Informal lecture notes for complex analysis

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I'll assume you're familiar with the review of complex numbers and their algebra as contained in Appendix G of Stewart's book, so we'll pick up where that leaves off.

1 Elementary complex functions

In one-variable real calculus, we have a collection of basic functions, like polynomials, rational functions, the exponential and log functions, and the trig functions, which we understand well and which serve as the building blocks for more general functions. The same is true in one complex variable; in fact, the real functions we just listed can be extended to complex functions.

1.1 Polynomials and rational functions

We start with polynomials and rational functions. We know how to multiply and add complex numbers, and thus we understand polynomial functions. To be specific, a degree n polynomial, for some non-negative integer n, is a function of the form

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

where the c_i are complex numbers with $c_n \neq 0$. For example, $f(z) = 2z^3 + (1 - i)z + 2i$ is a degree three (complex) polynomial. Polynomials are clearly defined on all of $\mathbb{C}$. A rational function is the quotient of two polynomials, and it is defined everywhere where the denominator is non-zero.

Example: The function $f(z) = \frac{z^2+1}{z^2-1}$ is a rational function. The denominator will be zero precisely when $z^2 = 1$. We know that every non-zero complex number has n distinct nth roots, and thus there will be two points at which the denominator is zero. It's easy to see that those points are 1 and -1, and so f is defined on $\mathbb{C} \setminus \{1, -1\}$. If we want to compute f at some point, we just use our rules for complex algebra. For instance, using that $(1 + i)^2 = 1 + 2i - 1 = 2i$, we have

$$\begin{aligned} f(1+i) &= \frac{(1+i)^2+1}{(1+i)^2-1} = \frac{1+2i}{-1+2i} = \frac{1+2i}{-1+2i} \cdot \frac{-1-2i}{-1-2i} \\ &= \frac{-1-2i-2i+4}{1+4} = \frac{3}{5} - \frac{4}{5}i. \end{aligned}$$

1.2 The exponential function and logarithm

We've already seen the (complex) exponential function. Recall that we have the useful formula

$$e^{x+iy} = e^x (\cos y + i \sin y) .$$

Here we use our usual convention of writing a complex number as $z = x + iy$ for real x and y. One reason this formula is useful is that it allows us to actually compute. We had originally defined e^z by a power series, and if we used that definition directly, then evaluating something like $e^{1+\pi i}$ would require summing an infinite series. Instead, using the above formula allows us to write

$$e^{1+\pi i} = e^1 (\cos \pi + i \sin \pi) = -e.$$

Another reason this formula is useful is that it allows us to give a somewhat intuitive interpretation of the exponential function. Notice that we have written our "input" variable in Cartesian coordinates, but our "output" variable is in polar coordinates. We see that e^{x+iy} is the complex number with modulus (or r, in polar coordinates) e^x and argument (or θ, in polar coordinates) y. If our input is a real number, then y is zero and the output will be real and positive (since the positive real axis has argument zero) and will have modulus e^x. Thus we confirm that this agrees with the real exponential function for real input. Looking at $e^{1+\pi i}$ again, we see that this number has modulus $e^1 = e$, and argument π, which corresponds to the negative real axis.

The relationship to polar coordinates also allows us to understand what happens when we try to take the inverse of e^z to obtain the (complex) logarithm. By definition, $\log z$ should be the number w with the property that $e^w = z$. If we write $w = u + vi$ for real u and v, then we find that $u = \log |z|$ and $v = \arg(z)$. So while u is well-defined as the real logarithm of a real number, in defining v we encounter exactly the same problem we encountered in trying to define the argument of a complex number. Namely, we cannot define v so as to be continuous and have a single value, as a true function must. As a result, we cannot define the logarithm to be continuous and single-valued. The solution we adopt is the same as for the argument (indeed, since $v = \arg(z)$, we should make sure that our convention for the argument and the logarithm are compatible). We consider $\log z$ to have infinitely many values, which differ by integer multiples of $2\pi i$. Thus, just like $\arg(z)$, $\log z$ is *not* really a function. However, we will cheat a little and act like it is, with the implicit understanding that it really takes on infinitely many values.

Examples: We've already seen that $e^{1+\pi i} = -e$, and thus one choice for $\log(-e)$ is $1 + \pi i$. Any other choice differs by an integer multiple of $2\pi i$, so all possible logarithms of $-e$ are given by $1 + (2n + 1)\pi i$ for all integers n. For a second example, we consider $\log i$. Since the modulus of i is 1, we have $u = \log 1 = 0$. Further, the argument of i is $\pi/2$ (up to multiples of 2π), so one choice of v is $v = \pi/2$. Thus all possible values of $\log i$ are given by $(\pi/2)i + 2\pi n i = (1/2 + 2n)\pi i$ for all integers n. You can check that these give i after exponentiation.

We have a few final comments about the exponential and log functions before moving on. It is clear that e^z is defined on all of $\mathbb{C}$. As for $\log z$, the above discussion shows that it is defined for any non-zero z (unsurprisingly), and thus has domain $\mathbb{C} \setminus \{0\}$. It follows that the range of e^z is $\mathbb{C} \setminus \{0\}$ while the range of $\log z$ is all of $\mathbb{C}$. After this discussion of $\arg(z)$ and $\log z$, you might wonder if it would be nice to develop a theory of many-valued functions. In a full complex analysis course, this is often done. However, for our purposes, it will be enough just to understand these two functions as explained above. Finally, this might seem like a lot of hassle to deal with one function. However, when we get to complex integration, we will see that the fact that we can't define a single-valued, continuous logarithm is quite important.

