x) - \cos x - 1}{1-\cos^2 x} && \text{reciprocal identity}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\dfrac{1}{\cos x} + 1 - \dfrac{\cos^2 x}{\cos x} - 1}{1 - \cos^2 x} && \text{Simplify and find common denominator} \\
&= \frac{\dfrac{1 - \cos^2 x}{\cos x}}{1 - \cos^2 x} && \text{Simplify} \\
&= \frac{1 - \cos^2 x}{(\cos x)(1 - \cos^2 x)} && \text{Simplify} \\
&= \frac{1}{\cos x} && \text{Simplify} \\
&= \sec x
\end{aligned}
$$

Example 3.2.3

Verify the identity $\dfrac{\sec x + \tan x}{\sec x - \tan x} = (\sec x + \tan x)^2$

Solution: Here we will work with the left side and multiply by the conjugate of the denominator. We need to multiply by $1 = \dfrac{\sec x + \tan x}{\sec x + \tan x}$ and then use identity (3.6).

$$
\begin{aligned}
\frac{\sec x + \tan x}{\sec x - \tan x} &= \left(\frac{\sec x + \tan x}{\sec x - \tan x}\right)\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) && \text{multiply by 1} \\
&= \frac{(\sec x + \tan x)^2}{\sec^2 x - \tan^2 x} && \text{simplify} \\
&= \frac{(\sec x + \tan x)^2}{1} && \text{identity (3.6)} \\
&= (\sec x + \tan x)^2
\end{aligned}
$$

Example 3.2.4

Verify the identity $\dfrac{\sin x \cos x}{\sin x - \cos x} = \cos x - \dfrac{\cos x}{1 - \tan x}$

Solution: Neither side of this problem looks simple but the right hand side involves two fractions. That is more complicated than the one on the left so we will begin there.

$$
\cos x - \frac{\cos x}{1 - \tan x} = \cos x - \frac{\cos x}{1 - \dfrac{\sin x}{\cos x}} \qquad \text{Convert to sine and cosine}
$$

$$= \cos x - \frac{(\cos x)(\cos x)}{\left(1 - \dfrac{\sin x}{\cos x}\right)(\cos x)} \qquad \text{Multiply by } 1 = \frac{\cos x}{\cos x}$$

$$= \cos x - \frac{\cos^2 x}{\cos x - \sin x} \qquad \text{Simplify}$$

$$= \frac{(\cos x)(\cos x - \sin x)}{\cos x - \sin x} - \frac{\cos^2 x}{\cos x - \sin x} \qquad \text{Find a common denominator}$$

$$= \frac{\cos^2 x - \cos x \sin x - \cos^2 x}{\cos x - \sin x} \qquad \text{Combine the fractions}$$

$$= \frac{(-\cos x \sin x)(-1)}{(\cos x - \sin x)(-1)} \qquad \text{Simplify and multiply by } \frac{-1}{-1}$$

$$= \frac{\sin x \cos x}{\sin x - \cos x}$$

Example 3.2.5

Prove that $\dfrac{\tan^2\theta + 2}{1 + \tan^2\theta} = 1 + \cos^2\theta$.

Solution: Expand the left side:

$$\frac{\tan^2\theta + 2}{1 + \tan^2\theta} = \frac{(\tan^2\theta + 1) + 1}{1 + \tan^2\theta}$$

$$= \frac{\sec^2\theta + 1}{\sec^2\theta} \qquad \text{by identity (3.6)}$$

$$= \frac{\sec^2\theta}{\sec^2\theta} + \frac{1}{\sec^2\theta} \qquad \text{separate fractions.}$$

$$= 1 + \cos^2\theta \qquad \text{reciprocal identity}$$

Example 3.2.6

Verify the identity $\dfrac{1}{\sec x \tan x} = \csc x - \sin x$

Solution: We will begin on the left side by converting to sines and cosines

$$\frac{1}{\sec x \tan x} = \frac{1}{\sec x} \cdot \frac{1}{\tan x} \qquad \text{write as two fractions}$$

$$\begin{aligned}
&= \cos x\left(\frac{\cos x}{\sin x}\right) && \text{Convert to sine and cosine}\\
&= \frac{\cos^2 x}{\sin x} && \text{Simplify}\\
&= \frac{1-\sin^2 x}{\sin x} && \text{identity (3.5)}\\
&= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x} && \text{Write as separate fractions}\\
&= \csc x - \sin x && \text{Reciprocal identities}
\end{aligned}$$

Example 3.2.7

Find all solutions to

$$\cos x + \sin x \tan x = 2$$

Solution: We need to be able to either factor this expression or write it in terms of a single trigonometric function. We saw in **Example 3.2.1** that this equation can be simplified to $\cos x + \sin x \tan x = \sec x$. Now we can solve it.

$$\sec x = 2 \qquad \Longrightarrow \qquad \cos x = \frac{1}{2} \qquad \Longrightarrow \qquad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The general solution is

$$\boxed{x = \frac{\pi}{3} + 2n\pi, \qquad x = \frac{5\pi}{3} + 2n\pi, \qquad \text{where} \qquad n \in \mathbb{Z}}$$

3.2 Exercises

For Exercises 1-6 simplify each expression to an expression involving a single trigonometric function with no fractions.

1. $\dfrac{1+\tan x}{1+\cot x}$

2. $\dfrac{1+\csc t}{1+\sin t}$

3. $\dfrac{1-\sin^2 x}{1+\sin x}$

4. $\dfrac{\sec\theta - \cos\theta}{\sin\theta}$

5. $\dfrac{\tan\theta}{\sec\theta - \cos\theta}$

6. $\dfrac{\sin x}{1+\cos x} + \dfrac{\cos x}{\sin x}$

For Exercises 7 - 28, use trigonometric identites to show the identity is true. Remember, you may only work with one side of the equation at a time so do not cross multiply.

7. $\csc\theta\,(\sin\theta+\cos\theta)=1+\cot\theta$

8. $\cos\theta\sec\theta-\sin^2\theta=\cos^2\theta$

9. $\sec\alpha-\tan\alpha=\dfrac{\cos\alpha}{1+\sin\alpha}$

10. $\dfrac{\cos^2\beta-\sin^2\beta}{1-\tan^2\beta}=\cos^2\beta$

11. $\dfrac{1-\tan^2 x}{1+\tan^2 x}=2\cos^2 x-1$

12. $\sec\theta-\dfrac{1}{\sec\theta}=\sin\theta\tan\theta$

13. $\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=\csc\theta\sec\theta$

14. $\dfrac{\sin\theta}{1+\sin\theta}-\dfrac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

15. $2\tan x-(1+\tan x)^2=-\sec^2 x$

16. $\tan^2\theta-3\sin\theta\tan\theta\sec\theta=-2\tan^2\theta$

17. $\dfrac{1}{\cos^2 x}-\dfrac{1}{\cot^2 x}=1$

18. $\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=\sec\theta\csc\theta$

19. $\tan x\,(\cot x-\cos x)=1-\sin x$

20. $\dfrac{\sec\theta\sin\theta}{\tan\theta}-1=0$

21. $\dfrac{\cos^2\theta}{1+\sin\theta}=1-\sin\theta$

22. $\cos x=1-\dfrac{\sin^2 x}{1+\cos x}$

23. $\dfrac{1+\sin x}{\cos x}=\dfrac{\cos x}{1-\sin x}$

24. $\tan^2 x=\dfrac{-\sin^2 x}{\sin^2 x-1}$

25. $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

26. $\csc x-\sin x=\cot x\cos x$

27. $\tan x-\cot x=\dfrac{1-2\cos^2 x}{\sin x\cos x}$

28. $\cos\theta+\dfrac{\sin^2\theta}{\cos\theta}=\sec\theta$

For Exercises 29 - 34, use trigonometric identites to simplify each equation, then find all solutions on $[0,2\pi)$. Leave your answers in radians.

29. $\cos^2 x\tan^2 x=1$

30. $\sin\theta=\cos\theta$

31. $2\cos^2 x-\sin x-1=0$

32. $\cos^2 x=-6\sin x$

33. $\tan x-3\sin x=0$

34. $2\tan^2\theta=3\sec\theta$

3.3 Sum and Difference Formulas

In this section we will study the use of several trigonometric identities and formulas. Some of the formulas will be proved but most will not. The proofs of the others are very similar. The proofs are found at the end of the section.

Sum and Difference Formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{3.8}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{3.9}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3.10}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{3.11}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3.12}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{3.13}$$

These formulas are very useful but it is important to understand that these are not algebraic properties like distributing or factoring. These are identities so you can either use the left side or the right side but you are not really doing algebra on the problem. In particular:

$$\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$$

These formulas can be used to rewrite expressions in other forms, or to rewrite an angle in terms of simpler angles.

Example 3.3.1

Find the exact value of $\cos 75°$.

Solution: Since $75° = 30° + 45°$ we can evaluate as $\cos 75° = \cos(30° + 45°)$

$$\begin{aligned}
\cos 75° &= \cos(30° + 45°) \\
&= \cos(30°)\cos(45°) - \sin(30°)\sin(45°) \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

We leave our answers in an exact form. If you want to verify the answer you can use your calculator to see that $\boxed{\cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}}$. This is not the only way to solve this problem. We could have used $75° = 120° - 45°$ and used the difference formula instead. The answer would of course be the same.

Example 3.3.2

Find the exact value of $\sin\left(\dfrac{7\pi}{6}-\dfrac{\pi}{3}\right)$ using the difference formula.

Solution: The difference formula is

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

where $\alpha=\dfrac{7\pi}{6}$ and $\beta=\dfrac{\pi}{3}$. Since α is in QIII and it has values that we can easily find, we will draw a reference triangle (**Figure 3.1**) so we can evaluate the sine and cosine. Alternatively we could have used the Unit Circle.

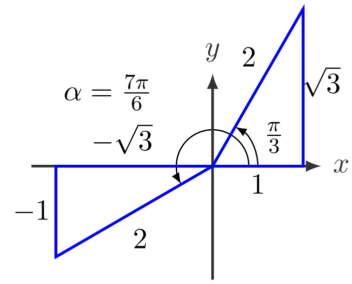


Figure 3.1

So $\sin\left(\dfrac{7\pi}{6}-\dfrac{\pi}{3}\right)=\sin\left(\dfrac{7\pi}{6}\right)\cos\left(\dfrac{\pi}{3}\right)-\cos\left(\dfrac{7\pi}{6}\right)\sin\left(\dfrac{\pi}{3}\right)$

Using our reference triangle we can find the values we want:

$$\sin\left(\frac{7\pi}{6}\right)=-\frac{1}{2},\quad \cos\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2},\quad \sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2},\ \text{and}\ \cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

.

We substitute them into our equation to find

$$\sin\left(\frac{7\pi}{6}-\frac{\pi}{3}\right)=\left(-\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)=\boxed{\frac{1}{2}}$$

Notice that this is the same answer we get from $\sin\frac{5\pi}{6}=\frac{1}{2}$. The angle $\frac{5\pi}{6}$ is in the QII so the answer should be positive, and it is.

Example 3.3.3

Find the exact value of $\cos\left(\dfrac{\pi}{16}\right)\cos\left(\dfrac{3\pi}{16}\right)-\sin\left(\dfrac{\pi}{16}\right)\sin\left(\dfrac{3\pi}{16}\right)$.

Solution: Neither angle here is one of the nice angles that we can evaluate exactly with a reference triangle so we need to try something else. This formula is the sum of cosines so we can apply formula (3.10)

$$\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)-\sin\left(\frac{\pi}{16}\right)\sin\left(\frac{3\pi}{16}\right)=\cos\left(\frac{\pi}{16}+\frac{3\pi}{16}\right)=\cos\frac{\pi}{4}=\boxed{\frac{\sqrt{2}}{2}}$$

Example 3.3.4

Verify the cofunction identity $\sin\left(x+\frac{\pi}{2}\right)=\cos x$

Solution: We saw that this was true in Section 1.4 by looking at values on the unit circle. We can now show that it is true using the addition formula for sine.

$$\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\left(\frac{\pi}{2}\right)+\cos x\sin\left(\frac{\pi}{2}\right)$$

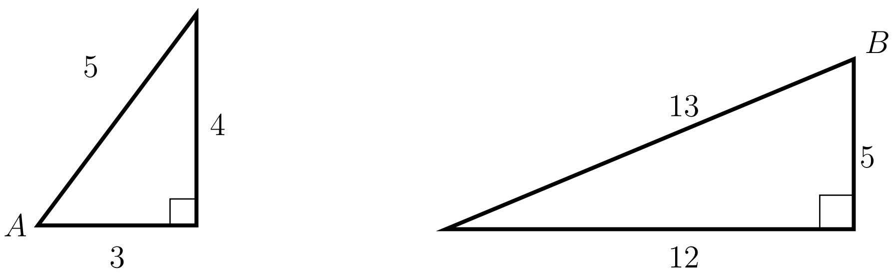
$\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ so

$$\boxed{\sin\left(x + \frac{\pi}{2}\right) = \cos x}$$

Example 3.3.5

Given angles A and B such that $\sin A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$ with $0 \leq A,\ B \leq \frac{\pi}{2}$ find the exact values of $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$.

Solution: We need to find the values of the other trigonometric functions so we will draw triangles for A and B. The missing sides are found using the Pythagorean theorem. See **Figure 3.2**.



(a) Angle A **(b)** Angle B

Figure 3.2: Example 3.3.5

Using the addition formula for sine, we get:

$$\begin{aligned} \sin(A+B) &= \sin A\ \cos B + \cos A\ \sin B \\ &= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13} \quad \Rightarrow \quad \boxed{\sin(A+B) = \frac{56}{65}} \end{aligned}$$

Using the addition formula for cosine, we get:

$$\begin{aligned} \cos(A+B) &= \cos A\ \cos B - \sin A\ \sin B \\ &= \frac{3}{5} \cdot \frac{5}{13} - \frac{4}{5} \cdot \frac{12}{13} \quad \Rightarrow \quad \boxed{\cos(A+B) = -\frac{33}{65}} \end{aligned}$$

Instead of using the addition formula for tangent, we can use the results above:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\frac{56}{65}}{-\frac{33}{65}} \quad \Rightarrow \quad \boxed{\tan(A+B) = -\frac{56}{33}}$$

Example 3.3.6

Suppose $\sin u = \frac{5}{13}$ with u in quadrant II and $\tan v = -\frac{x}{3}$ with v in quadrant III. Find an algebraic expression for $\cos(u+v)$.

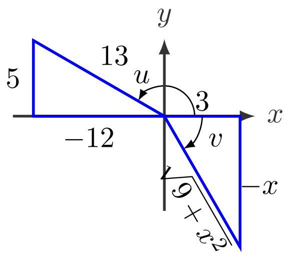


Figure 3.3

Solution: The cosine sum formula is

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

where u and v are the angles drawn in **Figure 3.3**. We can evaluate the sine and cosine using these reference triangles.

$$\begin{aligned}\cos(u+v) &= \cos u \cos v - \sin u \sin v \\ &= \left(-\frac{12}{13}\right)\left(\frac{3}{\sqrt{9+x^2}}\right) - \left(\frac{5}{13}\right)\left(\frac{-x}{\sqrt{9+x^2}}\right) \\ &= \boxed{\frac{5x-36}{13\sqrt{9+x^2}}}\end{aligned}$$

Example 3.3.7

Write

$$\sin\left(\tan^{-1} 1 + \cos^{-1} x\right)$$

as an algebraic expression.

Solution: This expression is in the form $\sin(\alpha+\beta)$ so we let $\alpha = \tan^{-1} 1$ and $\beta = \cos^{-1} x$. Those triangles are shown in **Figure 3.4**. We will use the formula and read the values of the sines and cosines off the triangles.

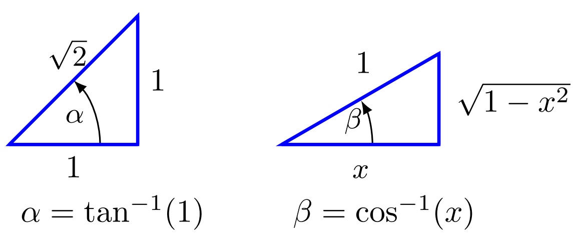


Figure 3.4

$$\begin{aligned}\sin\left(\tan^{-1} 1 + \cos^{-1} x\right) &= \sin(\tan^{-1} 1)\cos(\cos^{-1} x) + \cos(\tan^{-1} 1)\sin(\cos^{-1} x) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{x}{1}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{1-x^2}}{1}\right) \\ &= \boxed{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}}\end{aligned}$$

You can check to see if this is a reasonable answer by trying some values for x using your calculator.

We will prove the difference of angles identity for cosine

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

The formula for $\cos(\alpha + \beta)$ is derived by replacing $-\beta$ with $-(-\beta)$ in the formula and applying the negative angle identities $\sin(x) = -\sin(-x)$ and $\cos(x) = \cos(-x)$.

Consider two points on the unit circle in **Figure 3.5**:

Point P at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$.

Point Q at an angle β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$.

The triangle $\triangle OPQ$ has angle $\angle POQ$ of size $\alpha - \beta$. Triangle $\triangle OCD$ is $\triangle OPQ$ rotated β degrees clockwise so the length of the two red segments $\overline{PQ}$ and $\overline{CD}$ are the same lengths. We also know the coordinates of points C and D:

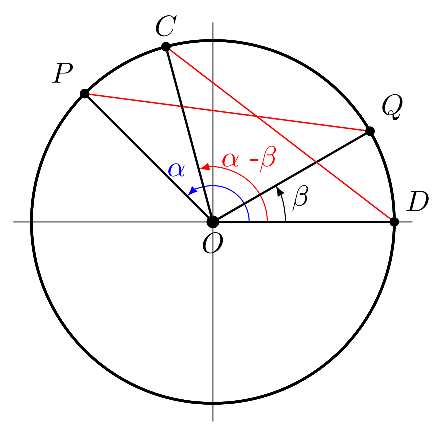


Figure 3.5: A unit circle

Point C is at an angle $\alpha - \beta$ from the positive x-axis with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ and point D is at $(1, 0)$

We can calculate the lengths of $\overline{PQ}$ and $\overline{CD}$ using the formula for the distance between two points (x_1, y_1) and (x_2, y_2):

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We can expand and simplify using the Pythagorean identity.

$$\begin{aligned}
\text{length } \overline{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\
&= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin^2\beta + \sin^2\beta} \\
&= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= \sqrt{2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta)}
\end{aligned}$$

Similarly we calculate the length of $\overline{CD}$

$$\begin{aligned}
\text{length } \overline{CD} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta))^2} \\
&= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} \\
&= \sqrt{2-2\cos(\alpha-\beta)} \\
&= \sqrt{2(1-\cos(\alpha-\beta))}
\end{aligned}$$

If we set the two lengths equal we see that

$$2(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta)=2(1-\cos(\alpha-\beta))$$

and with a bit of algebra

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

which is what we were trying to show. A similar calculation can produce $\sin(\alpha-\beta)$.

Formulas for $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$ are found by applying the identity $\tan x = \frac{\sin x}{\cos x}$ and the addition formulas for sine and cosine.

Example 3.3.8

Show that $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Solution:

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} && \text{formulas (3.8) and (3.10)} \\
&= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} && \text{divide everything by } \cos\alpha\cos\beta \\
&= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}} && \text{cancel common terms} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

3.3 Exercises

For Exercises 1-8 use the sum and difference formulas to find the exact values.

1. $\sin(45^\circ - 30^\circ)$ **2.** $\cos(45^\circ + 30^\circ)$ **3.** $\tan(135^\circ - 30^\circ)$ **4.** $\sin(135^\circ + 150^\circ)$

5. $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$ **6.** $\cos\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)$ **7.** $\tan\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$ **8.** $\cos\left(\frac{7\pi}{4} + \frac{\pi}{3}\right)$

For Exercises 9-16 use the sum and difference formulas to find the exact values.

9. $\sin 75^\circ$ **10.** $\cos 255^\circ$ **11.** $\tan(-165^\circ)$ **12.** $\sin 345^\circ$

13. $\sin\left(\frac{\pi}{12}\right)$ **14.** $\cos\left(\frac{5\pi}{12}\right)$ **15.** $\tan\left(\frac{23\pi}{12}\right)$ **16.** $\cos\left(-\frac{\pi}{12}\right)$

For Exercises 17 - 22, find the exact value of the expression.

17. $\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right) + \cos\left(\frac{\pi}{16}\right)\sin\left(\frac{7\pi}{16}\right)$

18. $\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right) - \cos\left(\frac{3\pi}{16}\right)\sin\left(\frac{7\pi}{16}\right)$

19. $\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right) - \sin\left(\frac{\pi}{16}\right)\sin\left(\frac{7\pi}{16}\right)$

20. $\sin\left(\frac{3\pi}{16}\right)\sin\left(\frac{7\pi}{16}\right) + \cos\left(\frac{3\pi}{16}\right)\cos\left(\frac{7\pi}{16}\right)$

21. $\dfrac{\tan\left(\frac{\pi}{16}\right) + \tan\left(\frac{7\pi}{16}\right)}{1 - \tan\left(\frac{\pi}{16}\right)\tan\left(\frac{7\pi}{16}\right)}$ **22.** $\dfrac{\tan\left(\frac{13\pi}{12}\right) - \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{13\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)}$

For Exercises 23 - 30, use the sum and difference formulas to rewrite each expression in terms of one trigonometric function.

23. $\cos\left(x + \frac{\pi}{2}\right)$ **24.** $\sin\left(x - \frac{\pi}{2}\right)$ **25.** $\cos(x + \pi)$ **26.** $\tan(x - \pi)$

27. $\csc\left(\frac{\pi}{2} - x\right)$ **28.** $\sec\left(\frac{\pi}{2} - t\right)$ **29.** $\cot\left(\frac{\pi}{2} - x\right)$ **30.** $\tan\left(\frac{\pi}{2} - \theta\right)$

For Exercises 31 - 34, given angles A and B such that $0 \le A,\ B \le \frac{\pi}{2}$ find the exact values of $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$.

