## Sections 6.3 and 6.4: Adding and Subtracting Angles, Double-Angle Formulas

In this class, we'll introduce a few new formulas for adding and subtracting angles. I can't really explain where these formulas come from... the course textbook has some elaborate pictures for explaining the $\cos(\alpha-\beta)$ formula though. If you want more background, talk to me.

**Sine Addition/Subtraction Formulas:**

$$\boxed{\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)}$$

Things to remember:

- This formula has what I call "mixed terms". Each term has one sin and one cos.
- Both sides use the same symbol (+ or −).

**Cosine Addition/Subtraction Formulas:**

$$\boxed{\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)}$$

Things to remember:

- This formula does not mix terms. Both cosines come first, then both sines.
- The sides use OPPOSITE symbols.

These formulas are useful when you already know trig values of $\alpha$ and $\beta$, and you want to combine them to get new trigs. We should also recall two useful skills:

- If you're given $x$ and $y$, get $r$ to get trig values.
- If you're given a trig value, I recommend reference triangles to get other trigs!

**Ex 1:** Compute $\sin(\pi/12)$ and $\cos(\pi/12)$ exactly. (Use $\pi/12 = \pi/3 - \pi/4$.)

**Ex 2:** Suppose $\cos(\alpha) = 8/17$ and $\tan(\beta) = 4/3$, where $\alpha$ is in Quad IV and $\beta$ is in Quad III. Find

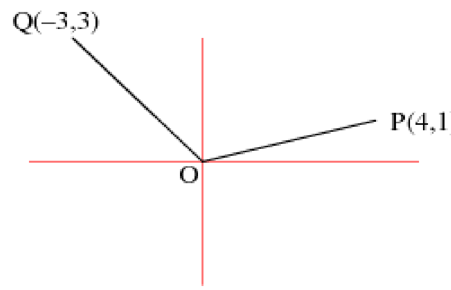
(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha + \beta)$ (c) which quadrant has $\alpha + \beta$

NOTE: You're going to need sin and cos for both $\alpha$ and $\beta$. By drawing reference triangles, you can find all those values. Once you've done (a) and (b), the signs should tell you (c)'s answer.

NOTE 2: If you need $\tan(\alpha + \beta)$, remember than tan is sin / cos.

**Ex 3:** If $\sin(\alpha) = -5/13$ and $\tan(\alpha) > 0$, compute $\sin(\alpha - \pi/6)$.

HINT: You're using the difference formula with $\beta = \pi/6$; you'll need $\cos(\alpha)$. Since $\sin(\alpha)$ is negative and $\tan(\alpha)$ is positive, $\alpha$ must be in Quad III. Thus, when you find $\cos(\alpha)$, it must be negative!




**Ex 4:** Find the cosine of $\angle POQ$ in the figure.

NOTE: If $\alpha$ is the angle to $Q$ (the bigger angle), and $\beta$ is the angle to $P$, then the angle we want is $\alpha - \beta$.

### Double-Angle Formulas

When you use the addition formulas with $\beta = \alpha$, you get formulas for $\sin(\alpha + \alpha)$ and $\cos(\alpha + \alpha)$.

**Sine Double-Angle Formula:** $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

**Cosine Double-Angle Formulas:** There are several variants... you convert between them by using $\sin^2 + \cos^2 = 1$. I mainly memorize the first one.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$$

**Ex 5:** Suppose the angle $T$ has $(7, y)$ on its terminal side. Find $\cos(2T)$ as a function of $y$.

GETTING STARTED: Once you have $x$, $y$, and $r$ for $T$, you can find $\cos(T)$ and $\sin(T)$. Once you have those, get $\cos(2T)$.

### Checking Identities

Remember that, if you have to find if an identity is true or false, it is usually easiest to simplify both sides. Our new formulas give us new options to try.