1.3 Trig functions

To finish our discussion of elementary complex functions, we turn to the trig functions. In order to define $\sin z$ and $\cos z$, we use the same method as for the exponential function. Namely, we start with the power series from real variables, and we assert that that power series continues to make sense for complex numbers (this is true, but we won't prove it). Further, it turns out that after some algebraic manipulation of power series, we can express both of these trig functions in terms of the exponential function. In particular, we have

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ &= \frac{e^{iz} - e^{-iz}}{2i} \\ \text{and} \quad \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\ &= \frac{e^{iz} + e^{-iz}}{2}. \end{aligned}$$

Examples: We compute $\sin i$ and $\cos i$. Using the above representations in terms of the exponential function, we have

$$\begin{aligned} \sin i &= \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{1}{2}\left(e - \frac{1}{e}\right) i \\ \text{and} \quad \cos i &= \frac{e^{i^2} + e^{-i^2}}{2} = \frac{1}{2}\left(e + \frac{1}{e}\right). \end{aligned}$$

Both $\sin z$ and $\cos z$ have all of $\mathbb{C}$ as their domain. The other four trig functions can be defined in terms of sine and cosine in the usual way. Observe that, in complex analysis, the trig functions can all be defined in terms of the exponential function. For this reason, it is common to focus attention on the exponential function and push the trig functions to the sidelines. We will follow this approach to some extent.

2 Complex functions in general

In the last section, we approached complex functions via concrete examples. Now we approach them from the other direction, asking what they are in general. By definition, a complex function is a function from an subset D of $\mathbb{C}$ to $\mathbb{C}$. In other words, it takes as "input" a complex number and gives as "output" a complex number, and we allow for the possibility that it isn't defined everywhere in $\mathbb{C}$ (as is the case for rational functions, for example). Just as a complex number can be written in terms of its real and imaginary parts, a complex function can always be written in terms of its real and imaginary parts as

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are both functions from $\mathbb{R}^2$ to $\mathbb{R}$. We have already seen one example of this way of writing complex functions; we saw that e^z can be written like this with $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Similar representations can be given for the other functions we discussed in the last section.

Examples: Consider $f(z) = z^2$. In terms of its real and imaginary parts, we have

$$f(x + iy) = (x + iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + (2xy)i.$$

Thus we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. For a second example, consider $f(z) = 1/z$, defined on $\mathbb{C} \setminus \{0\}$. We have

$$f(x + iy) = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2}i,$$

and so $u(x, y) = x/(x^2 + y^2)$ and $v(x, y) = -y/(x^2 + y^2)$.

This way of writing complex functions is not restricted to the examples from the last section (indeed, if it is supposed to be completely general, it can't be). Two obvious examples are $f(z) = \mathrm{Re}(z)$ for which $u(x, y) = x$ and $v(x, y) = 0$, and $f(z) = \mathrm{Im}(z)$ for which $u(x, y) = y$ and $v(x, y) = 0$.

Example: Consider $f(z) = \overline{z}$. Then $f(x + iy) = x - iy$, and so $u(x, y) = x$ and $v(x, y) = -y$.

Of course, we can also consider going the other way. That is, given two functions $u(x, y)$ and $v(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}$, we can try to write find a nice formula in terms of the algebra of complex numbers for the resulting f. We said "nice formula" because there is always a trivial way of doing this just by writing x and y as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, but that's not interesting. As an example, consider $u(x, y) = 2x - 1$ and $v(x, y) = 2y + 2$. A little work shows that these are the real and imaginary parts of $f(z) = 2z + (-1 + 2i)$. So in this case, our two real polynomials in two variables can be "combined" into a single complex polynomial. On the other hand, if we consider $u(x, y) = 2x$ and $v(x, y) = y$, then it turns out that there is no complex polynomial which has these as its real and imaginary parts. We'll talk more about how to show that and what it means in the next section.

The above makes it clear that we can take any function from $\mathbb{R}^2$ to $\mathbb{R}^2$, written as $f(x,y) = (u(x,y), v(x,y))$ and view it as a complex function $f(z) = u(x,y) + iv(x,y)$, and vice versa. This is not surprising, since we know that we can identify $\mathbb{C}$ with $\mathbb{R}^2$ just by viewing $\mathbb{C}$ as the complex plane. Thus, at this level of generality, whether we view a function as a function from one complex variable to one complex variable or as a function from two real variables to two real variables is really just a matter of perspective. This fact has numerous advantages. For example, we would like to talk about limits of complex functions and about complex functions being continuous. We can define these ideas in terms of what we already know for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. Thus, a sequence of complex numbers converges to some limit if the corresponding sequence of points in $\mathbb{R}^2$ converges to the corresponding limit. Similarly, a complex function is continuous if it is continuous when viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Finally, a subset of $\mathbb{C}$ is bounded, open, or closed if the corresponding subset of $\mathbb{R}^2$ is bounded, open, or closed. This allows us to take what we already know about real functions from Calc III and import it into our study of complex functions, which is certainly convenient. For example, we see that all of our elementary functions from the last section are continuous on their domains.

On the other hand, the fact that complex functions can be viewed as functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ calls into question the point of what we are doing. After all, if we already know how to work with real functions, why bother to reconceptualize them as complex functions? In order to answer that question, we need to introduce the notion of complex differentiation.

3 Complex differentiation

3.1 The definition

Our goal here is to define and begin to discuss the complex derivative of a complex function. In contrast to the ideas at the end of the last section (like continuity), it is not true that a function from $\mathbb{C}$ to $\mathbb{C}$ is complex differentiable if it is differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. So here we will have a difference between the real and the complex points of view.

To define the complex derivative, we simply mimic the definition of the derivative of a real function of one variable.

Definition: A function $f(z)$ from an open set $D \subset \mathbb{C}$ to $\mathbb{C}$ is complex differentiable at a point $c \in D$ if

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

exists. If so, the limit is the complex derivative of f at c, which we write $f'(c)$. Further, f is complex differentiable if it is complex differentiable at every point in D, in which case the complex derivative is a function which we write $f'(z)$.

Note that the difference quotient makes sense because the quotient of two complex numbers makes sense (unlike the quotient of two points in $\mathbb{R}^2$). Just as in the real case, we also use the notation $\frac{df}{dz}(z)$ for the complex derivative. Alternative terminology for "complex differentiable" is "holomorphic." We will generally use the term holomorphic in the following, and we will call the complex derivative just the derivative, it being clear from context what is meant (for example, the notation f' is meaningless if we think of f as a real function from $\mathbb{R}^2$ to $\mathbb{R}^2$). Holomorphic functions are also often called analytic or complex-analytic functions. If one is picky, "analytic" means something a bit different than "holomorphic," but there is a theorem which says that a (complex) function is analytic if and only if it is holomorphic, so people often use the two terms interchangeably. We won't be pursuing that here, so we'll just stick with holomorphic.