31. $\sin A = \dfrac{3}{5}$ and $\sin B = \dfrac{15}{17}$ **32.** $\sin A = \dfrac{24}{25}$ and $\cos B = \dfrac{5}{13}$

33. $\cos A = \dfrac{3}{5}$ and $\tan B = \dfrac{12}{5}$

34. $\sin A = \dfrac{5}{12}$ and $\sin B = \dfrac{3}{4}$

For Exercises 35 - 38, given angles A and B find the exact values of $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$.

35. $\sin A = \dfrac{5}{13}$ with A in quadrant II and $\cos B = -\dfrac{2}{3}$ with B in quadrant III.

36. $\sin A = -\dfrac{5}{13}$ with A in quadrant IV and $\sin B = \dfrac{2}{3}$ with B in quadrant I.

37. $\tan A = \dfrac{5}{13}$ with A in quadrant III and $\cos B = -\dfrac{5}{13}$ with B in quadrant III.

38. $\sin A = \dfrac{40}{41}$ with A in quadrant II and $\cos B = \dfrac{x}{41}$ with B in quadrant IV.

39. Write $\cos\left(\tan^{-1} 1 + \sin^{-1} x\right)$ as an algebraic expression.

40. Write $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}\left(\frac{x}{2}\right)\right)$ as an algebraic expression.

41. Prove the identity $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

42. Prove the identity $\cos(A+B)\cos(A-B) = \cos^2 A - \cos^2 B$

3.4 Multiple-Angle Formulas

Double Angle Formulas

Example 3.4.1

Find an expression for $\sin(2\theta)$.

Solution: We can find an expression for $\sin(2\theta)$ by rewriting it as $\sin(\theta + \theta)$ and using the addition formula.

$$\sin(\theta + \theta) = \sin\theta\cos\theta + \sin\theta\cos\theta = \boxed{2\sin\theta\cos\theta}$$

We can similarly find formulas for $\cos 2\theta$ and $\tan 2\theta$. The double angle formulas are summarized in the table below.

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{3.14}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{3.15}$$

$$= 2\cos^2\theta - 1 \tag{3.16}$$

$$= 1 - 2\sin^2\theta \tag{3.17}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{3.18}$$

Notice that there are three formulas for $\cos(2\theta)$. The first comes from applying the sum of angles for cosine formula (3.10). The other two are derived by the Pythagorean identity.

Example 3.4.2

Show that $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

Solution: Working with the left side and $\sin^2\theta = 1 - \cos^2\theta$ we get.

$$\cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

And so the identity is shown.

Example 3.4.3

Use a double angle formula to rewrite the equation

$$y = 4\cos^2 x - 2.$$

Then sketch the graph of the equation over the interval $[0,\ 2\pi]$.

Solution: We will factor a 2 and then use the double angle formula (3.16).

$$\begin{aligned} y &= 4\cos^2 x - 2 \\ &= 2\left(\cos^2 x - 1\right) \\ &= 2\cos(2x) \end{aligned}$$

This equation can be graphed in **Figure 3.6** using the techniques we saw in Section 2.1.

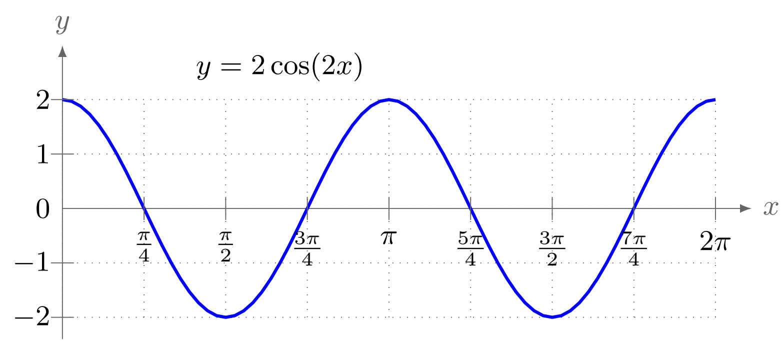


Figure 3.6

Example 3.4.4

Suppose $\cos\theta = -\frac{2}{3}$ with $\pi \le \theta \le \frac{3\pi}{2}$. Find the value of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

Solution: Since θ is in QIII and we know that $\cos\theta = -\frac{2}{3} = \frac{\text{adjacent}}{\text{hypotenuse}}$, we can find the missing side by the Pythagorean theorem and draw a reference triangle (**Figure 3.7**). From our reference triangle we can evaluate the sine, cosine and tangent. Now we can calculate

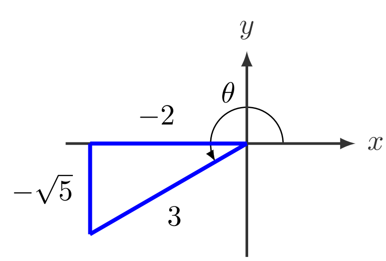


Figure 3.7

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = \boxed{\frac{4\sqrt{5}}{9}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{2}{3}\right)^2 - \left(\frac{-\sqrt{5}}{3}\right)^2 = \boxed{-\frac{1}{9}}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{\sqrt{5}}{2}\right)}{1-\left(\frac{\sqrt{5}}{2}\right)^2} = \frac{\sqrt{5}}{1-\frac{5}{4}} = \boxed{-4\sqrt{5}}$$

We could have calculated the tangent with the identity $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\frac{4\sqrt{5}}{9}}{-\frac{1}{9}} = \boxed{-4\sqrt{5}}$. Notice that this is the same answer we get from our original calculation.

Example 3.4.5

Express $\sin 3x$ in terms of $\sin x$.

Solution: We will have to use the sum formula on $3x = 2x + x$ and the double angle formulas. For the cosine we will use $\cos 2x = 1 - \sin^2 x$ because we want our answer entirely in terms of $\sin x$.

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x \\
&= 2\sin x\cos^2 x + \sin x - 2\sin^3 x \\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\
&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x
\end{aligned}$$

Example 3.4.6

Solve $\cos(2x) = \cos x$ for all solutions on $[0, 2\pi)$.

Solution: In general when solving a trigonometric equation it is more complicated if you have functions with different periods or different trigonometric functions. In this case we have $(2x)$ in one of the cosines and x in the other so they have different periods. We would like to have this equation in all in terms of $\cos x$ so we will use the double angle formula $\cos(2x) = 2\cos^2 x - 1$.

$$\begin{aligned}
\cos(2x) &= \cos x && \text{original equation} \\
2\cos^2 x - 1 &= \cos x && \text{double angle formula} \\
2\cos^2 x - \cos x - 1 &= 0 && \text{set quadratic equal to zero} \\
(2\cos x + 1)(\cos x - 1) &= 0 && \text{factor}
\end{aligned}$$

Now set each of the factors equal to zero and solve separately.

$$\begin{aligned}
2\cos x + 1 &= 0 & \text{or} \qquad & \cos x - 1 = 0 \\
\cos x &= -\frac{1}{2} & & \cos x = 1 \\
x = \frac{2\pi}{3} \text{ or } x &= \frac{4\pi}{3} & & x = 0
\end{aligned}$$

The solutions are

$$\boxed{x = \frac{2\pi}{3}, \qquad x = \frac{4\pi}{3}, \qquad \text{and} \qquad x = 0}$$

Power Reducing Formulas

Closely related to the double angle formulas are the **power-reducing formulas**. These are derived directly from the double angle formulas.

Example 3.4.7

Verify the identity $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$.

Solution: We will start with the double angle formula

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

and solve for $\sin^2\theta$.

$$\begin{aligned}
2\sin^2\theta &= 1 - \cos(2\theta) \\
\sin^2\theta &= \frac{1-\cos(2\theta)}{2}
\end{aligned}$$

We call this a power reducing formula because we take $\sin^2\theta$ and convert it to cosine to the first power. This formula is useful when you can't work with the square of the trigonometric function but you can work with the first power. In particular these power reducing formulas are used often in calculus. **Example 3.4.8** shows a typical power reduction used in calculus. We can similarly derive power reducing formulas for the cosine and tangent which are summarized in the following table.

Power-Reducing Formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Example 3.4.8

Rewrite $\cos^4\theta$ as a sum of first power of the cosines of multiple angles.

Solution:

$$\begin{aligned}
\cos^4\theta &= \left(\cos^2\theta\right)^2 && \text{exponent law}\\
&= \left(\frac{1+\cos(2\theta)}{2}\right)^2 && \text{power-reducing formula}\\
&= \frac{1}{4}\left(1+2\cos(2\theta)+\cos^2(2\theta)\right) && \text{algebra}\\
&= \frac{1}{4}\left(1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right) && \text{power-reducing formula on } \cos^2(2\theta)\\
&= \frac{1}{8}\left(3+4\cos(2\theta)+\cos(4\theta)\right) && \text{factor } \frac{1}{2} \text{ and simplify}
\end{aligned}$$

Note: In calculus it can be difficult to integrate sine and cosine powers greater than 1 but it is comparatively trivial to integrate the power-reduced equivalent.

Half-Angle Formulas

From the power reducing formulas we can derive **half-angle formulas** .

Example 3.4.9

Prove that $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$.

Solution: Start with the formula $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ and replace θ with $\frac{\theta}{2}$.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

Taking the square root provides the answer.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Note that we have a $\pm$ in front of the square root. The choice of sign depends on the quadrant of $\theta/2$.

The half-angle formulas are summarized here.

Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

The sign of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ depends on the quadrant of $\frac{\theta}{2}$.

Example 3.4.10

Use a half angle formula to find $\sin 165°$.

Solution Our answer will be positive because 165° is in the second quadrant and sine is positive in QII. Also notice that $165° = \dfrac{330°}{2}$ so we can use the half-angle formula for sine.

$$\sin 165° = \sin\left(\frac{330°}{2}\right) = +\sqrt{\frac{1-\cos 330°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$\boxed{\sin 165° = \sqrt{\frac{2-\sqrt{3}}{4}}}$$

3.4 Exercises

1. If $\sin x = \dfrac{1}{8}$ and x is in quadrant II, find exact values for (without solving for x):

(a) $\sin(2x)$ (b) $\cos(2x)$ (c) $\tan(2x)$ (d) $\sin(3x)$

2. If $\cos\theta = \dfrac{2}{5}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$, find exact values for (without solving for x):

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$ (c) $\tan(2\theta)$ (d) $\sin(3\theta)$

For Exercises 3-10 simplify each expression using the double angle formulas.

3. $\cos^2 x - \sin^2 x$

4. $2\cos^2\left(\frac{x}{2}\right) - 2\sin^2\left(\frac{x}{2}\right)$

5. $6\cos^2(3x) - 3$

6. $2\sin^2(2x) - 1$

7. $\sin^2(5x) - \cos^2(5x)$

8. $4\sin x\cos x$

9. $\sin x\cos x$

10. $1 - 2\sin^2(17°)$

For Exercises 11-15 solve for all solution on $[0, 2\pi)$. Leave exact answers.

11. $6\sin(2\theta) + 9\sin\theta = 0$

12. $2\sin(2\theta) + 3\cos\theta = 0$

13. $\sin(2\theta) = \cos\theta$

14. $\cos(2\theta) = \sin\theta$

15. $\sin(4\theta) = \sin(2\theta)$

For Exercises 16 - 21, use the power reducing formulas to rewrite the extpressions without exponents.

16. $\cos^2(2x)$

17. $\sin^4 x$

18. $\sin^4(3x)$

19. $\sin^2\left(\dfrac{x}{2}\right)\cos^2\left(\dfrac{x}{2}\right)$

20. $\cos^2 x\sin^4 x$

21. $\cos^4 x\sin^2 x$

For Exercises 22 - 30, use the half angle formula to find the exact value of each expression.

22. $\sin(75°)$

23. $\cos(75°)$

24. $\tan(75°)$

25. $\sin\left(\dfrac{\pi}{8}\right)$

26. $\cos\left(\dfrac{\pi}{8}\right)$

27. $\tan\left(\dfrac{\pi}{8}\right)$

28. $\sin\left(\dfrac{7\pi}{12}\right)$

29. $\cos\left(\dfrac{7\pi}{12}\right)$

30. $\tan(105°)$

For Exercises 31 - 33, given angles A find the exact values of **(a)** $\sin\left(\dfrac{A}{2}\right)$, **(b)** $\cos\left(\dfrac{A}{2}\right)$, and **(c)** $\tan\left(\dfrac{A}{2}\right)$.

31. $\cot A = 7$ with A in quadrant III.

32. $\sin A = -\dfrac{5}{13}$ with A in quadrant IV.

33. $\sec A = 4$ with $\frac{3\pi}{2} \le A \le 2\pi$.

Chapter 4

General Triangles

4.1 Law of Sines

Introduction

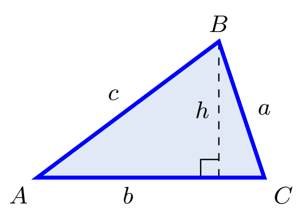


Figure 4.1: Oblique triangle

Up to now all the triangles we have looked at have been right triangles (one angle of 90°). If we knew two other pieces of information about the triangle, lengths of sides or angle measure, we could solve the triangle. Recall that to solve a triangle we wanted to find the lengths of all the sides and the measure of all the angles. Suppose we have a triangle with no right angles such as $\triangle ABC$ in **Figure 4.1**. A triangle with no right angles is called an **oblique triangle**. For our oblique triangle we label the angles with upper case letters A, B, and C and the sides opposite those angles with the corresponding lower case letter. Suppose we want to find a relationship between the $\sin A$ and the sides of triangle. We can't use our usual relationship of opposite over hypotenuse because that applies to right triangles. We will draw the height of the triangle h, (in this case from B), and divide the triangle into two right triangles. With the right triangles we can use our usual relationships:

$$\sin A = \frac{h}{c} \qquad \sin C = \frac{h}{a}$$

Solving each of the equations for h gives us

$$h = c\ \sin A \qquad h = a\ \sin C$$

Setting them equal

$$\begin{aligned} h &= h \\ c\ \sin A &= a\ \sin C \end{aligned}$$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

We can similarly find a relationship for $\sin B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

This is known as the **Law of Sines** and is summarized in the table below.

Law of Sinesθ

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The reciprocal is also true

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Note: The law of sines was proved for an acute triangle where all the angles were less than 90° but the law holds for all triangles.

There are 2 cases where we can use the law of sines. In each of these cases we need three pieces of information.

Case 1: One side and two angles (AAS or ASA)
Case 2: Two sides and an angle opposite one of them (**S**ide **S**ide **A**ngle SSA)

Example 4.1.1

Case 1: One side and two angles (AAS)

Solve the triangle in **Figure 4.2** where $B = 105°$, $C = 40°$, and $b = 20$ meters.

Solution: Recall that to solve the triangle we need to find the remaining sides and angles. We begin with the missing angle because the sum of the angles of a triangle is always 180°.

$$\begin{aligned} A &= 180 - B - C \\ &= 180 - 105° - 40° \\ &= 35° \end{aligned}$$

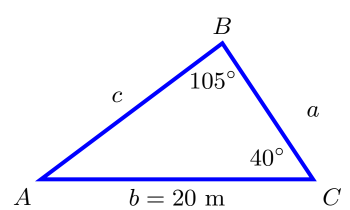


Figure 4.2

So $\boxed{A = 35°}$ and by the law of sines we can find the missing sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 35°} = \frac{20}{\sin 105°} = \frac{c}{\sin 40}$$

So we have the following two equations:

$$\frac{a}{\sin 35^\circ} = \frac{20}{\sin 105^\circ} \qquad \text{and} \qquad \frac{20}{\sin 105^\circ} = \frac{c}{\sin 40}$$

and we can solve for a and c

$$a = \left(\frac{20}{\sin 105^\circ}\right)(\sin 35^\circ) \qquad \text{and} \qquad c = \left(\frac{20}{\sin 105^\circ}\right)(\sin 40)$$

$$\boxed{a \approx 11.88 \text{ m}} \qquad \text{and} \qquad \boxed{c \approx 13.11 \text{ m}}$$

The Ambiguous Case (SSA)

In **Example 4.1.1** we knew two of the angles and one side. This amount of information determines one unique triangle. In the case where you know two sides and an angle opposite one of them there are 3 possible outcomes which are shown in **Figure 4.3**: no solutions, one solution or two solutions. This is called **the ambiguous case**.

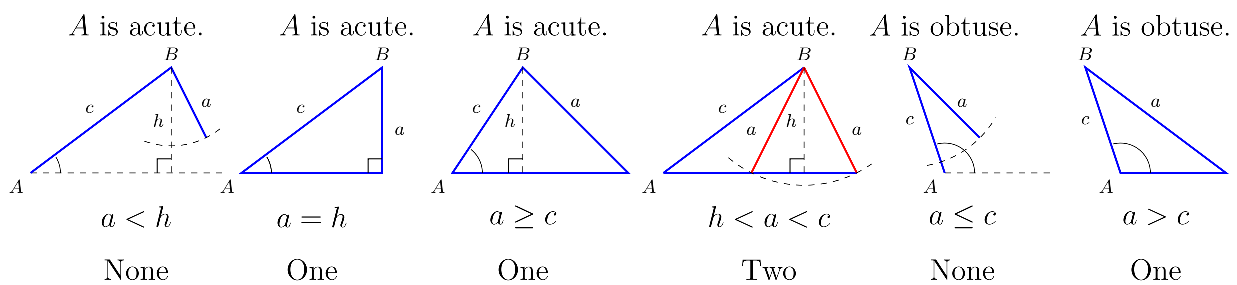


Figure 4.3: The Ambiguous Cases (SSA): Conditions and Possible Triangles

Example 4.1.2

Case 2: Two sides and one angle, two solutions (SSA)

Solve the triangle where $A = 60^\circ$, $a = 9$, and $c = 10$.

Solution: When you have an angle and two sides you want to draw what you know and then calculate the height. The height will let you know if you can make a triangle or not. The side opposite the angle you know has to be at least as long as the height or you can't make a triangle.

$$\sin 60^\circ = \frac{h}{10} \quad \Longrightarrow \quad h = 8.66$$

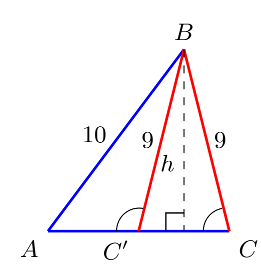


Figure 4.4

In **Figure 4.4** the red sides are the two possibilities because

$$(h = 8.66) < (a = 9) < (c = 10)$$

We start by solving the triangle where C is an acute angle. Using the law of sines, $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, we can solve for C

$$\frac{\sin 60}{9} = \frac{\sin C}{10} \implies C = \sin^{-1}\left(\frac{10\sin 60}{9}\right) = 74.21^\circ$$

and $B = 180^\circ - 60^\circ - 74.21^\circ = 45.79$. Then the final side can be found with the law of sines again.

$$\frac{9}{\sin 60^\circ} = \frac{b}{\sin(45.79^\circ)} \implies b = \frac{9\sin(45.79^\circ)}{\sin 60^\circ} = 7.45$$

The solution to the first triangle is $\boxed{C = 74.21^\circ,\ B = 45.79^\circ \text{ and } b = 7.45}$.

The second triangle has $C' > 90^\circ$ and is the supplementary to C. (Why?)

$$C' = 180^\circ - 74.21^\circ = 105.79$$

and $B' = 180^\circ - 60^\circ - 105.79^\circ = 14.21^\circ$. The final side can once again be calculated using the law of sines.

$$\frac{9}{\sin 60^\circ} = \frac{b'}{\sin(14.21^\circ)} \implies b' = \frac{9\sin(14.21^\circ)}{\sin 60^\circ} = 2.55$$

The solution to the second triangle is $\boxed{C = 105.79^\circ,\ B = 14.21^\circ \text{ and } b = 2.55}$.

Example 4.1.3

Case 3: Two sides and one angle, No solution (SSA)

Solve the triangle where $A = 30^\circ$, $a = 6$, and $b = 12.8$.

Solution: In this case we have no solution because the sides can't meet. Drawing a diagram of the information you know will help to see this as in **Figure 4.5**. Consider the height h of the this possible triangle.

$$\sin 30 = \frac{h}{12.8} \implies h = 6.4$$

Since the height is 6.4 but the side opposite A has length 6, there is no way to construct this triangle and hence there is $\boxed{\text{no solution.}}$

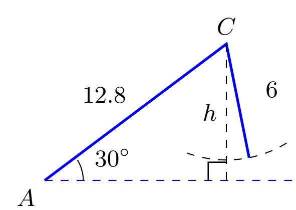


Figure 4.5

Example 4.1.4

Two radar stations located 10 km apart both detect a UFO located between them. The angle of elevation measured by the first station (A) is 36° and the angle of elevation measured by the second station (C) is 20°. What is the altitude (h) of the UFO? See **Figure 4.6**

Solution: The triangle formed by the radar stations and the UFO is not a right triangle. If we call the angle at the UFO B then we can see that $B = 180^\circ - 36^\circ - 20^\circ = 124^\circ$. To find the altitude we would need to know one side of a right triangle. Since the height h makes

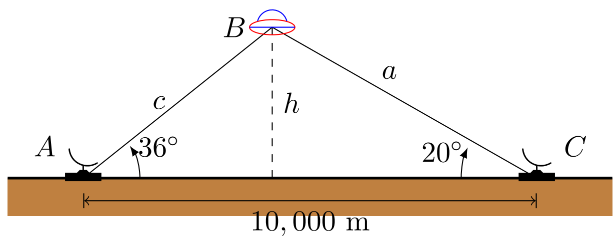


Figure 4.6: UFO and radar stations

two right triangles we can use either side a or c to solve the problem. We will use side a but you can verify that you arrive at the same answer if you use side c.