**Ex 6:** Is the formula $\sin(\theta + \pi/4) = \sqrt{2}/2(\sin(\theta) + \cos(\theta))$ correct?

HINT: Use the $\sin(\alpha + \beta)$ formula with $\alpha = \theta$ and $\beta = \pi/4$.

**Ex 7:** Is the formula $\cos(u + v) - \cos(u - v) = 2\sin(u)\sin(v)$ correct?

HINT: Use the identities twice on the left side, then distribute the − and simplify.

## Solving Equations using our Identities

When you have to solve a trig equation and multiple unknown angles are involved, our earlier tactics are not enough. We have to use identities to turn our problem into a problem with ONE unknown angle. This tends to come up in one of two ways:

- *Reversing an addition/subtraction identity*: Be on the lookout for this if you have terms looking like $\text{trig}(\alpha) * \text{trig}(\beta)$.

  **Sample:** If $\sin(5t)\cos(2t) - \cos(5t)\sin(2t) = 0$, then you recognize the left side as coming from the sine difference formula. Thus, it becomes $\sin(5t - 2t) = 0$, or $\sin(3t) = 0$. Continue from here by setting $\theta = 3t$.
- *Double-angle*: If you have both $2x$ and $x$, use a double-angle identity on the $\text{trig}(2x)$. Try to use a formula so that you either can (a) factor, or (b) get a quadratic.

  **Sample:** If $\cos(2x) - \cos(x) = 0$, we want to use a cosine double-angle formula. We can get $\cos^2(x) - \sin^2(x) - \cos(x) = 0$... but it would be better to have all trigs be the same. Instead, we'll use a different version of the identity and get $2\cos^2(x) - 1 - \cos(x) = 0$, which is quadratic in $u = \cos(x)$.

**Ex 8:** Solve the equation $\cos(4t)\cos(2t) = -\sin(4t)\sin(2t)$ for all $t$ in $[0, 2\pi)$.
HINT: This uses the cosine difference identity once you get everything on one side.

**Ex 9:** Solve the equation $\sin(u) - \sin(2u) = 0$ for $u$ in $[0, 2\pi)$.
HINT: After using double-angle, you should factor.

**Ex 10:** Solve the equation $\cos(2x) + \cos(x) = 0$ for $x$ in $[0, 2\pi)$.
HINT: The double-angle formula that uses $2\cos^2(x) - 1$ is most useful, so you get a quadratic in cos.

## More Practice

We won't have class time for this, but here are more problems you can try. Go over these in office hours, for instance!

**Ex 11:** If $\cot(\theta) = 4/3$ and $180° < \theta < 270°$, find the values of $\sin(\theta)$, $\cos(\theta)$, and $\sin(2\theta)$.
GETTING STARTED: The value of cot tells you enough to make a reference triangle. You're told the quadrant for $\theta$, which you'll need for the correct values of $\sin(\theta)$ and $\cos(\theta)$.

**Ex 12:** Express the following as a trig function of a single angle: $\sin(63°)\cos(19^{\text{deg}}) + \cos(63^{\text{deg}})\sin(19^{\text{deg}})$.
GETTING STARTED: Which of the addition/subtraction formulas works here? You can tell it must be a sin formula because of the mix of sin and cos parts. Next, you figure out whether it's a + formula or a − formula.

**Ex 13:** Solve the equation $\sec(x)\tan(x) + \tan(x) - \sec(x) - 1 = 0$ for all solutions $x$ in $[0, 2\pi)$.
GETTING STARTED: This one can be factored, but it's difficult. The key technique used is *factoring by grouping*. Factor $\sec(x)$ from the first two terms, then factor $-1$ from the last two terms:

$$\sec(x)(\tan(x) + 1) - (\tan(x) + 1) = 0$$

Notice that both parts have $\tan(x) + 1$ as a common factor? Pull it out!

$$(\tan(x) + 1)(\sec(x) - 1) = 0$$

This gives you two basic trig equations: $\tan(x) = -1$ and $\sec(x) = 1$.