To get a feel for our new notion, we will compute a few simple derivatives directly from the definition. The most obvious place to start is with a constant function, $f(z) = C$ for some $C \in \mathbb{C}$. Then at any point $c \in \mathbb{C}$ we have

$$f'(c) = \lim_{z \to c} \frac{C - C}{z - c} = 0,$$

and thus $f(z)$ is holomorphic with derivative $f'(z) = 0$. This is reassuring, since any reasonable notion of derivative should have something to do with measuring how a function changes, and so a constant function should, intuitively, have derivative zero. As a second example, we take $f(z) = z$. Then

$$f'(c) = \lim_{z \to c} \frac{z - c}{z - c} = 1,$$

and so we see that $f(z)$ is holomorphic with derivative $f'(z) = 1$. This is also reassuring, since if we restrict this function to the real axis it becomes $f(x) = x$ (which has derivative 1), and we would hope that our notion of complex differentiation would be, in some sense, compatible with our notion of real differentiation. We will have more to say about this later on.

Before discussing the general properties of the complex derivative, we give an example to show that not every simple complex function is complex differentiable. The example we have in mind is $f(z) = \mathrm{Re}(z)$. To see that this is not (complex) differentiable at any point we reason as follows. In order for the limit of the difference quotient to exist, we must get the same limit no matter how we approach the point c. We make the change of variables $h = z - c$ so that we can rewrite the definition of the derivative as

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

We write $c = a + bi$. We first consider what happens when we approach c along the real axis by letting $h = t$ for real t and observing that the difference quotient becomes

$$\frac{f(c + h) - f(c)}{h} = \frac{f(a + t + bi) - f(a + bi)}{t} = \frac{(a + t) - a}{t} = 1.$$

Thus, as $t \to 0$, we get 1. Next we approach c along the imaginary axis by letting $h = ti$ for real t so that the difference quotient becomes

$$\frac{f(c+h)-f(c)}{h} = \frac{f(a+ti+bi)-f(a+bi)}{ti} = \frac{a-a}{ti} = 0.$$

As $t \to 0$, we get 0. Since we get different limits approaching along the real and imaginary axis, the limit of the difference quotient as $h \to 0$ does not exist, and therefore $f(z) = \mathrm{Re}(z)$ is nowhere (complex) differentiable.

3.2 Complex differentiation rules

Simple functions like constant functions and $f(z) = z$ can be differentiated just using the definition, as we saw. However, just as in the real case, for more complicated functions we rely on various differentiation rules. The following rules extend the familiar rules from real calculus to the complex setting. In fact, they are proved in essentially the same way, so we won't give the proofs here.

Theorem: Let $f(z)$ and $g(z)$ be holomorphic functions. Then the following are true:

1. (Linearity) For any constants $c, d \in \mathbb{C}$, $cf(z) + dg(z)$ is holomorphic and $(cf(z) + dg(z))' = cf'(z) + dg'(z)$.
2. (Product rule) The product $(fg)(z)$ is holomorphic and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.
3. (Quotient rule) The quotient $f(z)/g(z)$ is holomorphic wherever $g(z) \neq 0$ and
$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}.$$
4. (Chain rule) The composite function $f(g(z))$ is holomorphic whenever $g(z)$ is in the domain of $f(z)$, and $(f(g))'(z) = f'(g(z))g'(z)$.

Of course, these rules are are only useful if we already have functions f and g, the derivatives of which we know. In the last lecture, we showed that the derivative of any constant function is 0 everywhere and the derivative of $f(z) = z$ is 1 everywhere. Using this and the product rule, we see that

$$\left(z^2\right)' = (z \cdot z)' = 1 \cdot z + z \cdot 1 = 2z.$$

Iterating this procedure, we see that $(z^n)' = nz^{n-1}$. Finally, using the above and linearity, we see that any (complex) polynomial, which we write

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

is holomorphic on all of $\mathbb{C}$ and has derivative

$$f'(z) = nc_n z^{n-1} + (n-1)c_{n-1} z^{n-2} + \cdots 2c_2 z + c_1.$$

Going beyond polynomials, observe that, using the quotient rule, we can write

$$\left(\frac{1}{z^n}\right)' = \frac{0\cdot z^n - 1\cdot nz^{n-1}}{z^{2n}} = \frac{-n}{z^{n+1}}.$$

(On the homework, you are asked to show that $(1/z)' = -1/z^2$ by a different method.) More generally, the quotient rule and the above formula for polynomials allow us to differentiate any rational function.

These computations for polynomials and rational functions are a good start. However, it's not immediately clear what to do about differentiating e^z, for example. Moreover, we would like to go beyond elementary functions and look at general complex functions $f(x+iy) = u(x,y) + iv(x,y)$. How to do this is the subject of the next section.

3.3 The Cauchy-Riemann equations

Consider a general complex function $f(x+iy) = u(x,y) + iv(x,y)$. As we saw last lecture, a necessary condition for it to be complex differentiable is that the difference quotient gives the same limit along both the real and imaginary axis (this is how we showed that $\mathrm{Re}(z)$ is not holomorphic). Assume that $f(z) = f(x+iy)$ is holomorphic. Then it must be true that, for real t,

$$f'(z) = \lim_{t\to 0}\frac{f(z+t)-f(z)}{t} = \lim_{t\to 0}\frac{f(z+it)-f(z)}{it}.$$

In terms of u and v, we see that the approach along the real axis gives

$$\begin{aligned}\lim_{t\to 0}\frac{f(z+t)-f(z)}{t} &= \lim_{t\to 0}\frac{u(x+t,y)+iv(x+t,y)-u(x,y)-v(x,y)}{t}\\ &= \lim_{t\to 0}\frac{u(x+t,y)-u(x,y)}{t} + i\cdot\lim_{t\to 0}\frac{v(x+t,y)-v(x,y)}{t}\\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},\end{aligned}$$

while the approach along the complex axis gives

$$\begin{aligned}\lim_{t\to 0}\frac{f(z+it)-f(z)}{it} &= \lim_{t\to 0}\frac{u(x,y+t)+iv(x,y+t)-u(x,y)-v(x,y)}{it}\\ &= -i\cdot\lim_{t\to 0}\frac{u(x,y+t)-u(x,y)}{t} + \lim_{t\to 0}\frac{v(x,y+t)-v(x,y)}{t}\\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.\end{aligned}$$