Since we do not have a right triangle and this situation is AAS we will use the law of sines and we know we have only one possible solution.

$$\frac{10,000}{\sin 124^\circ} = \frac{a}{\sin 36^\circ} \quad \Longrightarrow \quad a = \frac{10,000 \sin 36^\circ}{\sin 124^\circ} = 7090\text{m}$$

Now we can use the standard relationship for the sine to calculate the height.

$$\sin 20^\circ = \frac{h}{a}$$

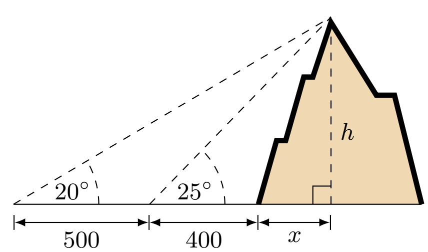
$$\sin 20^\circ = \frac{h}{7089}$$

$$\boxed{h = 2425 \text{ m}}$$

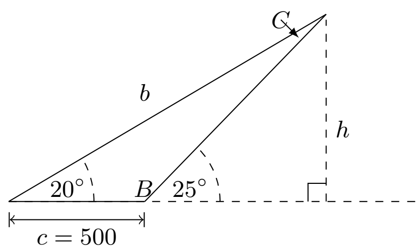
Example 4.1.5

A person standing 400 ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be 25°. She then walks 500 ft straight back and measures the angle of elevation to now be 20°. How tall is the mountain?

Solution: This is the same problem (**Example 1.5.6**) that we had when were were looking at applications of trigonometric functions in Section 1.5. In that problem we used the tangent function and a bit of algebra to do the calculation. This time we will use the law of sines.



Once again we assume that the ground is flat and not inclined relative to the base of the mountain and we let h be the height of the mountain as in the picture on the right. To use the law of sines we will use the following simplified triangle.



We know that angle B is supplementary to $25°$ so $B = 180° - 25° = 155°$. The angles in a triangle add up to $180°$ so $C = 5°$. Now we have enough information to use the law of sines to calculate the distance from the second observation point to the top of the mountain, length b in the diagram.

$$\begin{aligned} \frac{b}{\sin 155°} &= \frac{500}{\sin 5°} \\ b &= \frac{500 \sin 155°}{\sin 5°} \\ b &\approx 2424\text{ft} \end{aligned}$$

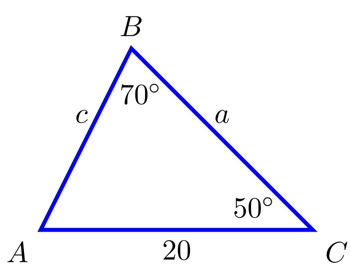
Now we can use the right triangle with the height h as the opposite side to the $20°$ and $b = 2424$ ft as the hypotenuse.

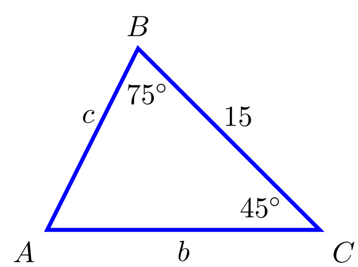
$$h = 2424 \sin 20° = \boxed{829\text{ft}}$$

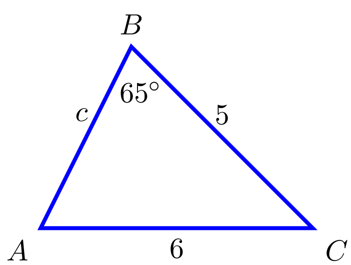
This is the same height we had calculated earlier but the calculations were simpler.

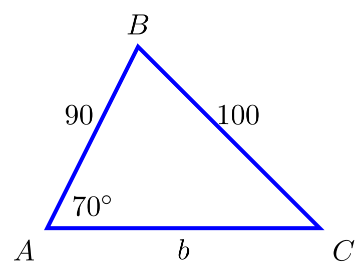
4.1 Exercises

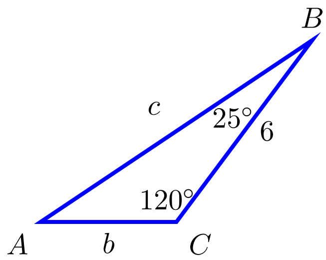
For Exercises 1-6 use the law of sines to solve the triangle $\triangle ABC$.

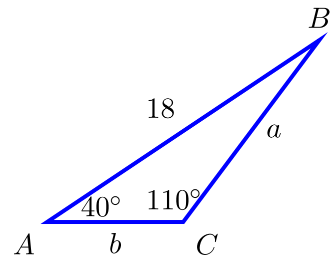
1. 

2. 

3. 

4. 

5. 

6. 

For Exercises 7-16 use the law of sines to solve the triangle $\triangle ABC$. If there is more than one possible solution, give both. If there is no answer state that there is no possible triangle.

7. $a = 10$, $A = 35°$, $B = 25°$

8. $b = 40$, $B = 75°$, $c = 35$

9. $A = 40°$, $B = 45°$, $c = 15$

10. $a = 5$, $A = 42°$, $b = 7$

11. $a = 40$, $A = 25°$, $c = 30$

12. $a = 5$, $A = 47°$, $b = 9$

13. $a = 12$, $A = 94°$, $b = 5$

14. $a = 12$, $A = 94°$, $b = 15$

15. $a = 12.3$, $A = 41°$, $b = 15.6$

16. $a = 22$, $A = 50°$, $c = 27$

For Exercises 17-19 solve for the unknown quantity in **Figure: 4.7**. (Not to scale)

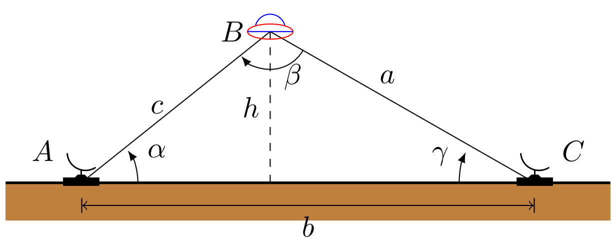


Figure 4.7: UFO and radar stations

17. Two radar stations located $b = 17$ km apart both detect a UFO located between them. The angle of elevation measured by the first station (A) is $\alpha = 72°$ and the angle of elevation measured by the second station (C) is $\gamma = 51°$. What is the altitude (h) of the UFO?

18. Two radar stations located $b = 17$ km apart both detect a UFO located between them. The angle of elevation measured by the first station (A) is $\alpha = 19°$ and the angle of elevation measured by the second station (C) is $\gamma = 151°$. What is the altitude (h) of the UFO? (Note: The UFO is to the right of station C.)

19. Two radar stations located $b = 107$ km apart both detect a UFO located between them. The angle of elevation measured by the first station (A) is $\alpha = 52°$ and the angle of elevation measured by the second station (C) is $\gamma = 32°$. What is the altitude (h) of the UFO?

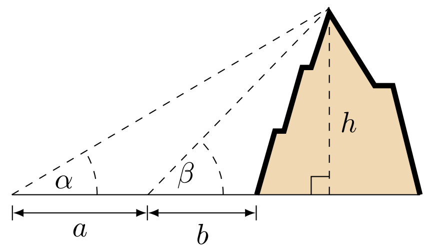


Figure 4.8: Mountain height

For Exercises 20-23 solve for the height of the mountain in **Figure: 4.8**. (Not to scale)

20. $\alpha = 31°$, $\beta = 87°$, $a = 10$ km, $b = 1$ km

21. $\alpha = 68°$, $\beta = 71°$, $a = 1000$ m, $b = 250$ m

22. $\alpha = 37°$, $\beta = 50°$, $a = 2.5$ km, $b = 2$ km

23. $\alpha = 50°$, $\beta = 57°$, $a = 5.0$ km, $b = 50$ km

4.2 Law of Cosines

Introduction

In Section 4.1 we were able to solve triangles with no right angles using the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The law of sines works in two cases:

Case 1: One side and two angles (AAS or ASA)
Case 2: Two sides and an angle opposite one of them (SSA)

There are two cases for which the law of sines does not work because we only have one piece of information in each of our ratios. To use the law of sines you have to have all the information to evaluate one of the fractions, an angle and its opposite side, and that is not true for these last two cases.

Case 3: Three sides (SSS)
Case 4: Two sides and the included angle (SAS)

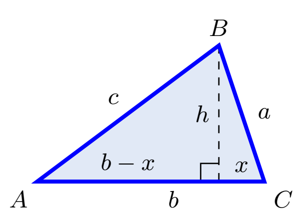


Figure 4.9: Law of Cosines diagram

To find another equation to solve the last two cases we will once again construct an oblique triangle and label the angles with upper case letters A, B, and C and the sides opposite those angles with the corresponding lower case letter. We draw the height of the triangle h, (in this case from B), and divide the triangle into two right triangles. Now side b is divided into two pieces, one with length x and the other with length $b - x$. Using the Pythagorean theorem we can write an equation for h for both triangles.

For the triangle on the right

$$h^2 = a^2 - x^2 \tag{4.1}$$

For the triangle on the left

$$\begin{aligned} h^2 &= c^2 - (b - x)^2 \\ h^2 &= c^2 - \left(b^2 - 2bx + x^2\right) \\ h^2 &= c^2 - b^2 + 2bx - x^2 \end{aligned} \tag{4.2}$$

Both of these equations involve x but we would like to use only the sides and angles originally given so using the cosine we see that $x = a\ \cos C$. Now set equation (4.1) equal to equation (4.2) and simplify.

$$h^2 = h^2$$

$$a^2 - x^2 = c^2 - b^2 + 2bx - x^2$$
$$c^2 = a^2 + b^2 - 2bx$$

Replace $x = a\cos\ C$

$$c^2 = a^2 + b^2 - 2ab\ \cos\ C. \tag{4.3}$$

This is known as the **Law of Cosines** And it relates the three sides of the triangle and one of the angles. This equation can be written in terms of any of the angles. The results are summarized here.

Law of Cosines

If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

Standard Form	Alternative Form
$a^2 = b^2 + c^2 - 2bc\ \cos\ A$	$\cos\ A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\ \cos\ B$	$\cos\ B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\ \cos\ C$	$\cos\ C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Note: The law of cosines was proved for an acute triangle where all the angles were less than 90° but the law holds for all triangles.

Example 4.2.1

Case 3: Three sides (SSS)

Solve the triangle in **Figure 4.10** where $a = 3$, $b = 9$, and $c = 8$.

Solution: Recall that to solve the triangle we need to find all sides and angles. We have three sides so we can't use the law of sines but we can use the law of cosines. We will use the alternate form so we can find one of the angles. We will start with the largest angle, which is opposite the longest side, $\angle B$.

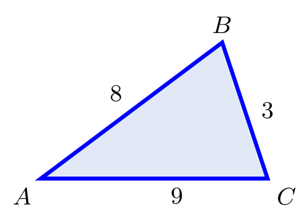


Figure 4.10

$$\begin{aligned}\cos\ B &= \frac{a^2 + c^2 - b^2}{2ac}\\ &= \frac{8^2 + 3^2 - 9^2}{2 \cdot 3 \cdot 8}\\ &= -\frac{1}{6}\end{aligned}$$

So $\boxed{B = 99.59°}$. Generally if you can use the law of sines it is easier than the law of cosines. Now that we have one of our angles we can use the law of sines to find another angle, say $\angle A$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin A}{3} = \frac{\sin 99.59°}{9}$$
$$A = \sin^{-1}\left(\frac{3(\sin 99.59°)}{9}\right)$$

Then $\boxed{A = 19.19°}$ and $C = 180° - A - B = 180 - 19.19° - 99.59° \implies \boxed{C = 61.22}$.

Example 4.2.2

Case 4: Two sides and the included angle (SAS)

Solve the triangle where $A = 55°$, $b = 3$, and $c = 10$.

Solution: Figure 4.11 is a sketch of the given information. Once again we can't use the law of sines because we don't know an angle and the length of its opposite side. We will start by calculating the length of a with the law of cosines and then use the law of sines to find another angle. While we could use the law of cosines to do solve for the angle, it is easier to use the law of sines whenever you have the choice.

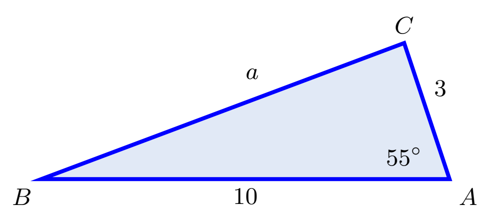


Figure 4.11

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\ A \\ &= 3^3 + 10^2 - 3 \cdot 3 \cdot 10\cos(55°) \\ &= 74.5854 \end{aligned}$$

so $a = 8.64$. Using the law of sines, $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, we can solve for C. (**NOTE:** Always solve for the largest angle first.)

$$\frac{\sin 55°}{8.64} = \frac{\sin C}{10} \implies C = \sin^{-1}\left(\frac{10\sin 55°}{8.64}\right) = 108.48°$$

and $B = 180° - 55° - 108.48° = 16.52°$.

The solution to the first triangle is $\boxed{C = 108.48°,\ B = 16.52° \text{ and } a = 8.64}$.

Example 4.2.3

Two radar stations located 10 km apart both detect a UFO located between them. Station Alpha calculates the distance to the object to be 7500 m and Station Beta calculates the distance as 9200 m. Find the angle of elevation measured by both stations (α) and (β). See **Figure 4.12**

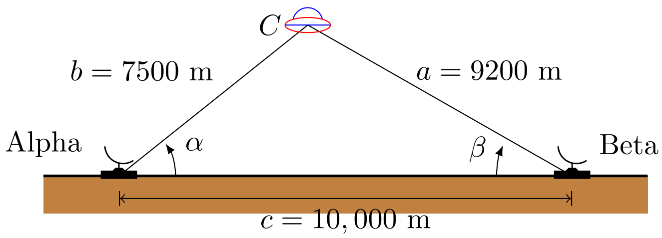


Figure 4.12: UFO and radar stations

Solution: The triangle formed by the radar stations and the UFO is not a right triangle and we know three sides (SSS). This means we need to use the law of cosines to calculate one of the angles. As before we will use the law of sines to calculate the second angle. Since we are looking for the angle we need the alternate form of the law of cosines:

$$\begin{aligned}\cos\beta &= \frac{a^2+c^2-b^2}{2ac}\\ &= \frac{9200^2+10000^2-7500^2}{2(9200)(10000)}\\ &= 0.697772\end{aligned}$$

So $\beta = \cos^{-1}(0.697772) = 45.75^\circ$ and we can use the law of sines to find α.

$$\begin{aligned}\frac{\sin\alpha}{a} &= \frac{\sin\beta}{b}\\ \frac{\sin\alpha}{9200} &= \frac{\sin 45.75^\circ}{7500}\\ \sin\alpha &= \frac{9200\sin 45.75^\circ}{7500}\\ \alpha &= 61.48^\circ\end{aligned}$$

Then $\boxed{\alpha = 61.48^\circ \text{ and } \beta = 45.75^\circ}$

Example 4.2.4

A baseball diamond is a square with 90 foot sides, with a pitchers mound 60.5 feet from home plate. How far is it from the pitchers mound to third base? A diagram of the dimensions of a baseball diamond is in **Figure 4.13**.

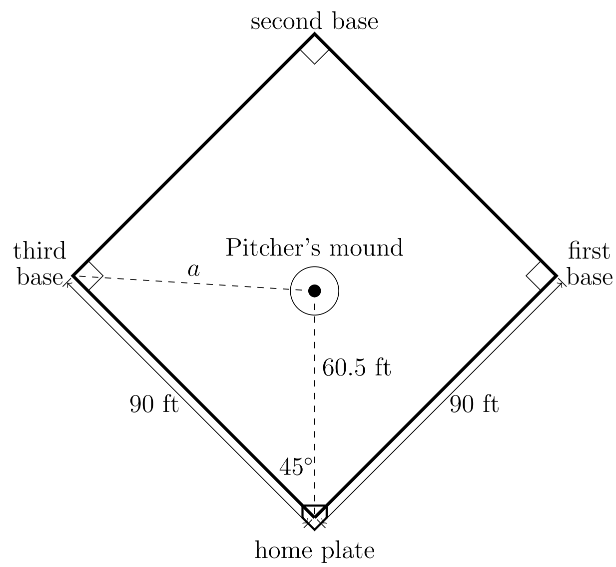


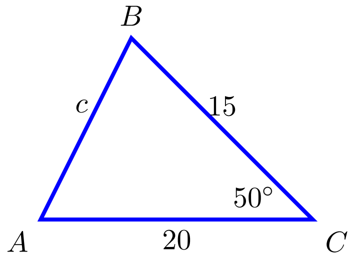
Figure 4.13: Dimensions on a baseball diamond

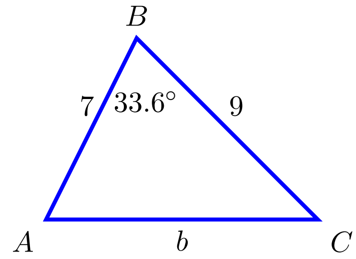
Solution: It is tempting to assume the pitcher's mound is in the center of the baseball diamond but it is not. It is located about 3 feet closer to home plate than the center. The distance to third base will therefore be different than the distance to home plate. We do have two sides of a triangle and the angle between them. The triangle is drawn on the diagram and the angle is 45° (why?). Using the law of cosines we can find the missing length.

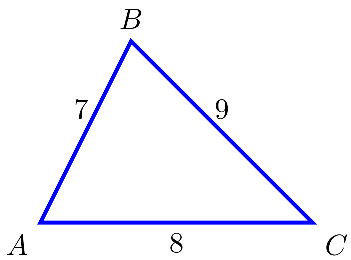
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\ A \\ a^2 &= 90^2 + 60.5^2 - 2(90)(60.5)\cos\ 45^\circ \\ a^2 &= 4060 \\ a &= 63.72 \text{ ft} \end{aligned}$$

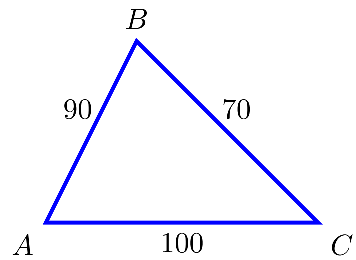
4.2 Exercises

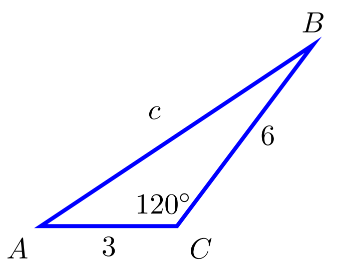
For Exercises 1-6 use the law of cosines to solve the triangle $\triangle ABC$.

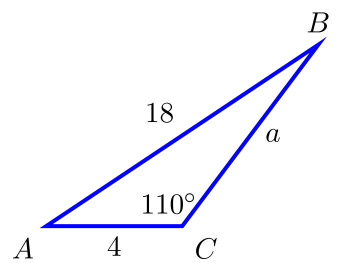
1. 

2. 

3. 

4. 

5. 

6. 

For Exercises 7-12 use the law of cosines to solve the triangle $\triangle ABC$. If there is more than one possible solution, give both. If there is no answer state that there is no possible triangle.

7. $a = 10$, $b = 35$, $c = 30$

8. $b = 40$, $A = 75°$, $c = 35$

9. $a = 40$, $B = 25°$, $c = 30$

10. $a = 5$, $B = 47°$, $c = 9$

11. $a = 12$, $C = 94°$, $b = 15$

12. $a = 22$, $b = 40$, $c = 27$

For Exercises 13-16 solve for the unknown quantity in **Figure: 4.14**. (Not to scale)

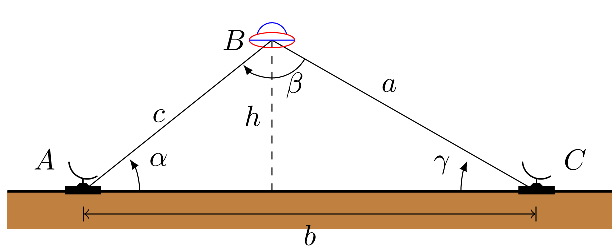


Figure 4.14: UFO and radar stations

13. To find the distance between two radar installations a UFO calculates the distance to installation A to be $c = 370$ km, the distance to installation C to be $a = 350$ km, and the angle between them $\beta = 2.1°$. Find the distance between the installations.

14. To find the distance between two radar installations a UFO calculates the distance to installation A to be $c = 200$ km, the distance to installation C to be $a = 300$ km, and the angle between them $\beta = 5.0°$. Find the distance between the installations.

15. Two radar stations located 80 km apart both detect a UFO located between them. Station A calculates the distance to the object to be 20 km and Station C calculates the distance as 92 km. Find the angles of elevation (α and γ) measured by both stations.

16. To find the distance between two radar installations a UFO calculates the distance to installation A to be $c = 420$ km, the distance to installation C to be $a = 150$ km, and the angle between them $\beta = 4.0°$. Find the distance between the installations.

17. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10 degrees to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

18. Two planes leave the same airport at the same time. One flies at 20 degrees east of north at 500 miles per hour. The second flies at 30 east of south at 600 miles per hour. How far apart are the planes after 2 hours?

19. To find the distance across a small lake, a surveyor has taken the measurements shown in **Figure** 4.15. Find the distance across the lake.

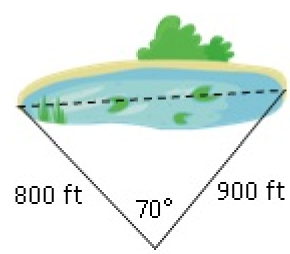


Figure 4.15: Lake width

20. A 127 foot tower is located on a hill that is inclined 38° to the horizontal. A guy-wire is to be attached to the top of the tower and anchored at a point 64 feet downhill from the base of the tower as seen in **Figure** 4.16. Find the length of wire needed.

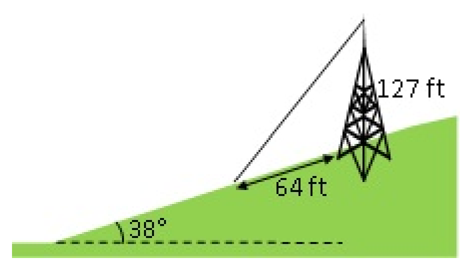


Figure 4.16: Wire Length

21. A 113 foot tower is located on a hill that is inclined 34° to the horizontal. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower as seen in **Figure 4.17**. Find the length of wire needed.