These two quantities must be equal, and thus their real and imaginary parts must be equal. We conclude that if $f(z)$ is holomorphic, $u(x,y)$ and $v(x,y)$ must satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are called the *Cauchy-Riemann equations*. So if $f(z)$ is holomorphic, its real and imaginary parts satisfy the Cauchy-Riemann equations, and furthermore, its derivative can be written

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

The obvious question now is whether the converse holds, that is, if the real and imaginary parts of $f(z)$ satisfy the Cauchy-Riemann equations, is $f(z)$ holomorphic? The answer is yes, assuming that the partial derivatives are sufficiently smooth. This is the content of the next theorem.

Theorem: Suppose $u(x,y)$ and $v(x,y)$ have continuous first order partial derivatives. Then the complex function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic if and only if u and v satisfy the Cauchy-Riemann equations, and if so, its derivative is $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

We won't give a proof of this theorem here; one can be found in almost any complex analysis book. This is a significant result. It gives us a way to determine whether an arbitrary function is holomorphic and, if so, what its derivative is, in terms of the (real) partial derivatives of u and v, which we understand from Calc III.

3.4 Applications of the Cauchy-Riemann equations

As a first application, we show that e^z is holomorphic and compute its derivative. Recall that the real and imaginary parts are given by $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Thus we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, & \frac{\partial u}{\partial y} &= -e^x \sin y, \\ \frac{\partial v}{\partial x} &= e^x \sin y, & \frac{\partial v}{\partial y} &= e^x \cos y. \end{aligned}$$

It's easy to see that these partial derivatives satisfy the Cauchy-Riemann equations, and that $\frac{d}{dz}e^z = e^z$.

Our next objective is to determine the derivative of $\log z$. This might seem strange, since $\log z$ isn't a true function because it takes infinitely many values. However, we will see that its derivative is a function. Recall that the real and imaginary parts of $\log z$ are $u(x,y) = \log\sqrt{x^2+y^2}$ and $v(x,y) = \arg(x+iy)$. So u is a true function, and finding its partials is easy; we have

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

As for $v(x,y)$, we see that it is just the θ coordinate of (x,y). It's partials are

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}.$$

To see where these come from, first observe that a formula for $\arg(x+iy)$ which is valid in, say, the first quadrant is $\arg(x+iy) = \operatorname{Arctan}\left(\frac{y}{x}\right)$. Taking the partials of this expression gives the above. The problem is that this formula breaks down at $x = 0$. To get around this, one can change coordinates by rotating the plane and then check that you get the same formulas everywhere in the plane. This isn't particularly hard, but it is tedious, so we won't provide the details. Also, note that even though $\arg(x+iy)$ takes infinitely many values, its partial derivative are true functions. To understand this, note that in a small, simply connected ball around a point (that is, the ball doesn't contain the origin), we can choose unique, continuous values for $\arg(x+iy)$. Any other choice differs by addition of a multiple of 2π, but a multiple of 2π is a constant, and so when we take the derivative we get the same answer no matter what multiple of 2π we choose. At any rate, now that we've found the partials of the real and imaginary parts of $\log z$, we see that they satisfy the Cauchy-Riemann equations and that

$$\frac{d}{dz}\log z = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{1}{z}.$$

Before moving on, we note that there is another approach to differentiating $\log z$. One can prove that if $f(z)$ is holomorphic and has non-zero derivative in a small ball around a point c, then the inverse of f exists on some ball around $f(c)$, is holomorphic there, and has derivative $(f^{-1})'(z) = 1/f'(f^{-1}(z))$. Proving the existence of the inverse and that it is holomorphic takes a little work, but the formula for the derivative is just an application of the chain rule. In particular, by the definition of the inverse function we have $f(f^{-1}(z)) = z$. Differentiating both sides gives

$$f'(f^{-1}(z)) \cdot (f^{-1})'(z) = 1,$$

and solving for $(f^{-1})'(z)$ gives the above formula. (This is also how you find the derivatives of the inverse trig functions in real calculus.) If we let $f(z) = e^z$ and $f^{-1}(z)\log z$, we get

$$(\log)'(z) = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

We chose to use the Cauchy-Riemann equations in order to get more practice with them and also to see the connection of the derivative with the "gradient" of θ (recall that, back when we were studying Green's theorem, this was our main example of a vector field with curl zero but with nonzero integral over closed curves around the origin). This connection will be important when we get to complex integration.

Finally, we consider the derivatives of the trig functions. Unsurprisingly, $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$. Deriving the first of these formulas is a problem on your homework, and the second is done in the same way. Once we know these, we can see that the other four trig functions have the expected derivatives just by using the quotient rule.

At this point, we have a grasp of how to compute complex derivatives roughly equal to what we know about computing one dimensional real derivatives. Indeed, we've seen that basically all of our elementary real functions extend to be elementary complex functions and that they obey exactly the same differentiation rules. So given a complex function built out of elementary functions, we can differentiate it exactly the way we would the corresponding real function. In some sense, this shouldn't be too surprising, since the complex derivative of an elementary function should, intuitively, agree with the real derivative when z is real.

Example: Let $f(z) = iz^3 + \cos^2 z$. Then $f'(z) = 3iz^2 - 2\cos z \sin z$. Note that, to be a bit oversimplistic, the difference between this and taking the corresponding real derivative is that we have a z instead of an x.