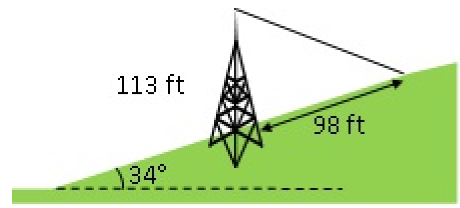


Figure 4.17: Wire Length

4.3 Area of a Triangle

Introduction

The formula for the area of a triangle is

$$Area = \frac{1}{2}(base) \cdot (height) = \frac{1}{2} b \cdot h$$

Any leg of the triangle can be used as the base but unless you have a right triangle the height is not obvious. The proof of the law of sines provides a way to find the height. Consider either of the triangles in **Figure 4.18** where we know the lengths of the sides and the angles. Now we can calculate the height $h = c \sin A$ so

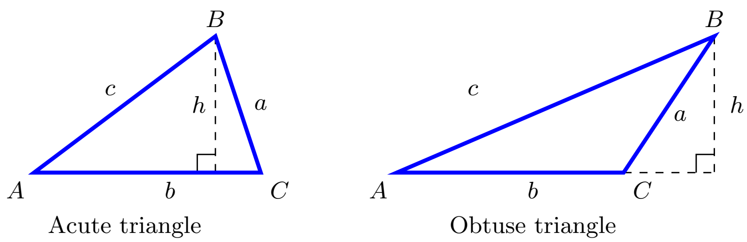


Figure 4.18

$$Area = \frac{1}{2} b \cdot h = \frac{1}{2} b \cdot c \cdot \sin A$$

This formula works any time you know two sides and the included angle (SAS). The shape of the triangle does not matter.

Formula for the Area of a Triangle

Given a triangle with angles A, B and C and sides a, b and c opposite those angles

$$Area = \frac{1}{2} b \cdot h = \frac{1}{2} b \cdot c \cdot \sin A = \frac{1}{2} b \cdot a \cdot \sin C = \frac{1}{2} a \cdot c \cdot \sin B.$$

Example 4.3.1

Find the area of a triangular lot having two sides of lengths 150 meters and 100 meters with included angle of 99°

Solution: Draw a diagram to represent the problem. **Figure 4.19** Then apply the formula

$$\begin{aligned} Area &= \frac{1}{2} b \cdot c \cdot \sin A \\ &= \frac{1}{2} \cdot 150 \cdot 100 \cdot \sin 99^\circ \\ &= 7407 \text{m}^2 \end{aligned}$$

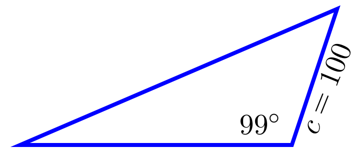


Figure 4.19

Heron's Formula

When you have 3 sides of a triangle and do not know an angle **Heron's formula**[1] (sometimes Hero's formula) can be used. Heron's formula will not be proved here but can be derived using the law of cosines, the Pythagorean identiy and some clever factoring.

Heron's Formula

Heron's formula states that the area of a triangle whose sides have lengths a, b and c is given by

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semiperimeter

$$s = \frac{1}{2}(a + b + c)$$

Example 4.3.2

A surveyor measures the sides of a tiangular parcel of land to be 206 feet, 293 feet and 187 feet. Find the area of the parcel.

Solution: When using Heron's formula find the semiperimeter s first.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(206 + 293 + 187) = 343\text{ft}$$

Then calculate the area

$$\begin{aligned} Area &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{343(343-206)(343-293)(343-187)} \\ &= 19100\text{ft}^2 \end{aligned}$$

[1]Named after Heron of Alexandria who wrote about it in 60 AD. The formula was discovered independently by the Chinese and their earliest known record of it is from Qin Jiushao in 1247 AD.

Example 4.3.3

Find the area of the triangle with side lengths $a = 1000000$, $b = 999999.9999979$ and $c = 0.0000029$.

Solution: The problem with this example is that many calculators will not provide the correct answer because of the number of decimal places in the calculation of

$$a + b + c = 2000000.0000008$$

which has 14 digits. While most calculators will store 14 digits internally for calculations they will only display 8 of them. Your calculator may round this to 2000000.0 so when calculating $(s - a)$ you get $(s - a) = (1000000 - 1000000) = 0$ which gives an area of 0. Clearly this is not the correct answer. The correct answer is

$$\boxed{Area = 0.99999999999895}$$

There are two alterative forms of Heron's formula. One from a 13$^{\text{th}}$ century Cinese text by Qin Jiushao

$$Area = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2 + c^2 - b^2}{2}\right)} \qquad \text{where} \qquad a \geq b \geq c$$

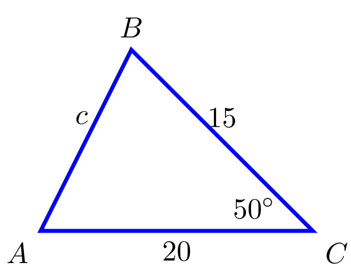
and one by William Kahan published in 2000. Arrange the sides so that $a \geq b \geq c$

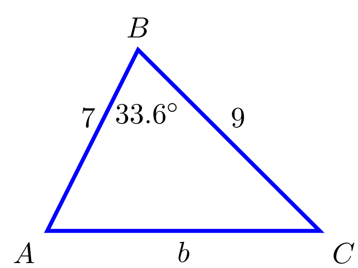
$$Area = \frac{1}{4}\sqrt{[a + (b + c)][c - (a - b)][c + (a - b)][a + (b - c)]}$$

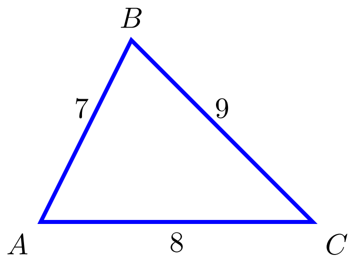
Both of these will provide the correct answer in your calculator if you use all the parentheses and brackets shown.

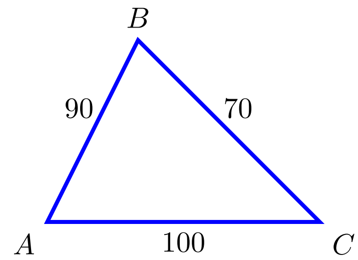
4.3 Exercises

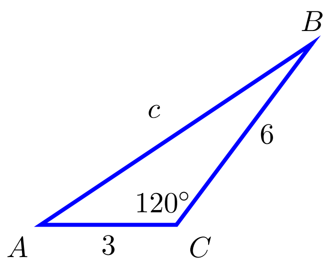
For Exercises 1-6 find the area of the triangle $\triangle ABC$.

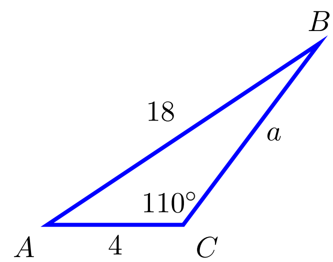
1. 

2. 

3. 

4. 

5. 

6. 

For Exercises 7-12 find the area of the triangle $\triangle ABC$.

7. $a = 10$, $b = 35$, $c = 30$

8. $b = 40$, $A = 75°$, $c = 35$

9. $a = 40$, $B = 25°$, $c = 30$

10. $a = 5$, $B = 47°$, $c = 9$

11. $a = 12$, $C = 94°$, $b = 15$

12. $a = 22$, $b = 40$, $c = 27$

13. Find the area of the quadrilateral in Figure 4.20 below.

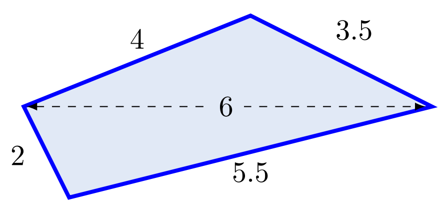


Figure 4.20: Exercise 13

Chapter 5

Additional Topics

5.1 Polar Coordinates

Introduction

Up to now we have done all our work in this course and previous courses in the **Cartesian Coordinate** system. This is the square grid where we have an x-axis and a y-axis and every point in the plane can be described by using two pieces of information: distance traveled in the x direction and distance traveled in the y-direction. The points and their distances from the origin are indicated as an ordered pair (x, y).

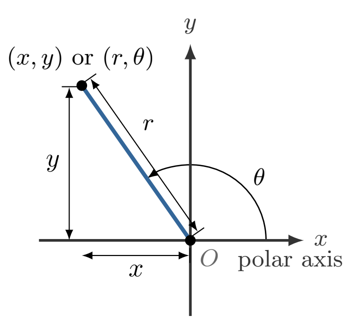


Figure 5.1: Point in the plane identified with Cartesian (x, y) and polar (r, θ) coordinates

While this system of identifying points on the plane is quite useful it is not the only way to do so. Another way is the use of **polar coordinates** . Polar coordinates are drawn in the plane starting at a fixed point O called the **pole** or **origin** and a ray in the positive x direction called the **polar axis** . Polar coordinates also use two pieces of information to identify a point in the plane:

θ: an angle measured from the polar axis

r: a directed distance from the pole.

Figure 5.1 shows a point in the plane identified with both coordinate systems. In polar coordinates the point is (r, θ). In Cartesian coordinates it is useful to draw a square grid to measure distances in the x and y directions but this grid is not what we need for polar coordinates. In polar coordinates we have concentric circles that represent the radii and lines extending out radially indicating the angles. See **Figures** 5.2 **and** 5.3 for two different versions. You can mark the angles in either degrees or radians but radians is the most common. We will primarily use radians for all our work with polar coordinates in this text.

The angle θ can be both positive and negative just as when constructing reference angles. When positive, it is measured starting at the polar axis traveling in the counter clockwise direction and, when negative, it is measured in the clockwise direction. The radius r is called a directed distance because it can also be positive or negative. If it is positive it is measured from the origin in the direction of the angle and if negative it is measured in the opposite direction. See **Example 5.1.1**

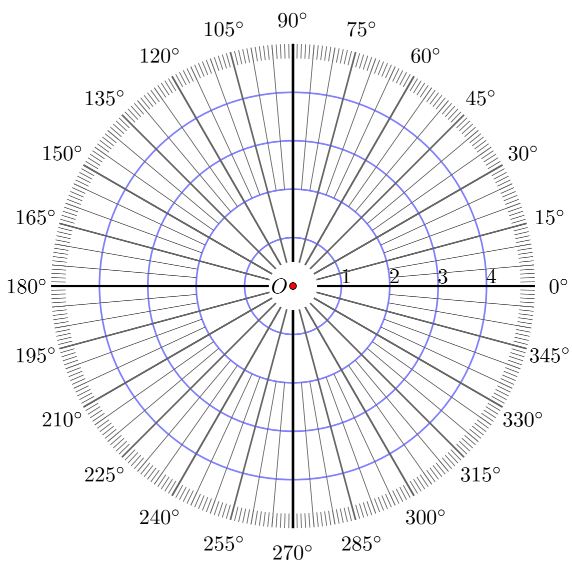


Figure 5.2: Polar graph paper in degrees

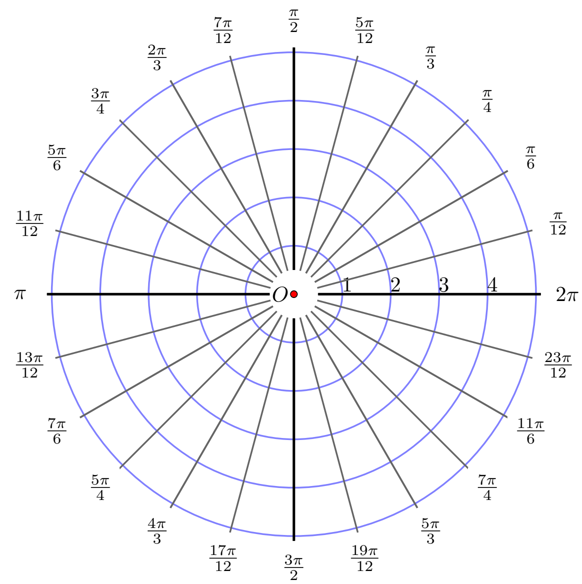


Figure 5.3: Polar graph paper in radians

Example 5.1.1

Plot the points $\left(1.5, \dfrac{5\pi}{6}\right)$ and $\left(-1.5, \dfrac{7\pi}{6}\right)$

Solution: When graphing in polar coordinates always find the angle first, then the radius. The line that represents the angle passes through the origin and extends indefinitely in both directions. The lines representing the angles in **Figure 5.4** are marked with an arrow in the positive direction. If the radius is positive you measure from the center in that direction. If the radius is negative you measure in the opposite direction starting from the center. Just as we could have an infinite number of representation for an angle drawn in standard position there are an infinite number of ways to represent every point in polar coordinates. Notice that if we plot $\left(1.5, \dfrac{\pi}{6}\right)$ it is the same point as $\left(-1.5, \dfrac{7\pi}{6}\right)$.

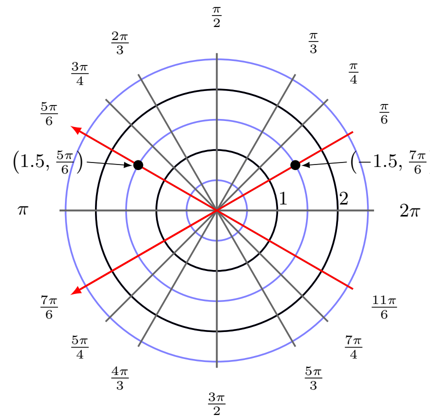


Figure 5.4

Converting between Cartesian and Polar Coordinates

To convert between the coordinate systems we will use a triangle. By drawing a triangle on our previous representation of a point on the plane we can use the trigonometric functions and the Pythagorean theorem to relate x, y, r and θ.

Converting Between Polar and Cartesian Coordinates

$$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

You need to be careful when calculating θ because $\tan^{-1}\left(\frac{y}{x}\right)$ only gives answers between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

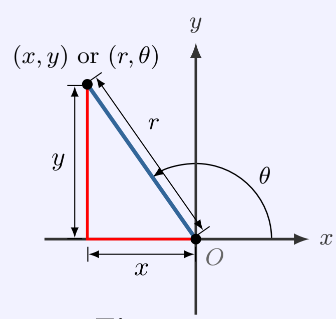


Figure 5.5

Example 5.1.2

Convert the Cartesian points $(1, 1)$ and $(-2, 3)$ to polar coordinates.

Solution It is often best to plot the point before converting. It will be easier to see if you answer makes sense. **Figure 5.6** is the plot of $(1, 1)$ and **Figure 5.7** is the plot of $(-2, 3)$

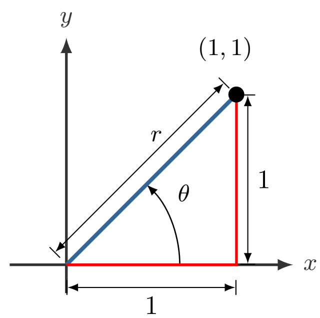


Figure 5.6

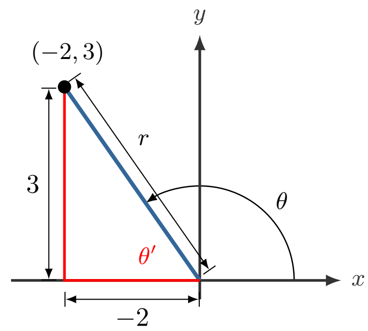


Figure 5.7

$(x, y) = (1, 1)$

This is the standard $45 - 45 - 90$ triangle so

$r = \sqrt{2}$ and

$\theta = \dfrac{\pi}{4}$

$$\boxed{(r, \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)}$$

$(x, y) = (-2, 3)$

$r = \sqrt{2^2 + 3^2}$

$r = \sqrt{13}$

$\theta' = \tan^{-1}\left(\frac{3}{2}\right)$

$\theta = \pi - \theta' = 2.16$

$$\boxed{(r, \theta) = \left(\sqrt{13}, 2.16\right)}$$

Example 5.1.3

Convert the polar points $\left(7, \dfrac{\pi}{3}\right)$ and $\left(7, -\dfrac{5\pi}{3}\right)$ to Cartesian coordinates.

Solution Again we will plot the points before converting. **Figure 5.8** is the plot of $\left(7, \dfrac{\pi}{3}\right)$ and $\left(7, -\dfrac{5\pi}{3}\right)$. These are both the same point so we only have to calculate the Cartesian coordinates for one of them.

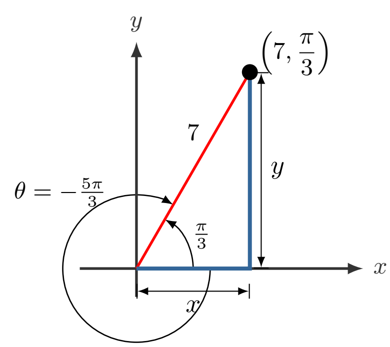


Figure 5.8

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 7\cos\frac{\pi}{3} & &= 7\sin\frac{\pi}{3} \\ &= \frac{7}{2} & &= \frac{7\sqrt{3}}{2} \end{aligned}$$

$$\boxed{(x, y) = \left(\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)}$$

Converting between Cartesian and Polar Equations

Example 5.1.4

Convert $r = 2\cos\theta$ to an equation in Cartesian coordinates and identify the shape of the graph.

Solution: The conversion equations are $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2+y^2$ so we would like our original equation to have pieces that look like these conversions. Since neither side of our original problem looks exactly like any of our conversion equations we will apply a trick to make it look correct. The trick is to multiply by r on both sides of the equation and then convert to cartesian. We will then complete the square to write it in the standard form of a circle.

$$\begin{aligned}
r &= 2\cos\theta && \text{original equation}\\
r^2 &= 2r\cos\theta && \text{Multiply on both sides by } r\\
x^2+y^2 &= 2x && \text{replace } x^2+y^2=r^2 \text{ and } x = r\cos\theta\\
x^2-2x+y^2 &= 0 && \text{move all variables to left}\\
x^2-2x+1+y^2 &= 1 && \text{complete the square}\\
(x-1)^2+y^2 &= 1 && \text{factor}
\end{aligned}$$

The converted equation is $\boxed{(x-1)^2+y^2 \ = \ 1}$ which is a circle with center at $(1,0)$ and radius 1.

Example 5.1.5

Convert $y = 3x+2$ to a polar equation.

Solution: Here we can use the two conversions $x = r\cos\theta$ and $y = r\sin\theta$. We would like to have an equation of the form $r = f(\theta)$ if possible so we will solve for r.

$$\begin{aligned}
y &= 3x+2\\
r\sin\theta &= 3r\cos\theta+2\\
r\sin\theta - 3r\cos\theta &= 2\\
r(\sin\theta - 3\cos\theta) &= 2
\end{aligned}$$

$$\boxed{r \ = \ \frac{2}{\sin\theta - 3\cos\theta}}$$

Graphing Polar Equations

Example 5.1.6

Graph the polar equation $r = \theta$

Solution: To graph this we will create a table of points by selecting θ values and calculating the corresponding r values. Then we connect the dots with a smooth line traveling in a clockwise direction around the circle (**Figure 5.9**). A more complicated graph will need more points.

θ	r	(r,θ)
0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4},\frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2},\frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4},\frac{3\pi}{4}\right)$
$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2},\frac{3\pi}{2}\right)$
π	π	(π,π)
$\frac{5\pi}{4}$	$\frac{5\pi}{4}$	$\left(\frac{5\pi}{4},\frac{5\pi}{4}\right)$

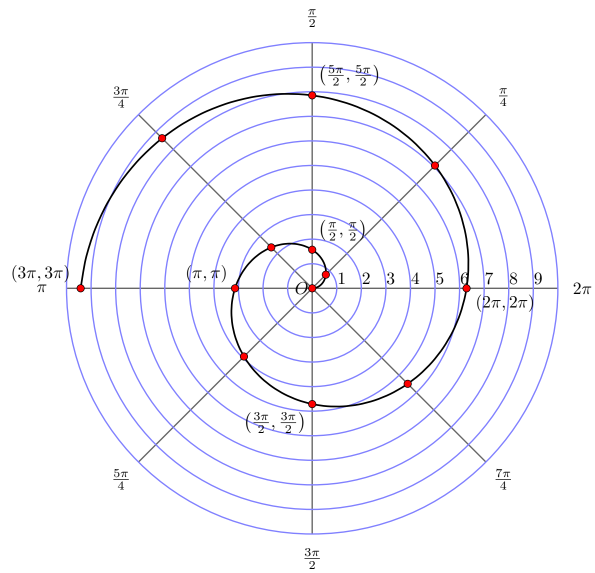


Figure 5.9: $r=\theta$

Example 5.1.7

Graph the polar equation $r = 2\cos\theta$.

Solution: To graph this we will create a table of points by selecting θ values and calculating the corresponding r values. Then we connect the dots with a smooth line traveling in a clockwise direction around the circle (**Figure 5.10**).

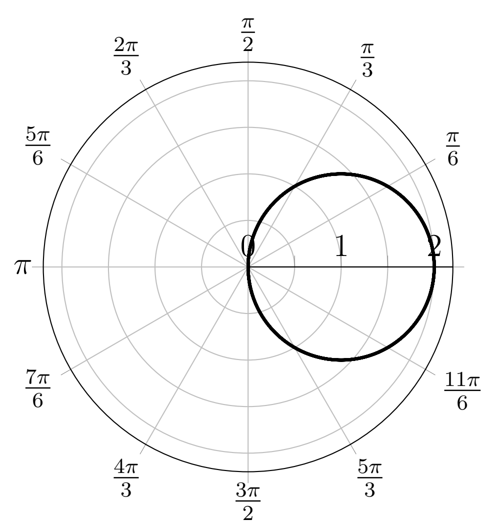


Figure 5.10: $r = 2\cos\theta$

θ	r	(r,θ)
0	1	$(0,1)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\pi}{6},\sqrt{3}\right)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3},1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2},0\right)$
$\frac{2\pi}{3}$	$-\frac{3\pi}{2}$	$\left(\frac{2\pi}{3},-\sqrt{3}\right)$
$\frac{5\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{5\pi}{6},-1\right)$
π	-1	$(\pi,-2)$

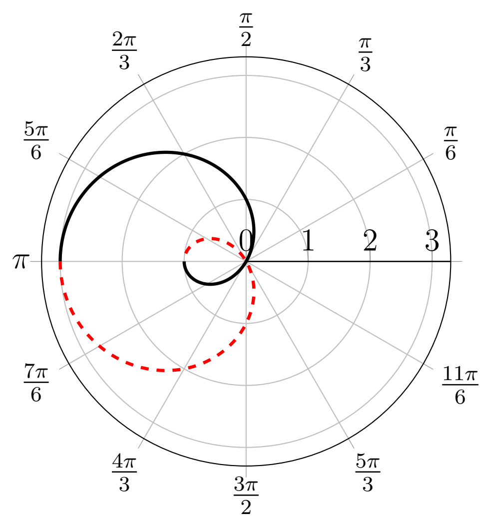
It is not necessary to plot any more points because any extra points will be repeats of the ones in the table. Notice that this is a circle of radius 1 centered at $(1, 0)$ and this is also what was calculated in **Example 5.1.4**. It is not always clear how many points you need to get an accurate graph. It is better to have too many points than too few. While most of the polar graphs are symmetric they can have interesting behavior.

Example 5.1.8

Graph the polar equation $r = 1 - 2\cos\theta$.

Solution: To graph this we will create a table of points by selecting θ values and calculating the corresponding r value. We will use symmetry for this because $\cos\theta = \cos(-\theta)$ so we can plot the values $0 \leq \theta \leq \pi$ and we have equal values for the negative values $-\pi \leq \theta \leq 0$. Notice that this graph is symmetric with respect to the polar axis.

θ	r	(r, θ)
0	-1	$(0, -1)$
$\frac{\pi}{6}$	≈ -0.7321	$\left(\frac{\pi}{6}, -0.7321\right)$
$\frac{\pi}{3}$	0	$\left(\frac{\pi}{3}, 0\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{2\pi}{3}$	2	$\left(\frac{2\pi}{3}, 2\right)$
$\frac{5\pi}{6}$	≈ 2.7321	$\left(\frac{5\pi}{6}, 2.7321\right)$
π	3	$(\pi, 3)$



Black: $r = 1 - 2\cos\theta$ for $0 \leq \theta \leq \pi$

Red: $r = 1 - 2\cos\theta$ for $-\pi \leq \theta \leq 0$

Figure 5.11

Then we connect the dots with a smooth line traveling in a clockwise direction around the circle (**Figure 5.11**). The solid black part of the graph is the table data and the red dashed part of the graph is the part plotted with symmetry.

Example 5.1.9

Graph the polar equation $r = 1 + 2\sin(2\theta)$.

Solution: Since this equation has a 2θ inside the sine we will use values of θ in increments of $\frac{\pi}{12}$ (or $15°$) because when we double those we are at multiples of $\frac{\pi}{6}$ $(30°)$ or $\frac{\pi}{4}$ $(45°)$. These are the values that will be easier to graph because they are our special angles. We will also start at $-\frac{\pi}{12}$ because that will be a point at the origin. We plot points to $\frac{7\pi}{12}$ because that brings the r values back to zero. This set of data produces the large lobe in the first quadrant. Continuing to plot points between $\frac{7\pi}{12} \leq \theta \leq \frac{11\pi}{12}$ produces the smaller lobe in the fourth quadrant.

θ	r	(r, θ)
$-\frac{\pi}{12}$	0	$\left(-\frac{\pi}{12}, 0\right)$
0	1	$(0, 1)$
$\frac{\pi}{12}$	2	$\left(\frac{\pi}{12}, 2\right)$
$\frac{\pi}{6}$	2.732	$\left(\frac{\pi}{6}, 2.732\right)$
$\frac{\pi}{4}$	3	$\left(\frac{\pi}{4}, 3\right)$
$\frac{3\pi}{3}$	2.732	$\left(\frac{\pi}{3}, 2.732\right)$
$\frac{5\pi}{12}$	2	$\left(\frac{5\pi}{12}, 2\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{7\pi}{12}$	0	$\left(\frac{7\pi}{12}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{\pi}{4}, -1\right)$
$\frac{11\pi}{12}$	0	$\left(\frac{11\pi}{12}, 0\right)$

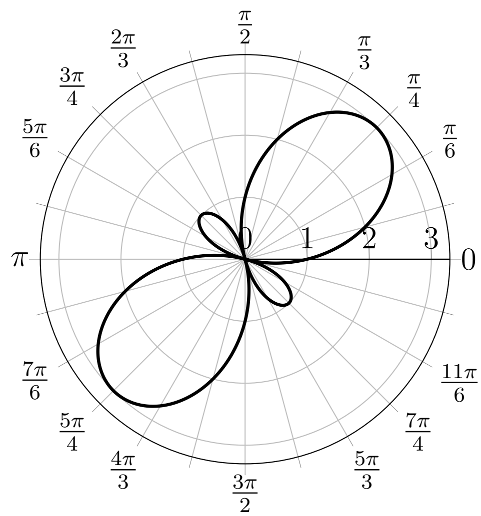
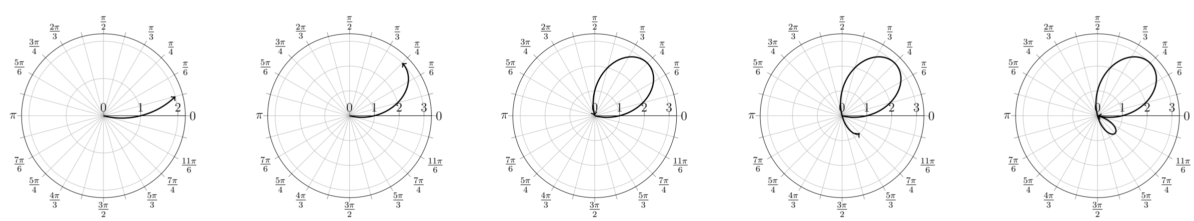
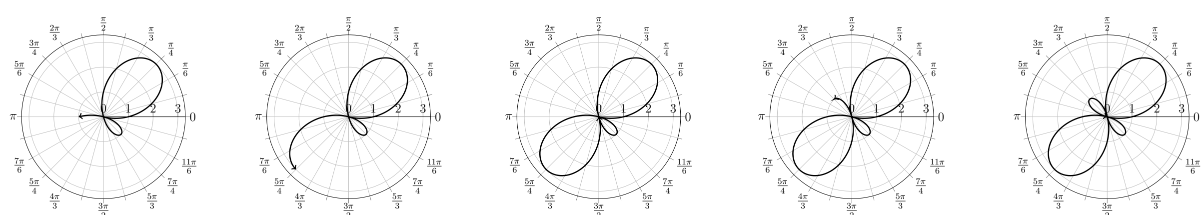


Figure 5.12: $r = 1 + 2\sin(2\theta)$

The process of sketching the this figure is shown below. The process of sketching the points in order, with a smooth curve, is demonstrated through the 10 diagrams.

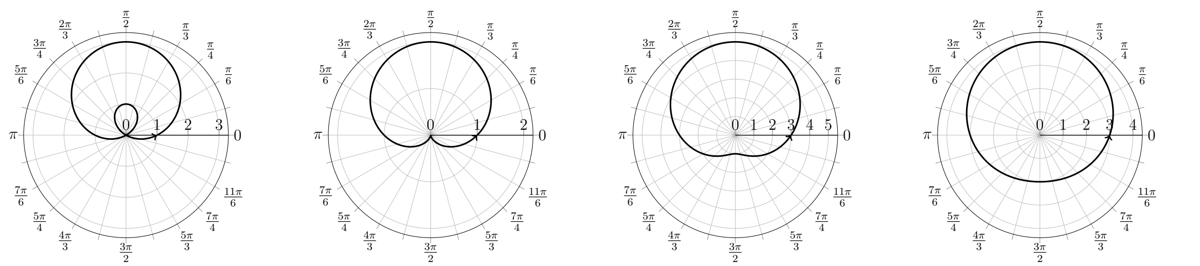


$-\frac{\pi}{12} \leq \theta \leq \frac{\pi}{12}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{\pi}{4}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{7\pi}{12}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{3\pi}{4}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{11\pi}{12}$



$-\frac{\pi}{12} \leq \theta \leq \pi$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{4}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{19\pi}{12}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{7\pi}{4}$ $\quad -\frac{\pi}{12} \leq \theta \leq \frac{23\pi}{12}$

There are some general shapes that the polar graphs can have. The figure drawn in **Example 5.1.8** is called a **limaçon** and is the name given to any curve with an equation of the form $r = a \pm b \sin\theta$ or $r = a \pm b\cos\theta$. The limaçon can take on one of four shapes depending on the relationship between a and b. See **Figure 5.13**. The limacon will be symmetric with the vertical axis if it is a sine graph and symmetric with the horizontal if a cosine graph.



$\frac{a}{b} < 1$ — $1 + 2\sin\theta$ — **(a)** With loop

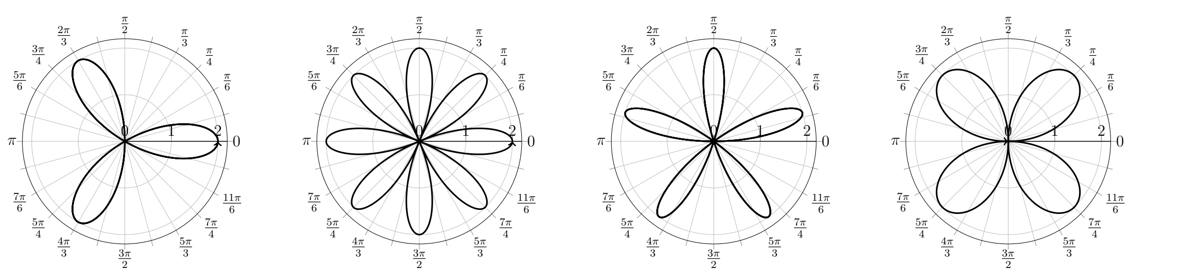
$\frac{a}{b} = 1$ — $1 + \sin\theta$ — **(b)** Cardioid

$1 < \frac{a}{b} < 2$ — $3 + 2\sin\theta$ — **(c)** Dimpled

$\frac{a}{b} \geq 2$ — $3 + \sin\theta$ — **(d)** Convex

Figure 5.13: Some basic limaçons: $r = a \pm b\sin\theta$ or $r = a \pm b\cos\theta$

Another typical shape with polar graphs is the rose shape. This was demonstrated in **Example 5.1.9**. The rose curve comes from equations of the form $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ and rose has n petals if n is odd and has $2n$ petals if n is even.



$r = 2\cos(3\theta)$ — $r = 2\cos(4\theta)$ — $r = 2\sin(5\theta)$ — $r = 2\sin(2\theta)$

Figure 5.14: Typical rose curves, both cosine and sine: $r = a\sin n\theta$ or $r = a\cos n\theta$

5.1 Exercises

For Exercises 1-8 plot the point and convert from polar to Cartesian coordinates.

1. $(4, 210°)$

2. $\left(5, \dfrac{7\pi}{6}\right)$

3. $\left(5, \dfrac{3\pi}{4}\right)$

4. $\left(3, \dfrac{-3\pi}{4}\right)$

5. $\left(4, \dfrac{7\pi}{3}\right)$

6. $\left(-5, \dfrac{11\pi}{4}\right)$

7. $\left(-3, \dfrac{-3\pi}{4}\right)$

8. $\left(2, \dfrac{\pi}{2}\right)$

For Exercises 9-16 convert from Cartesian to polar coordinates.

9. $(6, 2)$

10. $(-1, 3)$

11. $(1, 1)$

12. $(-3, -3)$

13. $(-7, -1)$

14. $(1, -\sqrt{3})$

15. $(-3\sqrt{3}, -3)$

16. $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

For Exercises 17-22 convert the Cartesian equation to a polar equation.

17. $y = 3$

18. $y = x^2$

19. $x^2 + y^2 = 9$

20. $x^2 + y^2 = 9y$

21. $y = \sqrt{3}x$

22. $5y + x + 2 = 0$

For Exercises 23-28 convert the polar equation to a Cartesian equation.

23. $\theta = \dfrac{\pi}{4}$

24. $r = 4\cos\theta$

25. $r = 5$

26. $r = -6\sin\theta$

27. $r = \dfrac{4}{\sin\theta + 7\cos\theta}$

28. $r = 2\sec\theta$

For Exercises 29-37 sketch the graph of the polar equation.

29. $r = 4\cos\theta$

30. $r = -6\sin(2\theta)$

31. $r = 3\sin(5\theta)$

32. $r = 4 + 4\cos\theta$

33. $r = 1 + 2\cos(2\theta)$

34. $r = 3\cos(3\theta)$

35. $r = 5$

36. $r = 2 + 4\sin\theta$

37. $\theta = \dfrac{\pi}{4}$

5.2 Vectors in the Plane

Introduction

We deal with many quantities that are represented by a number that shows their magnitude. These include speed, money, time, length and temperature. Quantities that are represented only by their **magnitude** or size are called **scalars** . When you travel in your car and you look at the speedometer it tells you how fast you are going but not where you are going. This is a scalar value and is called the **speed**.

A **vector** is a quantity that has both a *magnitude* (size) and a *direction*. To describe a vector you must have both parts. If you know that you are traveling at 150 mph north then that would be a vector quantity and it is called the **velocity**. It tells you how fast you are traveling, speed is 150 mph, as well as the direction, north.

Vector Representations

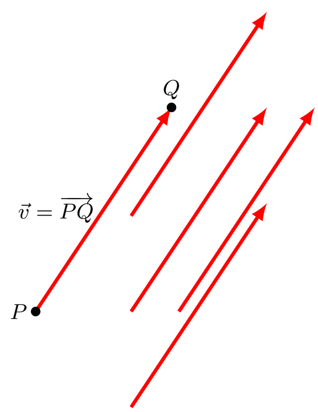


Figure 5.15: Equivalent vectors: same magnitude and direction

When we write a vector there are two common ways to do it. If we want to talk about "vector v" we can either write the **v** in bold or write the $\vec{v}$ with an arrow over it. In this text we will most often use the arrow notation but do be aware that the bold notation is also common.

To describe a vector we need to talk about both the magnitude and direction. The magnitude of a vector is represented by the notation $||\vec{v}||$. The direction can be described in different ways and depends on the application. For example you might say that a jet is traveling in the direction 10° north of east, or a force is applied at a particular angle or with a particular slope.

A vector can be represented by simply an arrow: in **Figure** 5.15 the vector $\vec{v} = \overrightarrow{PQ}$ which starts at point P and ends at point Q has magnitude equal to its length ($||\vec{v}||$) and direction as indicated. The vector can be moved around in the plane as long as the length and direction are unchanged. All the vectors in **Figure** 5.15 are equivalent because they all have the same length and point in the same direction. When the vector is drawn this way the length is always the magnitude. An accurate picture is necessary to accurately describe a vector this way. Sometimes it is called a **directed line segment**.

Example 5.2.1

Show that the directed segment $\vec{u}$ which starts at $P(-3, -2)$ and ends at $Q(1, 4)$ is equivalent to the directed segment $\vec{v}$ which starts at $R(3,1)$ and ends at $S(7,7)$.

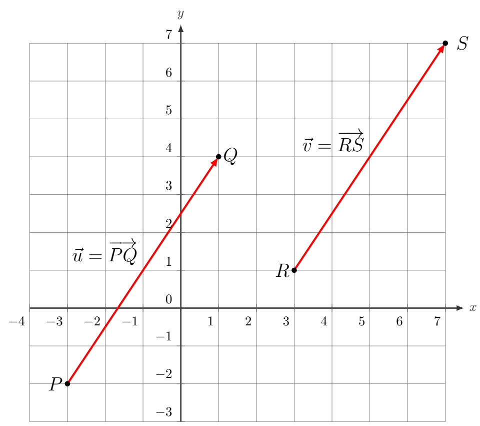


Figure 5.16

Solution: To show that the two vectors are equivalent we need to show that they have the same length and direction. Using the distance formula we can see they have the same length.

$$\begin{aligned} ||\vec{u}|| &= \sqrt{(1-(-3))^2 + (4-(-2))^2} \\ &= \sqrt{4^2+6^2} \\ &= 2\sqrt{13} \\ ||\vec{v}|| &= \sqrt{(7-3)^2+(7-1)^2} \\ &= \sqrt{4^2+6^2} \\ &= 2\sqrt{13} \end{aligned}$$

Both of these vectors have the same direction because they are both pointing to the upper right and have the same slope:

$$\frac{\Delta y}{\Delta x} = \frac{(4-(-2))}{(1-(-3))} = \frac{(7-3)}{(7-1)} = \frac{3}{2}$$

Thus they are equivalent.

A vector drawn starting at the origin is in **standard position** as shown in **Figure 5.17**. A vector in standard position has initial point at the origin $(0,0)$ and can be represented by the endpoint of the vector (a,b). This is known as representing the **vector by components**: $\vec{v} = \langle a, b\rangle$. It is common to see this written as $\vec{v} = \langle v_x, v_y\rangle$. See **Figure 5.17**. Notice the use of "angle brackets" $\langle\ \rangle$ to write the vector. This distinguishes it from the point at the end of the vector. Writing a vector as components is generally preferable because it is easier to perform calculations with components rather than directed line segments. Also, while all the work in this book is with two dimensional vectors you can also write vectors in three or even more dimensions. It is very difficult to draw a directed segment in three dimensions while writing it with components is quite straight forward.

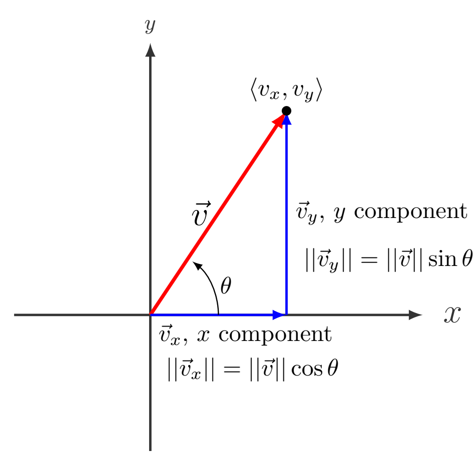


Figure 5.17: A vector split into the x and y components

If you want to write $\vec{v}$ from point P to point Q then $\vec{v} = P - Q$. For example in **Example 5.2.1** $\vec{u} = \overrightarrow{PQ} = (1,4) - (-3,-2) = \langle 4, 6\rangle$. It is important to subtract in the correct order. It is always "end point" minus "starting point". If you subtract in the wrong order you end up with a vector that has the same length but points in the opposite direction.

Component Form of a Vector

The component form of a vector $\vec{v}$ with initial point $P(p_1, p_2)$ and end point $Q(q_1, q_2)$ is

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2\rangle = \langle v_x, v_y\rangle = \vec{v}$$

The magnitude of $\vec{v}$, $||\vec{v}||$, is found by the Pythagorean theorem.

$$||\vec{v}|| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{(v_x)^2 + (v_y)^2}$$

A vector of magnitude (or length) 1 is called a **Unit Vector** . To create a unit vector you can divide any vector by its length. A unit vector in the direction of $\vec{v}$ is given by

$$\text{unit vector in the direction of } \vec{v} = \frac{\vec{v}}{||\vec{v}||}$$

Unit vectors are important because they can be used to represent the direction of the vector. Any vector $\vec{v}$ can be written as a product of the magnitude and direction, where the direction is the unit vector in the direction of $\vec{v}$

$$\vec{v} = ||\vec{v}|| \cdot \frac{\vec{v}}{||\vec{v}||} = \text{magnitude} \cdot \text{direction}$$

Example 5.2.2

Find the component form of the vector $\vec{v}$ that starts at $P(1,2)$ and ends at $Q(-3,4)$. Find the length of $\vec{v}$. Find a unit vector in the direction of $\vec{v}$. Write $\vec{v}$ as "magnitude · direction" where the direction is the unit vector. Sketch the vector in standard position.

Solution: $\vec{v} = \langle q_1 - p_1, q_2 - p_2\rangle = \langle(-3-1),(4-2)\rangle =$ $\boxed{\vec{v} = \langle -4, 2\rangle}$

The length: $||\vec{v}|| = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(-4)^2 + (2)^2} =$ $\boxed{||\vec{v}|| = 2\sqrt{5}}$

Unit vector: $\vec{u} = \dfrac{\vec{v}}{||\vec{v}||} = \dfrac{\langle -4, 2\rangle}{2\sqrt{5}} = \boxed{\left\langle \dfrac{-2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}} \right\rangle}$

"magnitude · direction" : $\boxed{\vec{v} = 2\sqrt{5} \cdot \left\langle \dfrac{-2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}} \right\rangle}$

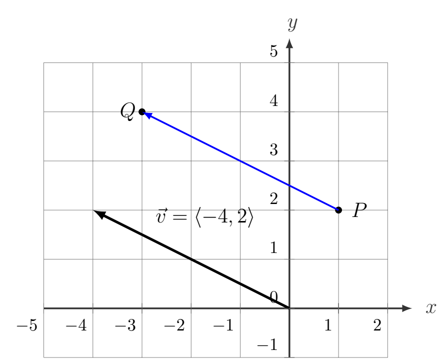


Figure 5.18

Example 5.2.3

Write the component form of the vector $\vec{v}$ with magnitude 7 and direction $\theta = 132°$ measured from the positive x axis. Sketch the vector in standard position. Find a unit vector in the direction of $\vec{v}$. Write $\vec{v}$ as "magnitude · direction" where the direction is the unit vector.

Solution: To write the vector in components we need to calculate the two legs of the triangle shown in **Figure 5.19**. We can do this by using the sine and cosine as was shown in **Figure 5.17**.

$$v_x = ||\vec{v}|| \cos\theta = 7\cos(132°) = -4.7$$

$$v_y = ||\vec{v}|| \sin\theta = 7\sin(132°) = 5.2$$

So $\boxed{\vec{v} = \langle -4.7, 5.2 \rangle}$. Notice that the signs of the trigonometric functions give the correct sign on the components.