On the other hand, when we deal with functions that aren't built out of elementary functions, things are different. Here the Cauchy-Riemann equations are especially helpful. We already saw that $\mathrm{Re}(z)$ is not holomorphic. If we consider $\mathrm{Im}(z)$, we have $u(x, y) = y$ and $v(x, y) = 0$. Then

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0, & \frac{\partial u}{\partial y} &= 1, \\ \frac{\partial v}{\partial x} &= 0, & \frac{\partial v}{\partial y} &= 0. \end{aligned}$$

Thus the Cauchy-Riemann equations are not satisfied anywhere (to say they are satisfied means that they are both satisfied), and $\mathrm{Im}(z)$ is not anywhere (complex) differentiable. By the same method, we can show that neither $|z|$ nor $\overline{z}$ is holomorphic.

This justifies our earlier claim that the complex function $f(z)$ given by $u(x, y) = 2x$ and $v(x, y) = y$ cannot be written as a complex polynomial. In particular, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2, & \frac{\partial u}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} &= 0, & \frac{\partial v}{\partial y} &= 1. \end{aligned}$$

These partial derivatives don't satisfy the Cauchy-Riemann equations, and thus $f(x + iy) = 2x + iy$ is not (complex) differentiable anywhere. Since we know that all complex polynomials are holomorphic, it follows that this f cannot be re-written as a complex polynomial.

As the above suggests, holomorphic complex functions (or functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ which can be viewed as holomorphic complex functions under the correspondence between $\mathbb{R}^2$ and $\mathbb{C}$) can be profitably understood as complex functions and studied using complex differentiation. On the other hand, non-holomorphic complex functions (or functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ which correspond to non-holomorphic functions) are, as a rule, better understood as real functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ using real calculus. In the next section, we describe the relationship between the real and complex derivatives in more detail.

4 Relation between real and complex derivatives

Before beginning, it is worth noting that the following description is a bit of a digression in the normal development of complex analysis. Most complex analysis courses (in my experience) don't bother to discuss it, and it isn't necessary in order to use complex analysis (or, in our case, to do the homework for this course). Thus, if you find it confusing or unhelpful, you can safely ignore it. I'm including it on the assumption that at least some people will find it helpful or interesting, and also because this is mostly a course in real calculus, and it provides connections between real calculus and out new notion of complex differentiation.

4.1 Real differentiation

We begin by recalling the "differential" point of view on one dimensional, real calculus. Suppose a function f from $\mathbb{R}$ to $\mathbb{R}$ has derivative $f'(x)$ at some point x. One way of viewing this is that it tells you how the "output" changes if you change the "input," at least in the linear approximation. In particular, if you change x by adding some Δx, then $f(x)$ changes by $f'(x)\Delta x$ (in the linear approximation). In other words, the changes in x and $f(x)$ (the "input" and "output") are related by (real) multiplication by the (real) derivative.

This point of view extends nicely to functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. If $f(x,y) = (u(x,y), v(x,y))$ is such a function, consider the matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

This matrix allows us to relate the change in (x,y) and the change in $f(x,y)$. In particular, suppose we change the input by adding some $(\Delta x, \Delta y)$ to (x,y). Then, in the linear approximation, the output changes by adding (here we write our vectors as column vectors and make use of matrix multiplication)

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\ \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \end{bmatrix}.$$

So we can view this matrix as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which takes changes in (x,y) and maps them to the corresponding changes in $f(x,y)$. Thus, for any point (x,y), we will think of this matrix of partial derivatives as the derivative of f at (x,y), in the sense that it completely describes (in the linear approximation) the change in $f(x,y)$ given a change in (x,y). Also note that, for any 2×2 matrix, we can find a function which has that as its derivative.

4.2 Complex differentiation

Now we consider how these ideas extend to the case of a complex function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. First, suppose that f is holomorphic with derivative $f'(z) = \alpha + i\beta$ at some point z. Then if we change the input by $\Delta z = \Delta x + i\Delta y$, the output changes (in the linear approximation) by

$$f'(z)\Delta z = (\alpha\Delta x - \beta\Delta y) + i(\alpha\Delta y + \beta\Delta x).$$

Looking at the left-hand side of this equation, we see that this situation is analogous to what we saw for one real variable. Namely, the change in input and output are related by multiplication by the derivative; the difference is that all of the quantities in question are complex numbers instead of real numbers and the multiplication involved is complex multiplication. On the other hand, we can also view this function as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and ask about its real derivative in the sense we were just talking about. If we think of Δz and $f'(z)\Delta z$ as vectors in $\mathbb{R}^2$, then the equation above describes a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which we can write as

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \alpha\Delta x - \beta\Delta y \\ \alpha\Delta y + \beta\Delta x \end{bmatrix}.$$

In other words, if we look at the map from the complex plane to the complex plane given by multiplying everything by $\alpha+i\beta$ and then think of this as a linear map from vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$, the corresponding matrix which gives this map is the one in the preceding equation. (That this is the right matrix can be seen just by comparing this with the result of the complex multiplication above and seeing that it gets both the real and imaginary parts right.) From this equation, or from direct computation with the real and imaginary parts, it follows that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Note that the relationships among the partial derivatives that this implies are the same as the ones from the Cauchy-Riemann equations. Further, this shows that the complex derivative of f as a complex function and the real derivative of f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ are compatible. More concretely, both give the same relationship between change in input and change in output; the only difference is whether those changes are thought of as complex numbers or as vectors in $\mathbb{R}^2$. All of this is true under the assumption that f is holomorphic.

To approach the relationship between real and complex differentiation from the other direction, let $f(x, y) = (u(x, y), v(x, y))$ be any real differentiable function from $\mathbb{R}^2$ to $\mathbb{R}^2$. If we want to think of it as having a complex derivative, then the reasoning from the previous paragraph indicates that its derivative, as map from $\mathbb{R}^2$ to $\mathbb{R}^2$, must correspond to complex multiplication by some complex number. Further, this will be the case if and only if its matrix of partials derivatives has the special from

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

in which case the corresponding complex multiplication is multiplication by $\alpha + i\beta$. Finally, note that the matrix of partial derivatives has this form if and only if the partials satisfy the Cauchy-Riemann equations, and if so, the complex derivative can be written in terms of the partials in exactly the way described in the previous section.

To summarize, we see that our notions of real and complex differentiation are compatible. More specifically, a holomorphic function is also differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the resulting linear map (which is what we're calling the derivative) corresponds to complex multiplication. In the other direction, a (real) differentiable function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is complex differentiable if and only if its derivative can be interpreted as complex multiplication. This won't be true for just any matrix of partial derivatives (there are plenty of linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which don't correspond to complex multiplication), but the condition that it is true is precisely the Cauchy-Riemann equations. So if our function can be seen as a holomorphic (complex) function, that is often an advantageous point of view to adopt, since it makes the differential calculus look "one variable," instead of the usual two variable real theory. But if our function isn't holomorphic, it is better viewed as a real function, since only the real calculus will apply.

5 Contour integrals

Now that we have a solid background in the basics of complex differentiation, it's time to move on to complex integration.

Consider a curve C in the complex plane parametrized by $z(t)$ for $a \leq z \leq b$. Of course, we can write this in terms of its real and imaginary parts as $z(t) = x(t)+iy(t)$. Now suppose we have a (complex) function $f(z) = u(x,y)+iv(x,y)$. We want to integrate f along the curve C. In order to define such an integral we start, as usual, with a Riemann sum approximation. If we subdivide the curve by subdividing the interval at points $a = t_0, t_1, \ldots, t_n = b$, then the corresponding Riemann sum looks like

$$\sum_{j=1}^{n} f(z(t)) \cdot (z(t_i) - z(t_{i-1}))$$

where the "dot" here refers, of course, to complex multiplication. (This is the difference with the definition of a real line integral, in which both factors are vectors and we take their dot product.) Then the *contour integral* of f along C, written $\int_C f(z)\,dz$, is defined to be the limit as $n \to \infty$, assuming that this limit exists.

This makes a good definition, but in order to compute with it, we will rewrite it in terms of the parametrization and the real and imaginary parts of f

and z. A little computation shows that

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b \left[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\right]\,dt \\ &\quad + i\int_a^b \left[v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)\right]\,dt \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy.\end{aligned}$$

An easy way to remember this is by writing the complex differential as $dz = dx + i\,dy$ and then multiplying by $f(z)$ written in terms of its real and imaginary parts.

Example: Let C be the quarter-circle from 1 to i and let $f(z) = z^2$. Then we can parametrize C as $z(t) = \cos t + i \sin t$ for $0 \le t \le \pi/2$. In terms of this parametrization, we have that $f(z(t)) = \cos^2 t - \sin^2 t + i(2\cos t \sin t)$. Further, we have $dz = dx + i\,dy = \sin t\,dt + i\cos t\,dt$, and thus, after some algebra, we find that

$$\begin{aligned}f(z(t))\cdot dz &= \left[\sin^3 t - 3\sin t\cos^2 t + i\left(\cos^3 t - 3\cos t\sin^2 t\right)\right]\,dt \\ &= \left[\sin t - 4\sin t\cos^2 t + i\left(\cos t - 4\cos t\sin^2 t\right)\right]\,dt.\end{aligned}$$

Note that the last line is obtained using $\cos^2 t = 1 - \sin^2 t$ and vice versa, and we have done this with an eye toward the fact that we're about to integrate this expression. We have

$$\begin{aligned}\int_C f(z)\,dz &= \int_0^{\pi/2} \left(\sin t - 4\sin t\cos^2 t\right)\,dt + i\int_0^{\pi/2}\left(\cos t - 4\cos t\sin^2 t\right)\,dt \\ &= \left[-\cos t + \frac{4}{3}\cos^3 t\right]_0^{\pi/2} + i\left[\sin t - \frac{4}{3}\sin^3 t\right]_0^{\pi/2} \\ &= -\frac{1}{3} - \frac{1}{3}i.\end{aligned}$$

Before leaving this example, it is worth noting that these computations can be streamlined by taking better advantage of complex functions. In particular, this parametrization of the curve C can be re-written as $z(t) = e^{it}$ for $0 \le t \le \pi/2$. Then we have $f(z(t)) = e^{2it}$. Further, we see that $dz = ie^{it}\,dt$, and thus

$$\begin{aligned}\int_C f(z)\,dz &= i\int_0^{\pi/2} e^{3it}\,dt \\ &= i\int_0^{\pi/2}\cos(3t)\,dt - \int_0^{\pi/2}\sin(3t)\,dt \\ &= i\left[\frac{1}{3}\sin(3t)\right]_0^{\pi/2} + \left[\frac{1}{3}\cos(3t)\right]_0^{\pi/2} \\ &= -\frac{1}{3}i - \frac{1}{3}.\end{aligned}$$

Note that here we wrote $f(z(t)) = e^{2it} = \cos(2t) + i\sin(2t)$, while above we had that this was equal to $\cos^2 t - \sin^2 t + i(2\cos t \sin t)$. However, these can be seen to be equal using the double-angle formulas. In fact (to continue this digression one more step), the relationship between the exponential and trig functions in complex analysis provides an efficient way to derive and/or remember various trig identities.

6 Antiderivatives

Recall the philosophy we developed when discussing complex differentiation that complex analysis should be, in some sense, analogous to real calculus. Based on this intuition, we might expect that it's possible to use the fact that $z^3/3$ is an antiderivative of z^2 in order to compute the integral from last section. (We say that $F(z)$ is an antiderivative of $f(z)$ if $F'(z) = f(z)$). Indeed, we might hope for some sort of fundamental theorem of calculus which allows us to evaluate integrals by evaluating antiderivatives at the endpoints of the curve. However, if this is true, then the corresponding contour integrals must be independent of path, since the fundamental theorem only uses the endpoints. Because of this, our first step in exploring the fundamental theorem of calculus in the complex setting will be to consider when contour integrals are independent of path.

Recall from the last section that the real and imaginary parts of a contour integral can be written

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy.$$

The contour integral will be independent of path precisely when the real and imaginary parts are each independent of path. Fortunately, the above equation gives the real and imaginary parts in terms of (real) line integrals, and so we can apply Green's theorem. For the moment, assume that $f(z)$ is defined on a simply-connected domain. Doing so, we see that the real part is independent of path if and only if $-\partial v/\partial x = \partial u/\partial y$ and that the imaginary part is independent of path if and only if $\partial u/\partial x = \partial v/\partial y$. These are just the Cauchy-Riemann equations. So, at least on a simply-connected domain, contour integrals of $f(z)$ are independent of path precisely when $f(z)$ is holomorphic. Moreover, if we take the potentials for the real and imaginary parts and combine them into a complex function, a little work shows that we get an antiderivative for $f(z)$.