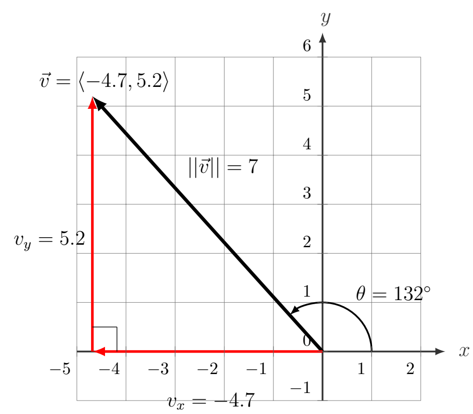


Figure 5.19

Unit vector: $\vec{u} = \dfrac{\vec{v}}{||\vec{v}||} = \dfrac{\langle -4.7, 5.2 \rangle}{7} = \boxed{\left\langle \dfrac{-4.7}{7}, \dfrac{5.2}{7} \right\rangle}$

"magnitude · direction" : $\boxed{\vec{v} = 7 \cdot \left\langle \dfrac{-4.7}{7}, \dfrac{5.2}{7} \right\rangle}$

Vector Operations

There are mathematical operations that we can do with vectors. The two most common are **multiplication by a scalar** and **vector addition** . Recall that a scalar is a number. If you want to multiply a vector $\vec{v}$ by a scalar k there are two ways to think about it. Multiplying by the scalar k does not change the direction of the vector but makes it longer or shorter by a factor of k. If you have your vector written in components $\vec{v} = \langle v_x, v_y \rangle$ then each component is multiplied by k:

$$k \cdot \vec{v} = k \cdot \langle v_x, v_y \rangle = \langle k \cdot v_x, k \cdot v_y \rangle$$

Example 5.2.4

Find the result when $\vec{u} = \langle 6, -1 \rangle$ is multiplied by 7.

Solution: $7\vec{u} = 7\,\langle 6, -1 \rangle = \boxed{\langle 42, -7 \rangle}$

Example 5.2.5

Find the result when $\vec{u} = \langle 6, -1 \rangle$ is multiplied by -1.

Solution: $(-1)\vec{u} = -\vec{u} = (-1)\,\langle 6, -1 \rangle = \boxed{\langle -6, 1 \rangle}$.

Note that $-\vec{u}$ is the same vector as $\vec{u}$ but pointing in the opposite direction. You can see this if you sketch both on the same set of axes.

Adding vectors can be done two ways. We can add vectors that are written as directed line segments or we can add them as components. If you wish to add $\vec{u}$ to $\vec{v}$ you draw $\vec{v}$ and then draw $\vec{u}$ so that the tail of $\vec{u}$ starts at the head of $\vec{v}$. You can see in **Figure 5.20** that it does not matter in which order you do this. $\vec{R} = \vec{u} + \vec{v} = \vec{v} + \vec{u}$

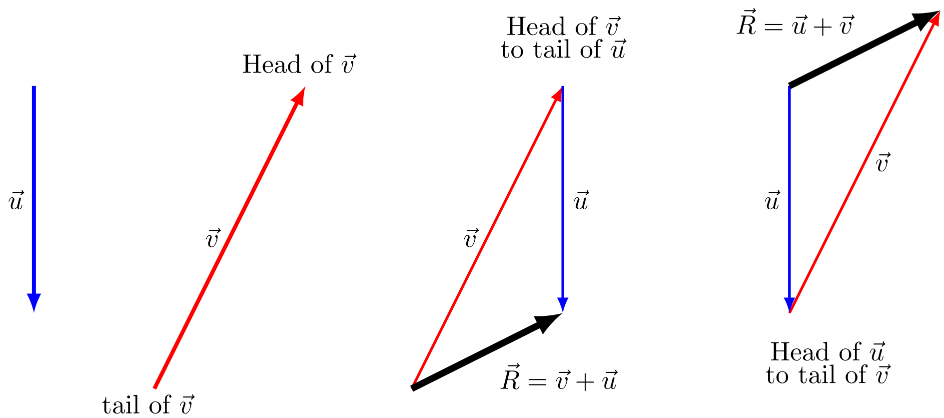


Figure 5.20: Adding vectors can be done in either order.

If the vectors are written as components you can add the x components and the y components separately. The component operations are summarized below.

Vector Addition and Scalar Multiplication

Given vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ and scalar k then the sum or difference of $\vec{u}$ and $\vec{v}$ is given by

$$\vec{u} + \vec{v} = \langle u_x, u_y \rangle + \langle v_x, v_y \rangle = \langle u_x + v_x, u_y + v_y \rangle$$

$$\vec{u} - \vec{v} = \langle u_x, u_y \rangle - \langle v_x, v_y \rangle = \langle u_x - v_x, u_y - v_y \rangle$$

The scalar multiple of k and $\vec{v}$ is

$$k \cdot \vec{v} = k \cdot \langle v_x, v_y \rangle = \langle k \cdot v_x, k \cdot v_y \rangle$$

Example 5.2.6

Let $\vec{u} = \langle 1, -2 \rangle$ and $\vec{v} = \langle -4, 2 \rangle$, and find

(a) $3\vec{u} + \vec{v}$

(b) $\vec{u} - \vec{v}$

(c) $\vec{v} - 2\vec{u}$

Solution: To add these we need to add the corresponding components. The order of operations is still valid here, perform the scalar multiplication first and then the vector addition.

(a) $3\vec{u} + \vec{v} = 3\langle 1, -2 \rangle + \langle -4, 2 \rangle = \langle 3, -6 \rangle + \langle -4, 2 \rangle = \boxed{\langle -1, -4 \rangle}$

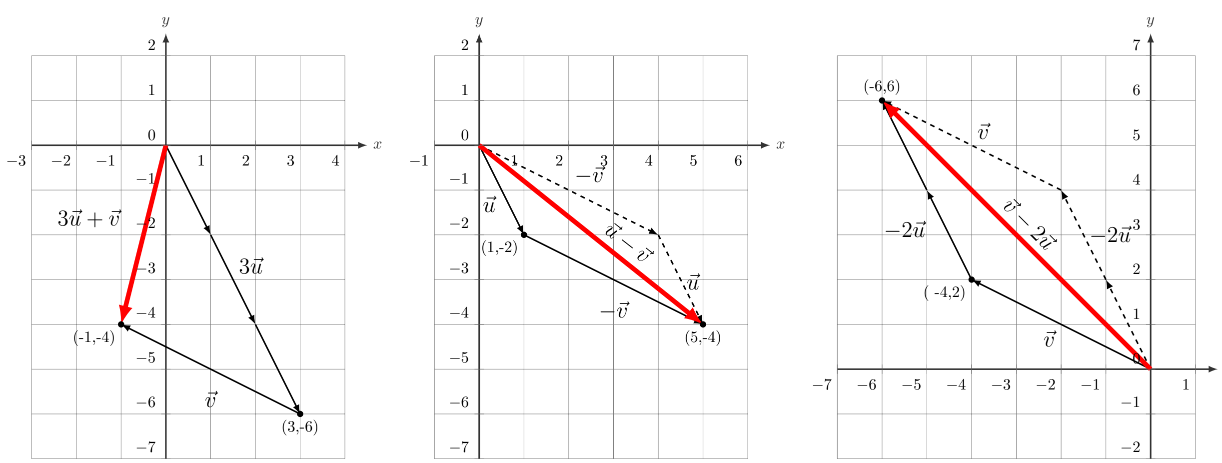
The solution is also shown in **Figure 5.21** (a)

(b) $\vec{u} - \vec{v} = \langle 1, -2 \rangle - \langle -4, 2 \rangle = \boxed{\langle 5, -4 \rangle}$

To do this with arrows on paper it is easiest to draw $-\vec{v}$ and then add that to $\vec{u}$. Remember that $-\vec{v}$ is the same as $\vec{v}$ but the arrow is on the other end of the vector. The solution is shown in **Figure 5.21** (b) Notice that we can add in either order, the dotted vectors are the result of $-\vec{v} + \vec{u}$

(c) $\vec{v} - 2\vec{u} = \langle -4, 2 \rangle - 2\langle 1, -2 \rangle = \langle -4, 2 \rangle + \langle -2, 4 \rangle = \boxed{\langle -6, 6 \rangle}$

Be careful with the sign when multiplying by the -2. The solution is shown in **Figure 5.21** (c). Notice that we can add in either order, the dotted vectors are the result of $-2\vec{u} + \vec{v}$



(a) $3\vec{u} + \vec{v}$ (b) $\vec{u} - \vec{v}$ (c) $\vec{v} - 2\vec{u}$

Figure 5.21

Example 5.2.7

To avoid a storm a jet travels 20° north of east from Winnipeg for 300 km and then turns to a heading 62° south of east for 1150 km to arrive at Chicago. Find the displacement from Winnipeg to Chicago.

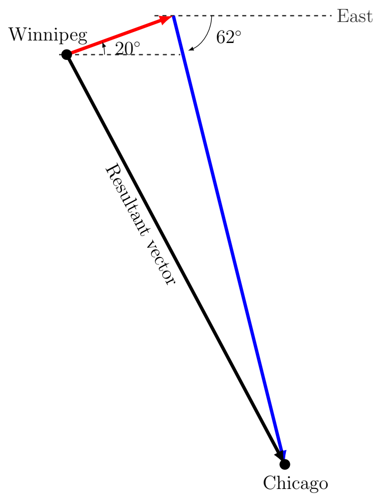


Figure 5.22

Solution: **Figure 5.22** shows the flight path. It is a good idea to draw a picture if possible. While it would be possible to try and measure the vectors and angles it will be easier to add these by components. We will calculate the components for each leg of the journey and then add them up. For the first leg $\overrightarrow{L1} = \langle L1_x, L1_y \rangle$ we have

$$L1_x = 300\cos(20^\circ) = 282$$

$$L1_y = 300\sin(20^\circ) = 103$$

For the second leg $\overrightarrow{L2} = \langle L2_x, L2_y \rangle$ we have

$$L2_x = 1150\cos(62^\circ) = 540$$

$$L2_y = -1150\sin(62^\circ) = -1015$$

It is important to notice that the y component is negative because it points in the negative y direction. The picture will help make sure the signs are correct on your components.

The resultant vector is $\overrightarrow{L1} + \overrightarrow{L2} = \langle 282, 103\rangle + \langle 540, -1015\rangle = \langle 822, -912\rangle$. The distance from Winnipeg to Chicago is the magnitude of the resultant vector. So the displacement is $\sqrt{822^2 + (-912)^2} = \boxed{1228 \text{ km}}$.

Example 5.2.8

An airplane is traveling with a ground speed of 750 km/hr at a bearing 37° west of north when it encounters a strong wind with a velocity 100 km/hr at a bearing of 60° north of east. Find the resultant speed and direction of the airplane. **Figure 5.23**

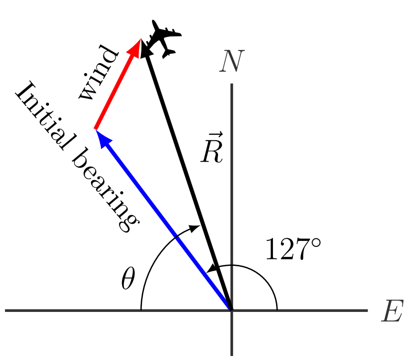


Figure 5.23: Not to scale

Solution: The resultant speed and direction of the airplane ($\vec{R}$) is the sum of the plane's ground speed velocity vector and the wind speed vector. **Figure 5.23** shows the relationship between the vectors. To add them we will first write them as components. Let $\vec{P} = \langle P_x, P_y\rangle$ be the airplane ground speed vector and $\vec{W} = \langle W_x, W_y\rangle$ be the wind speed vector.

$$\begin{aligned}\vec{P} &= 750\,\langle\cos(127^\circ), \sin(127^\circ)\rangle \\ &\approx \langle -451, 599\rangle\,\text{km/hr}\\ \vec{W} &= 100\,\langle\cos(60^\circ), \sin(60^\circ)\rangle \\ &\approx \langle 50, 87\rangle\,\text{km/hr}\end{aligned}$$

Note the signs on the components of the vectors and compare them to the figure. You expect the x component of the airplane's ground speed vector to be negative, and it is.

So the velocity of the plane in the wind is

$$\begin{aligned}\vec{R} &= \vec{P} + \vec{W} \\ &\approx \langle -451, 599\rangle + \langle 50, 87\rangle \\ &\approx \langle -401, 686\rangle \text{ km/hr}\end{aligned}$$

and the resultant speed of the airplane

$$\begin{aligned}||\vec{R}|| &\approx \sqrt{(-401)^2 + (686)^2} \\ &\approx 795\text{km/hr}\end{aligned}$$

For the bearing we will use the angle θ made with the negative x axis as shown in the figure.

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{686}{401}\right) \\ &\approx 59.7^\circ\end{aligned}$$

which we write as 59.7° north of west. And we can put them together to say the airplane is traveling at $\boxed{\text{795 km/hr bearing 59.7°north of west}}$

Example 5.2.9

A common use for vectors in physics and engineering applications is adding up forces acting on an object. Suppose there are three forces acting on an object as shown in **Figure 5.24**, a 40 Newton[1] force acting at 30°, a 30 Newton force acting at 300° and a 50 Newton force acting at 135°. Find the resultant force vector acting on the object.

Solution: The resultant force will be the sum of all the vectors. To add them we will first write them as components. Since we are measuring all the the angles from the horizontal x-axis the signs of each of the components will be correct because the sine and cosine functions will be positive and negative in the correct quadrants. You can verify this by noticing that the x component of F_2 and the y component of F_3 are both negative.

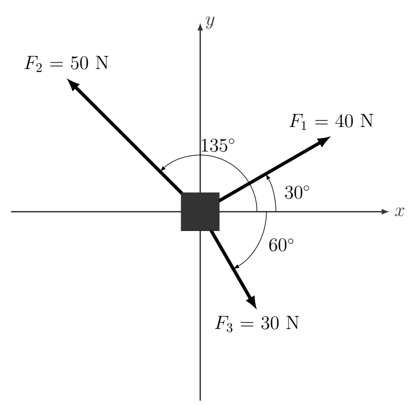


Figure 5.24

$$\begin{aligned}\vec{F_1} &= 40\langle\cos(30^\circ), \sin(30^\circ)\rangle \\ &\approx \langle 34.641, 20\rangle \text{ N} \\ \vec{F_2} &= 50\langle\cos(135^\circ), \sin(135^\circ)\rangle \\ &\approx \langle -35.355, 35.355\rangle \text{ N} \\ \vec{F_3} &= 30\langle\cos(300^\circ), \sin(300^\circ)\rangle \\ &\approx \langle 15, -25.981\rangle \text{ N}\end{aligned}$$

[1]A **Newton** (N) is a metric unit of force $N = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$

$$\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = \langle 34.641, 20 \rangle + \langle -35.355, 35.355 \rangle + \langle 15, -25.981 \rangle$$

$$\boxed{\vec{R} = \langle 14.286, 29.375 \rangle}$$

We can find the magnitude

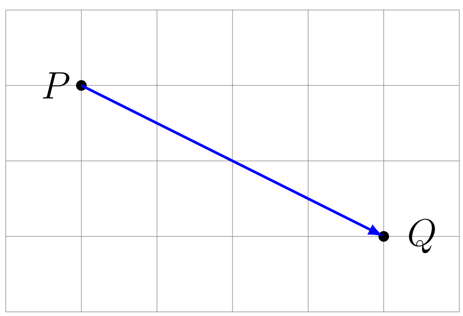
$$||\vec{R}|| = \sqrt{14.286^2 + 29.375^2} \approx 32.664 \text{ N}$$

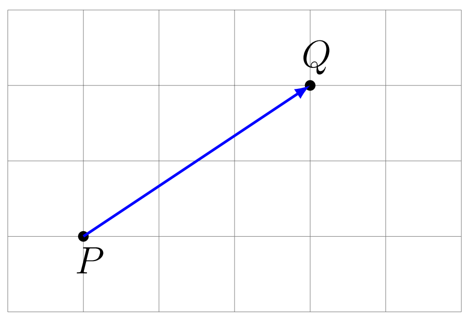
and direction of the resultant vector:

$$\theta = \tan^{-1}\left(\frac{29.375}{14.286}\right) \approx 64^\circ$$

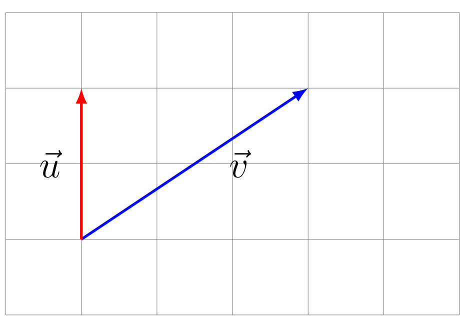
5.2 Exercises

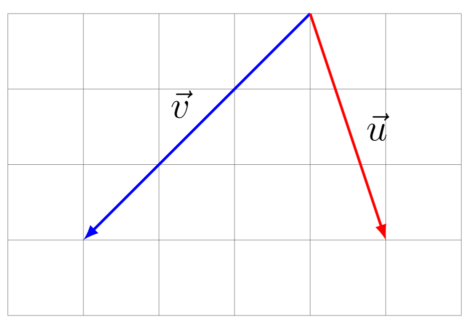
For Exercises 1-2 write the vector shown in component form.

1. 

2. 

For Exercises 3-4 given the vectors shown, sketch $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, and $2\vec{u}$.

3. 

4. 

For Exercises 5-10 write the vector $\vec{v} = \overrightarrow{PQ}$ in the form of $\vec{v} = (\text{magnitude}) \cdot (\text{direction})$ where the direction is a unit vector. See **Example 5.2.2**.

5. $P = (1, 2)$, $Q = (-2, 3)$

6. $P = (-3, 2)$, $Q = (-3, 3)$

7. $P = (0, 1)$, $Q = (-2, -7)$

8. $P = (-40, 23)$, $Q = (5, -5)$

9. $P = (-4, 2)$, $Q = (2, -3)$

10. $P = (1, 2)$, $Q = (0, 0)$

For Exercises 11-14 write the vector in component form from the given magnitude and direction.

11. Magnitude: 6; direction: 30°

12. Magnitude: 7; direction: 120°

13. Magnitude: 8; direction: 225°

14. Magnitude: 9; direction: 330°

For Exercises 15-18 given the vectors, compute $3\vec{u}$, $2\vec{u} + \vec{v}$, and $\vec{u} - 3\vec{v}$.

15. $\vec{u} = \langle 2, -2 \rangle$, $\vec{v} = \langle 3, 2 \rangle$

16. $\vec{u} = \langle 1, -2 \rangle$, $\vec{v} = \langle -4, 2 \rangle$

17. $\vec{u} = \langle 2, -3 \rangle$, $\vec{v} = \langle 1, 2 \rangle$

18. $\vec{u} = \langle 3, 4 \rangle$, $\vec{v} = \langle 5, -6 \rangle$

19. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

20. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

21. A person starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far have they walked? If they walked straight home, how far would they have to walk?

22. A person starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles north, 5 miles southwest, and 3 miles east. How far have they walked? If they walked straight home, how far would they have to walk?

23. Three forces act on an object: $\vec{F_1} = \langle 2, 5 \rangle$, $\vec{F_2} = \langle 8, 3 \rangle$ and $\vec{F_3} = \langle 0, -7 \rangle$. Find the net force acting on the object.

24. Three forces act on an object: $\vec{F_1} = \langle -2, 5 \rangle$, $\vec{F_2} = \langle -8, -3 \rangle$ and $\vec{F_3} = \langle 5, 0 \rangle$. Find the net force acting on the object.

25. Suppose there are three forces acting on an object, a 10 Newton force acting at 45°, a 20 Newton force acting at 210° and a 15 Newton force acting at 315°. Find the resultant force vector acting on the object.

26. A person starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If they walked straight home, how far would they have to walk, and in what direction?

27. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the planes speed relative to the ground?

28. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the planes speed relative to the ground?

29. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, the pilot will need to fly the plane how many degrees west of north?

30. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, the pilot will need to fly the plane how many degrees west of north?

31. As part of a video game, the point $\langle 5, 7\rangle$ is rotated counterclockwise about the origin through an angle of 35 degrees. Find the new coordinates of this point.

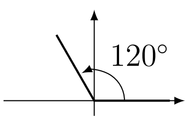
32. As part of a video game, the vector $\langle 7, 3\rangle$ is rotated counterclockwise about the origin through an angle of 40 degrees. Find the new coordinates of this point.

Appendix A

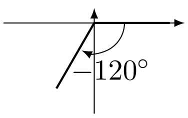
Answers and Hints to Selected Exercises

Chapter 1

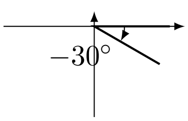
Section 1.1 (page 9)

1. 

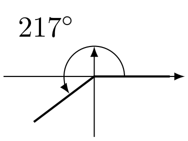
, $480°$, $-240°$

2. 

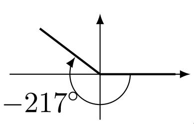
, $240°$, $-480°$

3. 

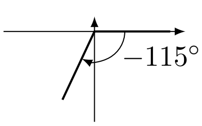
, $330°$, $-390°$

4. 

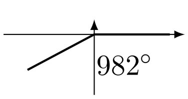
, $577°$, $-143°$

5. 

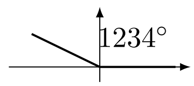
, $143°$, $-577°$

6. 

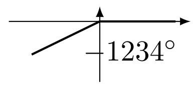
, $330°$, $-390°$

7. 

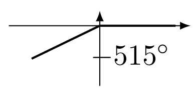
, $208°$, $-152°$

8. 

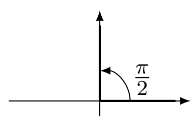
, $154°$, $-206°$

9. 

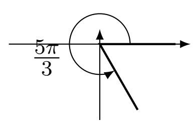
, $206°$, $-154°$

10. 

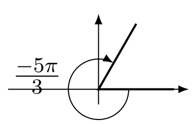
, $205°$, $-155°$

11. 

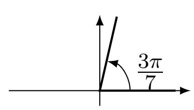
, $\frac{5\pi}{2}$, $\frac{-3\pi}{2}$

12. 

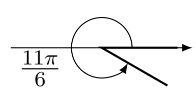
, $\frac{11\pi}{3}$, $\frac{-\pi}{3}$

13. 