This indicates that we should restrict our attention to holomorphic functions (so holomorphic functions are the natural objects for both complex differentiation and complex integration). By refining these considerations a bit, one can prove the following complex analogue of the fundamental theorem of calculus.

Theorem: Let $f(z)$ be a function defined on an open subset D of $\mathbb{C}$. If $f(z)$ has an antiderivative then $f(z)$ is holomorphic. Conversely, if $f(z)$ is holomorphic and D is simply-connected, then $f(z)$ has an antiderivative. Finally, if

$F(z)$ is an antiderivative of $f(z)$ and C is a curve from z_0 to z_1, then

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

Example: We return to the previous example, integrating z^2 along the quarter-circle from 1 to i. We know an antiderivative of z^2 is $z^3/3$, and thus

$$\int_C z^2\,dz = \int_1^i z^2\,dz = \left[\frac{1}{3}z^3\right]_1^i = -\frac{1}{3}i - \frac{1}{3}.$$

This agrees with our earlier computations. Obviously, this approach is easier than what we did in the previous section.

Just as for differentiation, we can now apply many of our techniques from real calculus to finding complex antiderivatives, and thus to computing complex integrals.

Example: Let C be any curve from 0 to πi, and let $f(z) = ze^z$. Then $f(z)$ is holomorphic on the whole complex plane, and so the contour integral is the same no matter what particular C we choose. Integration by parts (just as in the real case) shows that an antiderivative is $F(z) = (z-1)e^z$. Thus

$$\begin{aligned}\int_C f(z)\,dz &= [(z-1)e^z]_0^{\pi i} \\ &= (\pi i - 1)e^{\pi i} + e^0 \\ &= 2 - \pi i.\end{aligned}$$

7 Non-simply-connected domains

The theorem from the last section assures us that any holomorphic function on a simply-connected domain has an antiderivative, although not necessarily one that can be written in terms of elementary functions. However, as we know from our study of Green's theorem, things are more complicated on domains which are not simply-connected. First, note that the theorem applies on any domain if we know that $f(z)$ has an antiderivative. For example, this is the case for $f(z) = 1/z^2$. Although it is only defined on $\mathbb{C}\setminus\{0\}$, we know that $F(z) = -1/z$ is an antiderivative. In particular, the contour integral of $1/z^2$ over any closed curve is zero, even a closed curve around the 0.

To see this "by hand," we will integrate $1/z^2$ over the unit circle, oriented counterclockwise. To keep the computation manageable, we will use complex notation. We parametrize the circle as $z(t) = e^{it}$ for $0 \le t \le 2\pi$. Then $1/z^2(t) = e^{-2it}$ and $dz = ie^{it}\,dt$, so that

$$\begin{aligned}\int_C \frac{1}{z^2}\,dz &= \int_0^{2\pi} ie^{-it}\,dt \\ &= i\int_0^{2\pi} \cos(-t)\,dt - \int_0^{2\pi} \sin(-t)\,dt \\ &= 0.\end{aligned}$$

Similar considerations apply to $1/z^n$ for any $n \geq 2$. Namely, we know what the antiderivatives are, and so the corresponding contour integrals are independent of path. At this point, we have a pretty good understanding of integration for all of our elementary functions except for $1/z$. Of course, $1/z$ is only defined on $\mathbb{C} \setminus \{0\}$. If we look for an antiderivative, the obvious answer is $\log z$, and we saw in the last lecture that $\frac{d}{dz} \log z = 1/z$. However, $\log z$ is not a "true" function, and thus the previous theorem does not apply.

In order to see what is going on, we compute the integral of $1/z$ along the counterclockwise unit circle. We use the same parametrization as we did above for $1/z^2$; the difference is that now the integrand is $1/z(t) = e^{-it}$. We have

$$\begin{aligned} \int_C \frac{1}{z}\, dz &= \int_0^{2\pi} i\, dt \\ &= 2\pi i. \end{aligned}$$

This should not be too surprising in light of the connection between $\log z$ and polar coordinates. This clearly shows that contour integrals of $1/z$ are not independent of path. On the other hand, as Green's theorem shows, the value of an integral around a closed curve depends only on how many times the curve "goes around" the origin, and not on any other features of the curve (making this more precise is beyond the scope of this course, although it is typically done in courses devoted to complex analysis).

At this point, we've seen that contour integrals of holomorphic functions share many features of real, one-dimensional integrals. In particular, if our holomorphic function is built out of elementary functions then we can use the usual methods of calculus to try and find an antiderivative. On a more general level, any holomorphic function on a simply-connected domain has an antiderivative. On non-simply-connected domains, more complicated phenomena can occur, with a first example given by $1/z$. In the final section, we will look at some consequences of the behavior of $1/z$.

8 Applications of the behavior of $1/z$

At first, the fact that $1/z$ has non-zero integral over curves around the origin seems like an irritation. It is, basically, the one issue preventing the complex integration of elementary complex functions from completely mirroring the (one-dimensional, real) integration of elementary real functions. However, this fact is actually quite useful, and in this section we explore two applications, one somewhat theoretical and the other more concrete.

8.1 Cauchy's integral formula

Any simple, closed curve C divides the complex plane into two regions, an "inside" and an "outside." We say that such a C encloses a point z_0 if z_0 is contained in the "inside" region. The following theorem is almost a direct consequence of Green's theorem and the above computation for the integral of $1/z$ around the unit circle.

Theorem: (Cauchy's integral formula) Let D be an open, simply-connected region, let $f(z)$ be a holomorphic function on D, and let C be a simple, closed curve in D, oriented counterclockwise, that encloses a point z_0. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

We will outline a proof. By Green's theorem, we can replace C by any other simple, closed, counterclockwise curve which encloses z_0. (This is the same argument for changing the curve we've used in the real case; just apply it to the real and imaginary parts of the contour integral.) In particular, let C_r be a circle of radius r centered at z_0. For small r, $f(z)$ on C_r can be approximated by $f(z_0)$. So

$$\int_C \frac{f(z)}{z - z_0}\, dz \approx f(z_0) \int_{C_r} \frac{1}{z - z_0}\, dz.$$

The integral on the right is equivalent to the integral of $1/z$ over a circle after a simple change of coordinates, and thus we know it is equal to $2\pi i$ for any r. In the limit as $r \to 0$ this approximation becomes exact, and the theorem follows. In order to make this rigorous, one needs only to justify the approximation and be a bit more precise about our intuitive use of notions about curves.

Intuitively, this formula says that the value of $f(z)$ at any point inside C can be computed by a weighted average of the values of $f(z)$ on C. This has many important consequences, a few of which we'll mention here. First, this means that the values of $f(z)$ on the curve determine the values of $f(z)$ everywhere inside of the curve. More specifically, the values inside are averages of the values on the curve. This implies that, if we look at the image of D under f, points that are inside of C have images which are inside the image of C. Finally, we mention that this formula plays an important role in proving that any holomorphic function can actually be differentiated infinitely many times. (None of these facts are true for general real-differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.)

8.2 Computing real integrals

As a more concrete application of these ideas, we give an example of how they can be used to compute certain real integrals. This is not the most important application of complex analysis, but since this was primarily a class on real integration, perhaps it's appropriate to end by coming back to real integrals.

We start with a simple example. Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1 + x^2}\, dx.$$

The reason this is a simple example is that we know $\mathrm{Arctan}(x)$ is an antiderivative of $1/(1 + x^2)$, and the integral can be computed using that. However, we will present an alternative method using complex analysis. This can be viewed as a contour integral, namely

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz$$

where C_R is the straight line from $-R$ to R in the complex plane. This, of course, is not a closed curve. However, if we let $\tilde{C}_R$ be the semi-circle from R to $-R$ with positive imaginary part (that is, we choose the upper semi-circle), then the union of these two curves, which we write $C_R\cup\tilde{C}_R$ is a simple, closed curve. Further, we see that

$$\frac{1}{1+z^2}=\frac{1}{z+i}\cdot\frac{1}{z-i}.$$

Thus the domain is $\mathbb{C}\setminus\{i,-i\}$. If we look at the half-disk bounded by $C_R\cup\tilde{C}_R$, then $1/(z+i)$ is holomorphic on this region. We assume that $R\geq 1$ so that the half-disk contains i. Then by applying Cauchy's integral formula with $f(z)=1/(z+i)$ and $z_0=i$, we find that

$$\int_{C_R\cup\tilde{C}_R}\frac{1}{1+z^2}\,dz=2\pi i\frac{1}{i+i}=\pi.$$

The final step is to control the integral over $\tilde{C}_R$ in the limit. It's not hard to see that the integral of any complex function $f(z)$ over a curve C obeys the estimate

$$\left|\int_C f(z)\,dz\right|\leq\max_{z\in C}|f(z)|\cdot\text{length}(C).$$

In our case, $|z^2|=R^2$ everywhere on $\tilde{C}_R$ and $\text{length}(\tilde{C}_R)=\pi R$. Thus (using the triangle inequality on z^2+1), we have

$$\left|\int_{\tilde{C}_R}\frac{1}{1+z^2}\,dz\right|\leq\frac{\pi R}{R^2-1}$$

$$\text{and}\quad\lim_{R\to\infty}\int_{\tilde{C}_R}\frac{1}{1+z^2}\,dz=0.$$

Since

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz=\lim_{R\to\infty}\int_{C_R\cup\tilde{C}_R}\frac{1}{1+z^2}\,dz-\lim_{R\to\infty}\int_{\tilde{C}_R}\frac{1}{1+z^2}\,dz$$

it follows that

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx=\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz=\pi.$$

As mentioned, this could have been done more easily using regular one-dimensional calculus. However, consider the related problem of computing

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx=\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{1+x^2}\,dx.$$

In this case we don't have a nice antiderivative, and there's no obviously way to compute this integral using one-dimensional real calculus. However, we can determine the answer fairly easily by modifying our previous computation. Rewrite this integral as

$$\mathrm{Re}\left[\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{1+z^2}\,dz\right]$$

where C_R is defined as above. (You might have been expecting us to use $\cos z$ in the numerator instead of using e^{iz} and taking the real part. It turns out that this approach is noticeably easier.) Once again, we consider the simple, closed curve $C_R \cup \tilde{C}_R$ where $\tilde{C}_R$ is defined as above. Since e^{iz} is holomorphic everywhere, we can apply Cauchy's integral formula with $f(z) = e^{iz}/(z+i)$ and $z_0 = i$ to see that

$$\int_{C_R\cup\tilde{C}_R}\frac{e^{iz}}{1+z^2}\,dz = 2\pi i\frac{e^{i^2}}{i+i} = \frac{\pi}{e}$$

for any $R > 1$. Further, we see that $e^{iz} = e^{-y}(\cos x + i\sin x)$. Since $y \geq 0$ on $\tilde{C}_R$, it follows that $|e^{iz}| \leq 1$ on $\tilde{C}_R$. Then, in light of our previous work, we see that

$$\left|\int_{\tilde{C}_R}\frac{e^{iz}}{1+z^2}\,dz\right| \leq \frac{\pi R}{R^2-1}$$

$$\text{and}\quad \lim_{R\to\infty}\int_{\tilde{C}_R}\frac{e^{iz}}{1+z^2}\,dz = 0.$$

Just as above, we conclude that

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx = \mathrm{Re}\left[\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{1+z^2}\,dz\right] = \mathrm{Re}\left[\frac{\pi}{e}\right] = \frac{\pi}{e}.$$

The idea underlying this method is to replace the integral along the the real axis with the integral around a closed curve containing one point at which the integrand is not defined. This second integral can then be evaluated just using a little algebra and Cauchy's integral formula. Doing this in any concrete situation is something of an art, but when possible it's generally an efficient method.