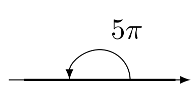
, $\frac{\pi}{3}$, $\frac{-11\pi}{3}$

14. 

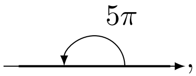
, $\frac{17\pi}{7}$, $\frac{-11\pi}{7}$

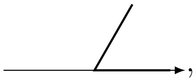
15. 

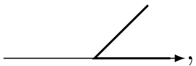
, $\frac{23\pi}{6}$, $\frac{-\pi}{6}$

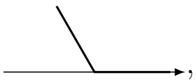
16. 

, π, $-\pi$

17. , 1.849, −4.434

18. , $\frac{\pi}{3}$, $\frac{-5\pi}{3}$

19. , $\frac{\pi}{4}$, $\frac{-7\pi}{4}$

20. , $\frac{2\pi}{3}$, $\frac{-4\pi}{3}$

21. $\frac{2\pi}{3}$ **22.** $\frac{23\pi}{36}$ **23.** $\frac{3\pi}{4}$ **24.** $\frac{-85\pi}{36}$

25. $\frac{-3\pi}{2}$ **26.** $\frac{\pi}{12}$ **27.** 90° **28.** 60°

29. 45° **30.** 36° **31.** -30° **32.** -330°

33. $12^\circ\ 30'$ **34.** $125^\circ\ 42'$

35. $539^\circ\ 15'$ **36.** $7352^\circ\ 7'\ 12''$

37. 12.203° **38.** 25.972°

39. 0.371° **40.** 1.017°

41. $52^\circ\ 7'\ 60''$, $106^\circ\ 40'\ 12''$

42. 35 mi **43.** 6 ft **44.** 25.1 cm

45. 31.4 mi **46.** 22.9°

47. 2.58 million miles

48. 120.5 km **49.** 3.373 km

50. $\frac{1}{2}$ radian **51.** 0.4 radian

52. $14.14\ \text{cm}^2$ **53.** $897.6\ \text{cm}^2$

Section 1.2 (page 20)

2. $\sin A = \frac{5\sqrt{61}}{61}$, $\cos A = \frac{6\sqrt{61}}{61}$, $\tan A = \frac{5}{6}$
$\csc A = \frac{\sqrt{61}}{5}$, $\sec A = \frac{\sqrt{61}}{6}$, $\cot A = \frac{6}{5}$
$\sin B = \frac{6\sqrt{61}}{61}$, $\cos B = \frac{5\sqrt{61}}{61}$, $\tan B = \frac{6}{5}$
$\csc B = \frac{\sqrt{61}}{6}$, $\sec B = \frac{\sqrt{61}}{5}$, $\cot B = \frac{5}{6}$

3. $\sin A = \frac{5}{6}$, $\cos A = \frac{\sqrt{11}}{6}$, $\tan A = \frac{5\sqrt{11}}{11}$
$\csc A = \frac{6}{5}$, $\sec A = \frac{6\sqrt{11}}{11}$, $\cot A = \frac{\sqrt{11}}{5}$
$\sin B = \frac{\sqrt{11}}{6}$, $\cos B = \frac{5}{6}$, $\tan B = \frac{\sqrt{11}}{5}$
$\csc B = \frac{6\sqrt{11}}{11}$, $\sec B = \frac{6}{5}$, $\cot B = \frac{5\sqrt{11}}{11}$

4. $\sin A = \frac{5\sqrt{34}}{34}$, $\cos A = \frac{3\sqrt{34}}{34}$, $\tan A = \frac{5}{3}$
$\csc A = \frac{\sqrt{34}}{5}$, $\sec A = \frac{\sqrt{34}}{3}$, $\cot A = \frac{3}{5}$
$\sin B = \frac{3\sqrt{34}}{34}$, $\cos B = \frac{5\sqrt{34}}{34}$, $\tan B = \frac{3}{5}$
$\csc B = \frac{\sqrt{34}}{3}$, $\sec B = \frac{\sqrt{34}}{5}$, $\cot B = \frac{5}{3}$

5. $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$
$\csc A = \frac{5}{3}$, $\sec A = \frac{5}{4}$, $\cot A = \frac{4}{3}$
$\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$, $\tan B = \frac{4}{3}$
$\csc B = \frac{5}{4}$, $\sec B = \frac{5}{3}$, $\cot B = \frac{3}{4}$

6. $\sin A = \frac{7}{25}$, $\cos A = \frac{24}{25}$, $\tan A = \frac{7}{24}$
$\csc A = \frac{25}{7}$, $\sec A = \frac{25}{24}$, $\cot A = \frac{24}{7}$
$\sin B = \frac{24}{25}$, $\cos B = \frac{7}{25}$, $\tan B = \frac{24}{7}$
$\csc B = \frac{25}{24}$, $\sec B = \frac{25}{7}$, $\cot B = \frac{7}{24}$

7. $\sin A = \frac{1}{2}$, $\cos A = \frac{\sqrt{3}}{2}$, $\tan A = \frac{\sqrt{3}}{3}$
$\csc A = 2$, $\sec A = \frac{2\sqrt{3}}{3}$, $\cot A = \sqrt{3}$
$\sin B = \frac{\sqrt{3}}{2}$, $\cos B = \frac{1}{2}$, $\tan B = \sqrt{3}$
$\csc B = \frac{2\sqrt{3}}{3}$, $\sec B = 2$, $\cot B = \frac{\sqrt{3}}{3}$

8. $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$
$\csc A = \frac{13}{5}$, $\sec A = \frac{13}{12}$, $\cot A = \frac{12}{5}$
$\sin B = \frac{12}{13}$, $\cos B = \frac{5}{13}$, $\tan B = \frac{12}{5}$
$\csc B = \frac{13}{12}$, $\sec B = \frac{13}{5}$, $\cot B = \frac{5}{12}$

9. $\sin A = \frac{\sqrt{5}}{3}$, $\cos A = \frac{2}{3}$, $\tan A = \frac{\sqrt{5}}{2}$
$\csc A = \frac{3\sqrt{5}}{5}$, $\sec A = \frac{3}{2}$, $\cot A = \frac{2\sqrt{5}}{5}$
$\sin B = \frac{2}{3}$, $\cos B = \frac{\sqrt{5}}{3}$, $\tan B = \frac{2\sqrt{5}}{5}$
$\csc B = \frac{3}{2}$, $\sec B = \frac{3\sqrt{5}}{5}$, $\cot B = \frac{\sqrt{5}}{2}$

10. $\sin A = \frac{3}{4}$, $\cos A = \frac{\sqrt{7}}{4}$, $\tan A = \frac{3\sqrt{7}}{7}$
$\csc A = \frac{4}{3}$, $\sec A = \frac{4\sqrt{7}}{7}$, $\cot A = \frac{\sqrt{7}}{3}$

11. $\sin A = \frac{\sqrt{7}}{4}$, $\cos A = \frac{3}{4}$, $\tan A = \frac{\sqrt{7}}{3}$
$\csc A = \frac{4\sqrt{7}}{7}$, $\sec A = \frac{4}{3}$, $\cot A = \frac{3\sqrt{7}}{7}$

12. $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$
$\csc A = \frac{5}{3}$, $\sec A = \frac{5}{4}$, $\cot A = \frac{4}{3}$

13. $\sin A = \frac{2\sqrt{2}}{3}$, $\cos A = \frac{1}{3}$, $\tan A = 2\sqrt{2}$
$\csc A = \frac{3\sqrt{2}}{4}$, $\sec A = 3$, $\cot A = \frac{\sqrt{2}}{4}$

14. $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$
$\csc A = \frac{13}{12}$, $\sec A = \frac{13}{5}$, $\cot A = \frac{5}{12}$

15. $\sin A = \frac{2\sqrt{5}}{5}$, $\cos A = \frac{\sqrt{5}}{5}$, $\tan A = 2$
$\csc A = \frac{\sqrt{5}}{2}$, $\sec A = \sqrt{5}$, $\cot A = \frac{1}{2}$

16. $\sin A = \frac{\sqrt{2}}{3}$, $\cos A = \frac{\sqrt{7}}{3}$, $\tan A = \frac{\sqrt{14}}{7}$
$\csc A = \frac{3\sqrt{2}}{2}$, $\sec A = \frac{3\sqrt{7}}{7}$, $\cot A = \frac{\sqrt{14}}{2}$

17. $\sin A = \frac{2\sqrt{34}}{17}$, $\cos A = \frac{3\sqrt{17}}{17}$, $\tan A = \frac{2\sqrt{2}}{3}$
$\csc A = \frac{\sqrt{34}}{4}$, $\sec A = \frac{\sqrt{17}}{3}$, $\cot A = \frac{3\sqrt{2}}{4}$

18. $\cos A = \sqrt{1-x^2}$, $\tan A = \frac{x}{\sqrt{1-x^2}}$

19. $\frac{\pi}{4}$; $\frac{\sqrt{2}}{2}$ **20.** $\frac{\pi}{3}$; 2 **21.** 30°; $\frac{\sqrt{3}}{3}$

22. 45°; $\sqrt{2}$ **23.** 45°; $\frac{\pi}{4}$ **24.** 45°; $\frac{\pi}{4}$

25. $a = 10$; $b = 20\sqrt{3}$

26. $\overline{DE} = 2(\sqrt{3} - \sqrt{2})$

27. $(x_1, y_1) = (82.272, 47.5)$
$(x_2, y_2) = (47.5, 82.272)$

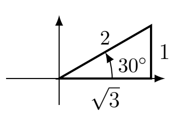
Section 1.3 (page 30)

7. 53° **8.** 70° **9.** 70° **10.** 18° **11.** 37°

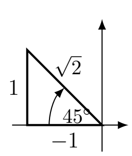
12. $\sin\theta = \frac{-3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{4}{3}$

13. $\sin\theta = \frac{-12}{13}$, $\cos\theta = \frac{-5}{13}$, $\tan\theta = \frac{12}{5}$

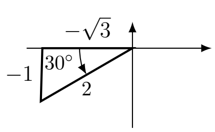
14. $\sin\theta = \frac{-15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = -\frac{15}{8}$

15.


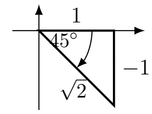
$\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$

16.


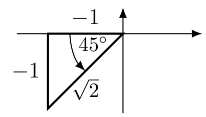
$\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{-\sqrt{2}}{2}$, $\tan\theta = -1$

17.


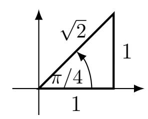
$\sin\theta = -\frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$

18.


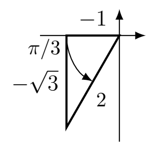
$\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$

19.


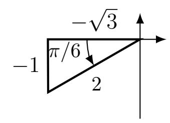
$\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$, $\tan\theta = 1$

20.


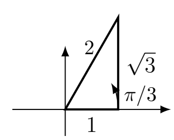
$\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = 1$

21.


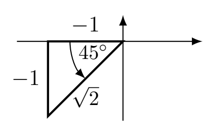
$\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = \sqrt{3}$

22.


$\sin\theta = -\frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$

23.


$\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$

24.


$\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$, $\tan\theta = 1$

25. $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$ and
$\sin\theta = -\frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{\sqrt{7}}{3}$

26. $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{\sqrt{7}}{3}$ and
$\sin\theta = -\frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$

27. $\sin\theta = \frac{\sqrt{15}}{4}$, $\tan\theta = \sqrt{15}$ and
$\sin\theta = -\frac{\sqrt{15}}{4}$, $\tan\theta = -\sqrt{15}$

28. $\sin\theta = 1$, $\tan\theta$ undefined
$\sin\theta = -1$, $\tan\theta$ undefined

29. $\sin\theta = 0$, $\tan\theta = 0$

30. $\cos\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{3}{\sqrt{7}}$ and
$\cos\theta = -\frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$

31. $\cos\theta = -\frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$ and
$\cos\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{3}{\sqrt{7}}$

32. $\cos\theta = \frac{\sqrt{15}}{4}$, $\tan\theta = \frac{\sqrt{15}}{15}$ and
$\cos\theta = -\frac{\sqrt{15}}{4}$, $\tan\theta = -\frac{\sqrt{15}}{15}$

33. $\cos\theta = 1$, $\tan\theta = 0$
$\cos\theta = -1$, $\tan\theta = 0$

34. $\cos\theta = 0$, $\tan\theta$ undefined

35. $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and
$\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$

36. $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$ and
$\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$

37. $\sin\theta = \frac{\sqrt{17}}{17}$, $\cos\theta = \frac{4\sqrt{17}}{17}$ and
$\sin\theta = -\frac{\sqrt{17}}{17}$, $\cos\theta = -\frac{4\sqrt{17}}{17}$

38. $\sin\theta = 0$, $\cos\theta = 1$
$\sin\theta = 0$, $\cos\theta = -1$

39. $\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$
$\sin\theta = -\frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$

40. $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$
$\csc\theta = \frac{17}{15}$, $\sec\theta = \frac{17}{8}$, $\cot\theta = \frac{8}{15}$

41. $\sin\theta = -\frac{9}{15}$, $\cos\theta = -\frac{12}{15}$, $\tan\theta = \frac{9}{12}$
$\csc\theta = -\frac{15}{9}$, $\sec\theta = -\frac{15}{12}$, $\cot\theta = \frac{12}{8}$

42. $\sin\theta = \frac{20}{29}$, $\cos\theta = \frac{21}{29}$, $\tan\theta = \frac{20}{21}$
$\csc\theta = \frac{29}{20}$, $\sec\theta = \frac{29}{21}$, $\cot\theta = \frac{21}{20}$

43. $\sin\theta = \frac{\sqrt{41}}{21}$, $\cos\theta = -\frac{20}{21}$, $\tan\theta = -\frac{\sqrt{41}}{20}$
$\csc\theta = \frac{21\sqrt{41}}{41}$, $\sec\theta = -\frac{21}{20}$, $\cot\theta = -\frac{20\sqrt{41}}{41}$

44. $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta =$ undef.
$\csc\theta = 1$, $\sec\theta =$ undef., $\cot\theta = 0$

45. 0.7986 **46.** −0.3420 **47.** 1.0642

48. −3.0777 **49.** 1.2521 **50.** 0.7265

51. −1.3764 **52.** 1.7013 **53.** undefined

54. −1.0613

55. 5.6222, 4.6852, 0.2885, −0.4997

Section 1.4 (page 37)

9. $(x, y) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\sin\alpha = \frac{1}{2}$, $\cos\alpha = -\frac{\sqrt{3}}{2}$, $\tan\alpha = -\frac{\sqrt{3}}{3}$
$\csc\alpha = \frac{2}{1}$, $\sec\alpha = -\frac{2\sqrt{3}}{3}$, $\cot\alpha = -\sqrt{3}$

10. $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{\sqrt{2}}{2}$, $\tan\theta = -1$
$\csc\theta = \sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = -1$

11. $(x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\sin\gamma = -\frac{\sqrt{2}}{2}$, $\cos\gamma = -\frac{\sqrt{2}}{2}$, $\tan\gamma = 1$
$\csc\gamma = -\sqrt{2}$, $\sec\gamma = -\sqrt{2}$, $\cot\gamma = 1$

12. $(x, y) = (1, 0)$
$\sin\beta = 0$, $\cos\beta = 1$, $\tan\beta = 0$
$\csc\beta$ undef., $\sec\beta = 1$, $\cot\beta$ undef.

13. $(x, y) = (-1, 0)$
$\sin\alpha = 0$, $\cos\alpha = -1$, $\tan\alpha = 0$
$\csc\alpha$ undef., $\sec\alpha = -1$, $\cot\alpha$ undef.

14. $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\sin\alpha = \frac{\sqrt{2}}{2}$, $\cos\alpha = -\frac{\sqrt{2}}{2}$, $\tan\alpha = -1$
$\csc\alpha = \sqrt{2}$, $\sec\alpha = -\sqrt{2}$, $\cot\alpha = -1$

15. $(x, y) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = -\sqrt{3}$
$\csc\theta = \frac{2\sqrt{3}}{3}$, $\sec\theta = -2$, $\cot\theta = -\frac{\sqrt{3}}{3}$

16. $(x, y) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\sin\gamma = \frac{\sqrt{3}}{2}$, $\cos\gamma = -\frac{1}{2}$, $\tan\gamma = -\sqrt{3}$
$\csc\gamma = \frac{2\sqrt{3}}{3}$, $\sec\gamma = -2$, $\cot\gamma = -\frac{\sqrt{3}}{3}$

17. $(x, y) = (-1, 0)$
$\sin\beta = 0$, $\cos\beta = -1$, $\tan\beta = 0$
$\csc\beta$ undef., $\sec\beta = -1$, $\cot\beta$ undef.

18. $(x, y) = (0, 1)$
$\sin\alpha = 1$, $\cos\alpha = 0$, $\tan\alpha$ undef.
$\csc\alpha = 1$, $\sec\alpha =$ undef., $\cot\alpha = 0$

19. $\frac{3}{4}$ **20.** $\frac{4}{3}$ **21.** $-\frac{4}{3}$ **22.** $\frac{3}{4}$

23. $\frac{3}{4}$ **24.** $\frac{4}{3}$ **25.** $\frac{4}{3}$ **26.** $\frac{4}{3}$

Section 1.5 (page 43)

1. $a = 4.201$, $B = 55°$, $c = 7.325$

2. $A = 84°$, $b = 0.526$, $c = 5.026$

3. $A = 54°$, $b = 0.727$, $c = 1.236$

4. $a = 1.045$, $B = 84°$, $b = 9.945$,

5. $A = 66°$, $a = 6.395$, $b = 2.847$

6. $B = 89°$, $b = 114.580$, $c = 114.597$

7. $a = 12$, $B = \frac{\pi}{4}$, $c = 12\sqrt{2}$

8. $A = \frac{\pi}{6}$, $a = 18$, $b = 18\sqrt{3}$

9. $x = 50.640$ **10.** $x = 15.655$

11. $x = 36.879$ **12.** $h = 16.629$ ft

13. $h = 836$ ft **14.** $h = 600$ ft

15. $h = 28.58$ ft **16.** $h = 21.61$ ft

17. $h = 241$ ft **18.** $c = 69.34$

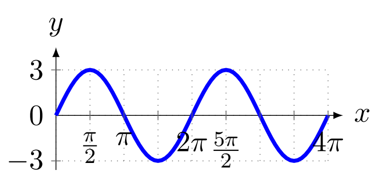
19. $c = 61.31$ ft **20.** $w = 396.3$ ft

21. $h = 330.5$ m **22.** $d = 80550000$ mi

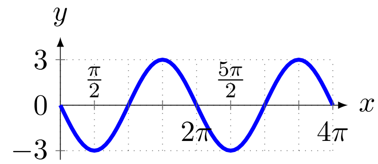
23. $h = 15434$ ft **24.** $d = 1.917 \times 10^{13}$ mi

Chapter 2

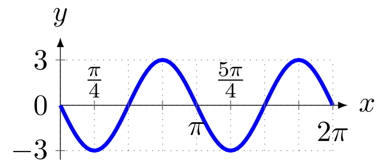
Section 2.1 (pg. 56)

1.


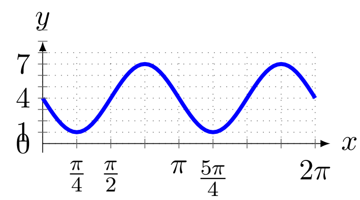
amp $= 3$, period $= 2\pi$, no shifts.

2.


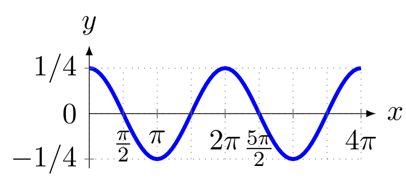
amp $= 3$, period $= 2\pi$, no shifts.

3.


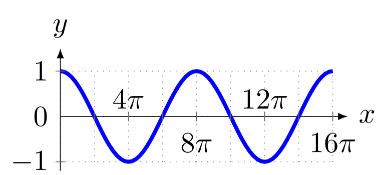
amp $= 3$, period $= \pi$, no shifts.

4.


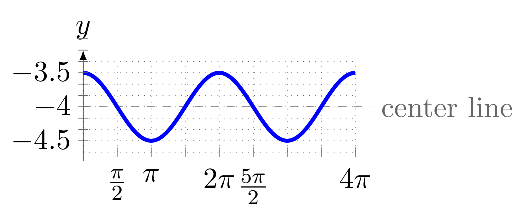
amp $= 3$, period $= \pi$, v. shift up 4.

5.


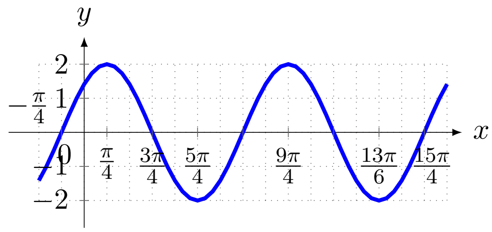
amp $= 1/4$, period $= 2\pi$, no shifts

6.


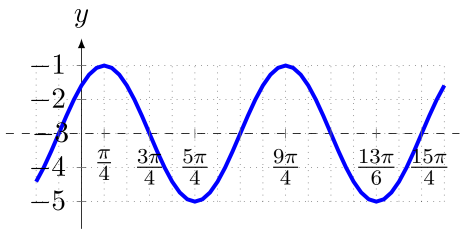
amp $= 1$, period $= 8\pi$, no shifts

7.


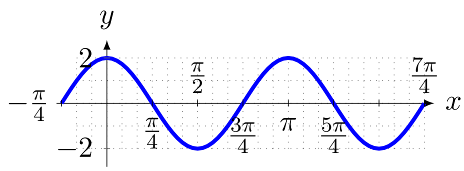
amp $= 1/2$, period $= 2\pi$, down 4

8. 

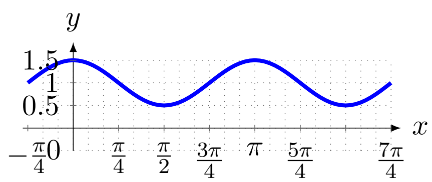
amp = 2, period = 2π, right $\frac{\pi}{4}$

9. 

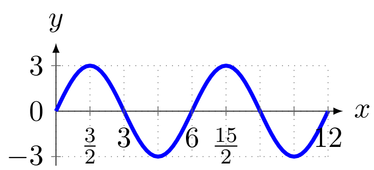
amp = 2, period = 2π, right $\frac{\pi}{4}$, down 3

10. 

amp = 2, period = π, left $\frac{\pi}{4}$

11. 

amp = 1/2, period = π, left $\frac{\pi}{4}$ up 1

12. 

amp = 3, period = 6, no shifts.

15. $y = 2\sin(5x)$ **16.** $y = 3 + 3\sin x$

17. $y = 3 - 2\sin\left(\frac{x}{2}\right)$

18. $f(t) = 50 + 7\sin\left(\frac{\pi t}{12}\right)$

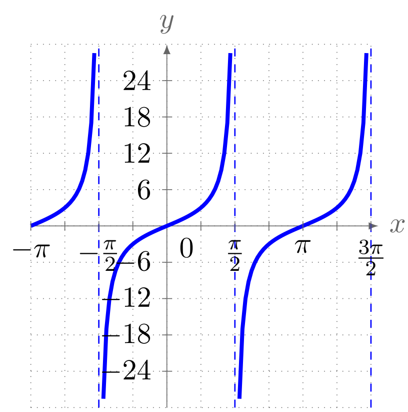
19. $f(t) = 68 + 12\sin\left(\frac{\pi t}{12}\right)$

20. $y = 4\sin\left(\frac{x}{3}\right) - 2$

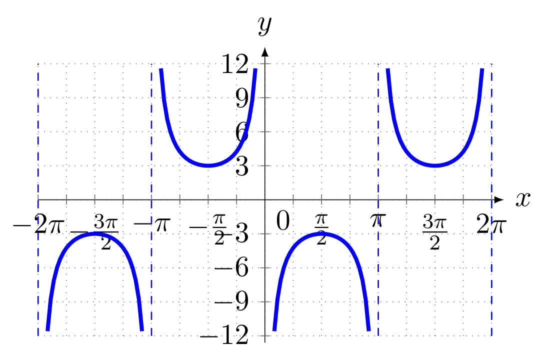
21. $y = 2\cos\left(\frac{\pi x}{2}\right)$

22. $x = L - R(1 - \cos\theta)$

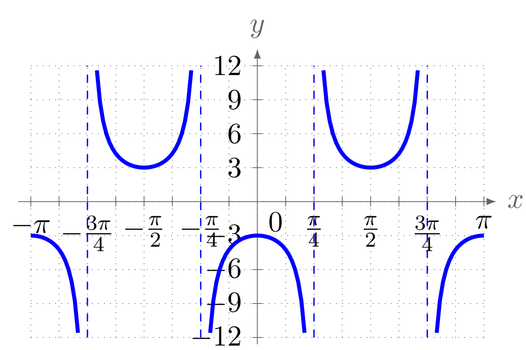
Section 2.2 (pg. 64)

1. 

period π, no shifts

2. 

period 2π, No shifts

3. 

period π, No shifts

Section 2.3 (pg. 73)

1. $\frac{\pi}{4}$ **2.** $-\frac{\pi}{4}$ **3.** 0 **4.** 0 **5.** π

6. $\frac{\pi}{2}$ **7.** $\frac{\pi}{2}$ **8.** $-\frac{\pi}{2}$ **9.** $\frac{\pi}{6}$ **10.** $\frac{5\pi}{6}$

11. $\frac{\pi}{4}$ **12.** $-\frac{\pi}{4}$ **13.** 0 **14.** $\frac{\pi}{3}$ **15.** $-\frac{\pi}{3}$

16. $\frac{\pi}{3}$ **17.** $-\frac{\pi}{5}$ **18.** $\frac{4\pi}{5}$ **19.** $\frac{\pi}{5}$ **20.** $-\frac{5\pi}{6}$

21. $\frac{5\pi}{6}$ **22.** $\frac{\pi}{3}$ **23.** $\frac{\pi}{3}$ **24.** $-\frac{\pi}{3}$ **25.** DNE

26. $\frac{4}{5}$ **27.** $\frac{4}{5}$ **28.** $\frac{3}{5}$ **29.** $\sqrt{1-x^2}$

30. $\frac{\sqrt{9-x^2}}{x}$ **31.** $\frac{\sqrt{9-x^2}}{3}$ **32.** $\frac{\sqrt{x^2+4}}{x}$ **33.** $40°$

Section 2.4 (pg. 80)

For all solutions $n \in \mathbb{Z}$

1. $\frac{\pi}{4}, \frac{3\pi}{4}$ **2.** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **3.** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **4.** $\frac{\pi}{2}, \frac{3\pi}{2}$

5. $\frac{2\pi}{3}, \frac{4\pi}{3}$ **6.** $\frac{\pi}{3}, \frac{4\pi}{3}$ **7.** $\frac{\pi}{4} + n\pi$

8. $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

9. $\frac{\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$

10. $\frac{\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$

11. $n\pi$ **12.** $\frac{\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi$

13. $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ **14.** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

15. $\frac{8\pi}{3}, \frac{10\pi}{3}$ **16.** $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

17. $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

18. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$

19. $0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}$ **20.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

21. $0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}$ **22.** $\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

23. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}$

24. $0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ **25.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

26. $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

27. $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

28. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **29.** $0.2898, -2.8518$

30. $1.8442, 4.4390$ **31.** $1.4633, 4.6049$

32. $0.1449, 1.4259, 3.2865, 4.5675$

33. $1.1832, 1.9584, 4.3248, 5.1000$

34. $0.4658, 1.4658, 2.4658, 3.4658, 4.4658, 5.4658$

Chapter 3

Section 3.1 (pg. 87)

1. 1 **2.** $\sec x$ **3.** $\tan t$

4. $\tan x$ **5.** $\csc t$ **6.** $1 - \sin x$

7. $\cot\theta \sec\theta$ **8.** $\tan\theta \sec\theta$ **9.** $\sec^2\theta$

10. $2\csc\theta$ **11.** $\csc\theta$ **12.** $\tan^2 x$

Answers will vary for exercises 13 - 18

Section 3.2 (pg. 92)

1. $\tan x$ **2.** $\csc t$ **3.** $1 - \sin x$

4. $\tan\theta$ **5.** $\csc\theta$ **6.** $\csc x$

Answers will vary for exercises 7 - 28

29. $\sin x = \pm 1, \;\; x = \frac{\pi}{2}, \frac{3\pi}{2}$

30. $\tan\theta = 1, \;\; \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

31. $2\sin^2 x + \sin x - 1 = 0, \;\; x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

32. $\sin^2 x - 6\sin x - 1 = 0$
Quadratic formula, $x = 3.3046, 6.1202$

33. $\sin x(\sec x - 3) = 0,$
$x = 0, \pi, 1.2310, 5.0522$

34. $2\sec^2\theta - 3\sec\theta - 1 = 0$
Quadratic formula $\theta = 1.2862, 4.9970$

Section 3.3 (pg. 100)

1. $\frac{\sqrt{6}-\sqrt{2}}{4}$ **2.** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **3.** $-2-\sqrt{3}$

4. $\frac{\sqrt{2}-\sqrt{6}}{4}$ **5.** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **6.** $\frac{\sqrt{6}-\sqrt{2}}{4}$

7. undefined **8.** $\frac{\sqrt{2}+\sqrt{6}}{4}$ **9.** $\frac{\sqrt{6}+\sqrt{2}}{4}$

10. $\frac{-\sqrt{2}-\sqrt{6}}{4}$ **11.** $2-\sqrt{3}$ **12.** $\frac{-\sqrt{6}-\sqrt{2}}{4}$

13. $\frac{\sqrt{6}-\sqrt{2}}{4}$ **14.** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **15.** $\sqrt{3}-2$

16. $\frac{\sqrt{6}+\sqrt{2}}{4}$ **17.** 1 **18.** $-\frac{\sqrt{2}}{2}$

19. 0 **20.** $\frac{\sqrt{2}}{2}$ **21.** undefined

22. 0 **23.** $-\sin x$ **24.** $-\cos x$

25. $-\cos x$ **26.** $\tan x$ **27.** $\sec x$

28. $\csc t$ **29.** $\tan x$ **30.** $\cot x$

31. $\frac{84}{85}, \frac{-13}{85}, \frac{-84}{13}$ **32.** $\frac{204}{325}, \frac{-253}{325}, \frac{-204}{253}$

33. $\frac{56}{65}, \frac{-33}{65}, \frac{-56}{33}$

34. $\frac{5\sqrt{7}+3\sqrt{119}}{48}, \frac{7\sqrt{17}-15}{48}, \frac{5\sqrt{7}+3\sqrt{119}}{7\sqrt{17}-15}$

35. $\frac{-10+12\sqrt{5}}{39}, \frac{24+5\sqrt{5}}{39}, \frac{-10+12\sqrt{5}}{24+5\sqrt{5}}$

36. $\frac{24-5\sqrt{5}}{39}, \frac{10+12\sqrt{5}}{39}, \frac{338\sqrt{5}-540}{620}$

37. $\frac{181\sqrt{194}}{2522}, \frac{5\sqrt{194}}{2522}, \frac{181}{5}$

38. $\frac{40x-9\sqrt{1681-x^2}}{41^2}, \frac{-9x+40\sqrt{1681-x^2}}{41^2}, \frac{40x-9\sqrt{1681-x^2}}{-9x+40\sqrt{1681-x^2}}$

39. $\frac{\sqrt{2}\left(\sqrt{1-x^2}-x\right)}{2}$ **40.** $\frac{2\sqrt{3}+x}{2\sqrt{x^2+4}}$

Section 3.4 (pg. 107)

1. (a) $\frac{\sqrt{63}}{32}$ (b) $\frac{31}{32}$ (c) $\frac{\sqrt{63}}{31}$ (d) $\frac{47}{128}$

2. (a) $\frac{-4\sqrt{21}}{25}$ (b) $\frac{-17}{25}$ (c) $\frac{4\sqrt{21}}{17}$ (d) $\frac{9\sqrt{21}}{125}$

3. $\cos(2x)$ **4.** $2\cos x$ **5.** $3\cos(6x)$

6. $-2\cos(4x)$ **7.** $\cos(10x)$ **8.** $2\sin(2x)$

9. $\frac{1}{2}\sin(2x)$ **10.** $\cos(34^\circ)$

11. $\theta = 0,\ \pi,\ 2.4189,\ 3.8643$

12. $\theta = \frac{\pi}{2},\ \frac{3\pi}{2},\ 3.9897,\ 5.4351$

13. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ **14.** $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

15. $\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

16. $\frac{1}{2}(1+\cos(4\theta))$ **17.** $\frac{3}{8} - \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8}$

18. $\frac{1}{8}\left(3 - 4\cos(6x) + \cos(12x)\right)$

19. $\frac{1}{8}(1-\cos(2x))$

20. $\frac{1}{16}\left(1 - \cos(2x) - \cos(4x) + \cos(4x)\cos(2x)\right)$

21. $\frac{1}{16}\left(1 + \cos(2x) - \cos(4x) - \cos(4x)\cos(2x)\right)$

22. $\frac{\sqrt{2+\sqrt{3}}}{2}$ **23.** $\frac{\sqrt{2-\sqrt{3}}}{2}$ **24.** $2+\sqrt{3}$

25. $\frac{\sqrt{2-\sqrt{2}}}{2}$ **26.** $\frac{\sqrt{2+\sqrt{2}}}{2}$ **27.** $\sqrt{2}-1$

28. $\frac{-\sqrt{2+\sqrt{3}}}{2}$ **29.** $\frac{-\sqrt{2-\sqrt{3}}}{2}$ **30.** $2+\sqrt{3}$

31. (a) $\frac{\sqrt{50+35\sqrt{2}}}{10}$, (b) $\frac{-\sqrt{50-35\sqrt{2}}}{10}$, (c) $5\sqrt{2}+7$

32. (a) $\frac{\sqrt{26}}{26}$, (b) $-\frac{5\sqrt{26}}{26}$, (c) $-\frac{1}{5}$

33. (a) $\frac{\sqrt{6}}{4}$, (b) $-\frac{\sqrt{10}}{4}$, (c) $-\frac{3\sqrt{5}}{5}$

Chapter 4

Section 4.1 (pg. 114)

1. $A = 60°$, $a = 18.43$, $c = 16.30$

2. $A = 60°$, $b = 16.73$, $c = 12.25$

3. $A = 49.05°$, $C = 65.95°$, $c = 6.05$

4. $B = 52.25°$, $C = 57.75°$, $b = 84.14$

5. $A = 35°$, $b = 4.42$, $c = 9.06$

6. $B = 30°$, $a = 12.31$, $b = 9.58$

7. $C = 120°$, $b = 7.37$, $c = 15.10$

8. $A = 47.31°$, $C = 57.69°$, $a = 30.44$

9. $C = 95°$, $a = 9.68$, $b = 10.65$

10. Two solutions
$B = 69.52°$, $C = 68.48°$, $c = 9.95$
$B = 110.48°$, $C = 27.52°$, $c = 3.45$

11. $B = 136.52°$, $C = 18.48°$, $b = 65.13$

12. No solution

13. $B = 24.56°$, $C = 61.44°$, $c = 10.57$

14. No solution

15. Two solutions
$B = 56.31°$, $C = 82.69°$, $c = 18.60$
$B = 123.69°$, $C = 15.31°$, $c = 4.95$

16. Two solutions
$B = 59.92°$, $C = 70.08°$, $b = 24.85$
$B = 20.08°$, $C = 109.92°$, $b = 9.86$

17. 14.98 km **18.** 3.20 km **19.** 44.93 km

20. 6.20 km **21.** 16750.8 m

22. 5.12 km **23.** 26.36 km

Section 4.2 (pg. 121)

1. $A = 47.97°$, $B = 82.03°$, $c = 15.47$

2. $B = 95.76°$, $C = 50.64°$, $b = 5.01$

3. $A = 58.41°$, $B = 73.40°$, $C = 48.19°$

4. $A = 42.83°$, $B = 76.23°$, $C = 60.94°$

5. $A = 40.90°$, $B = 19.10°$, $c = 7.94$

6. $A = 57.95°$, $B = 12.05°$, $a = 16.24$

7. $A = 15.36°$, $B = 112.02°$, $C = 52.62°$

8. $B = 57.47°$, $C = 47.53°$, $a = 45.83$

9. $A = 110.30°$, $C = 44.70°$, $b = 18.02$

10. $A = 33.18°$, $C = 99.81°$, $b = 6.68$

11. $A = 37.08°$, $B = 48.92°$, $c = 19.85$

12. $A = 31.33°$, $B = 109.01°$, $C = 39.66°$

13. $b = 23.96$ km **14.** $b = 102.26$ km

15. $\alpha = 121.22°$, $\gamma = 10.70°$

16. 271 km **17.** 2371 mi

18. 1996 mi **19.** 978.51 ft

20. 173.88 ft **21.** 99.94 ft

Section 4.3 (pg. 127)

1. 114.91 **2.** 17.43 **3.** 26.8

4. 3059.4 **5.** 7.8 **6.** 30.5

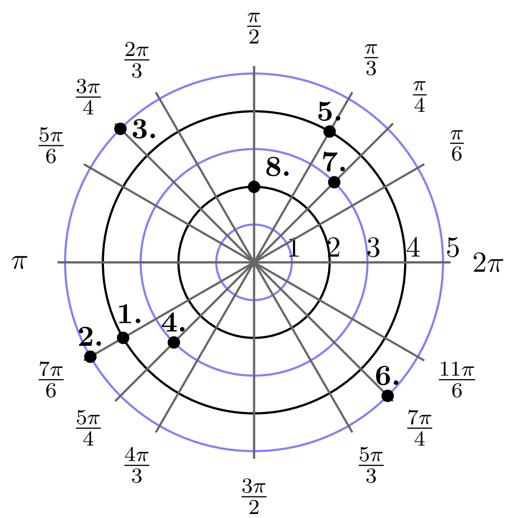
7. 139.1 **8.** 676.1 **9.** 253.6

10. 16.5 **11.** 89.8 **12.** 280.8

13. 12.2

Chapter 5

Section 5.1 (pg. 137)



1. $\left(-3\sqrt{3},-3\right)$ **2.** $\left(-\frac{5\sqrt{3}}{2},-\frac{5}{2}\right)$

3. $\left(-\frac{5\sqrt{2}}{2},\frac{5\sqrt{2}}{2}\right)$ **4.** $\left(-\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right)$

5. $\left(2,2\sqrt{3}\right)$ **6.** $\left(\frac{5\sqrt{2}}{2},-\frac{5\sqrt{2}}{2}\right)$

7. $\left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$ **8.** $(0,2)$

9. $\left(2\sqrt{10},18.43^\circ\right)$ **10.** $\left(\sqrt{10},108.43^\circ\right)$

11. $\left(2,\frac{\pi}{4}\right)$ **12.** $\left(3\sqrt{2},\frac{5\pi}{4}\right)$

13. $\left(5\sqrt{2},188.1^\circ\right)$ **14.** $\left(2,\frac{5\pi}{6}\right)$

15. $\left(6,\frac{7\pi}{6}\right)$ **16.** $\left(2,\frac{3\pi}{4}\right)$

17. $r=3\csc\theta$ **18.** $r=\tan\theta\sec\theta$

19. $r=3$ **20.** $r=9\sin\theta$

21. $\theta=\frac{\pi}{3}$ **22.** $r=\frac{-2}{5\sin\theta+\cos\theta}$

23. $y=x$ **24.** $(x-2)^2+y=4$

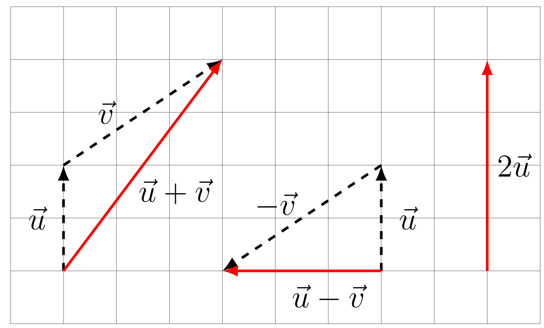
25. $x^2+y^2=25$ **26.** $x^2+(y+3)^2=9$

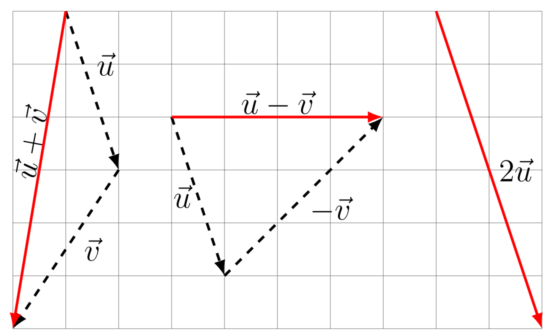
27. $y+7x=4$ **28.** $x=2$

For Exercises 29-37 a solution can be graphed using an online tool such as http://www.wolframalpha.com or https://www.desmos.com/calculator

Section 5.2 (pg. 147)

1. $\overrightarrow{PQ}=\langle -2,4\rangle$ **2.** $\overrightarrow{PQ}=\langle 3,2\rangle$

3.


4.


5. $\overrightarrow{PQ}=\sqrt{10}\cdot\left\langle\frac{-3}{\sqrt{10}},\frac{1}{\sqrt{10}}\right\rangle$

6. $\overrightarrow{PQ}=1\cdot\langle 0,1\rangle$

7. $\overrightarrow{PQ}=2\sqrt{17}\cdot\left\langle\frac{-1}{\sqrt{17}},\frac{-4}{\sqrt{17}}\right\rangle$

8. $\overrightarrow{PQ}=53\cdot\left\langle\frac{45}{53},\frac{-28}{53}\right\rangle$

9. $\overrightarrow{PQ}=\sqrt{37}\cdot\left\langle\frac{6}{\sqrt{37}},\frac{1}{\sqrt{37}}\right\rangle$

10. $\overrightarrow{PQ}=\sqrt{5}\cdot\left\langle\frac{-1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$

11. $\vec{v}=\left\langle 3\sqrt{3},3\right\rangle$ **12.** $\vec{v}=\left\langle\frac{-7}{2},\frac{7\sqrt{3}}{2}\right\rangle$

13. $\vec{v}=\left\langle 4\sqrt{2},-4\sqrt{2}\right\rangle$ **14.** $\vec{v}=\left\langle\frac{9\sqrt{3}}{2},\frac{-9}{2}\right\rangle$

15. $3\vec{u}=\langle 6,-6\rangle$, $2\vec{u}+\vec{v}=\langle 7,2\rangle$, $\vec{u}-2\vec{v}=\langle -7,-8\rangle$

16. $3\vec{u} = \langle 3, -6\rangle$, $2\vec{u} + \vec{v} = \langle -2, -2\rangle$, $\vec{u} - 2\vec{v} = \langle 13, -8\rangle$

17. $3\vec{u} = \langle 6, -9\rangle$, $2\vec{u} + \vec{v} = \langle 5, -4\rangle$, $\vec{u} - 2\vec{v} = \langle -1, -9\rangle$

18. $3\vec{u} = \langle 9, 12\rangle$, $2\vec{u} + \vec{v} = \langle 11, 2\rangle$, $\vec{u} - 2\vec{v} = \langle -12, 22\rangle$

19. distance 4.635 miles, direction 17.76° north of east

20. distance 7.548 miles, direction 79.2° south of west

21. total distance 17 miles, distance from home 10.3 miles

22. total distance 25 miles, distance from home 8.77 miles

23. $\vec{R} = \langle 10, 1\rangle$

24. $\vec{R} = \langle -5, 2\rangle$

25. $\vec{R} = \langle 0.357, -13.536\rangle$ N

26. distance 0.972 miles, direction 36° north of west

27. speed 658 km/h, 4.924° east

28. speed 465.7 km/h, 4.351° east

29. fly 4.424° west of north

30. fly 6.496° west of north

31. $(0.081, 8.602)$

32. $(3.434, 6.798)$

Index