Kronecker Commutation Matrices and Particle Physics

Christian Rakotonirina*
Civil engineering Department,
Institut Supérieur de Technologie d'Antananarivo, IST-T
Laboratoire de la Dynamique de l'Atmosphère, du Climat et des Océans, DYACO,
University of Antananarivo
Madagascar

December 5, 2016

Abstract

In this paper, formulas giving a Kronecker commutation matrices (KCMs) in terms of some matrices of particles physics and formulas giving electric charge operators (ECOs) for fundamental fermions in terms of KCMs have been reviewed. Physical meaning have been given to the eigenvalues and eigenvectors of a KCM.

Introduction

The Kronecker or tensor commutation matrices (KCMs) are matrices which commute Kronecker or tensor product of matrices. So, we can think of using them where Kronecker product is used. Kronecker product is used in many branches of physics and mathematics: in quantum information theory, optics, matrix equations and algebraic Bethe ansatz.
One can remark that a wave function of two identical fermions is eigenfunction of a KCM associated to the eigenvalue -1 and a wave function of two identical bosons is eigenfunction of a KCM associated to the eigenvalue -1. So, it is natural to think to what are the meaning we can give to these eigenvalues, their multiplicities and the eigenvectors associated, in particle physics.
The KCMs have already relations with some matrices of the particle physics and we can construct an electric charge operator (OCE) for fundamental fermions by using a KCM. After the first section which will speak about kronecker product and the mathematical definition of a KCM, in the second

*rakotonirinachristianpierre@gmail.com

section we will review the expression of the KCMs in terms of some matrices of particle physics, after that in the third section we will review the ECOs built with the KCMs. Finally, in the fourth section we will give the eigenvalues of KCMs with their multiplicities and the eigenvectors associated, and some examples giving their meaning in particle physics.

We will take as system of units the natural units $\hbar = c = 1$ and as unit of charge the charge of an electron e.

1 Kronecker Commutation Matrices

The Kronecker product of a matrix $\mathbf{A} = (A^i_j) \in \mathbb{C}^{m\times n}$ by other matrix $\mathbf{B}$:

$$\mathbf{A}\otimes\mathbf{B} = \begin{pmatrix} A^1_1 & \dots & A^1_j & \dots & A^1_n \\ \vdots & & \vdots & & \vdots \\ A^i_1 & \dots & A^i_j & \dots & A^i_n \\ \vdots & & \vdots & & \vdots \\ A^m_1 & \dots & A^m_j & \dots & A^m_n \end{pmatrix} \otimes\mathbf{B} = \begin{pmatrix} A^1_1\mathbf{B} & \dots & A^1_j\mathbf{B} & \dots & A^1_n\mathbf{B} \\ \vdots & & \vdots & & \vdots \\ A^i_1\mathbf{B} & \dots & A^i_j\mathbf{B} & \dots & A^i_n\mathbf{B} \\ \vdots & & \vdots & & \vdots \\ A^m_1\mathbf{B} & \dots & A^m_j\mathbf{B} & \dots & A^m_n\mathbf{B} \end{pmatrix}$$

is not commutative. That is

$$\mathbf{A}\otimes\mathbf{B} \neq \mathbf{B}\otimes\mathbf{A}$$

K the Kronecker commutation matrix (KCM)

$$K\left(\mathbf{a}\otimes\mathbf{b}\right) = \mathbf{b}\otimes\mathbf{a}$$

with $\mathbf{a}$, $\mathbf{b}$ are unicolumn matrices.

The KCM $K_{2\otimes 2}$ commutes two row and unicolumn matrices

$$K_{2\otimes 2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \otimes \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$K_{3\otimes 3}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \otimes \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and so on.

$$K_{2\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_{3\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2 KCMs and Matrices of Particle Physics

The KCMs $K_{2\otimes 2}$ and $K_{3\otimes 3}$ can be expressed respectively in terms of the Pauli matrices (See, for example, [1]) and the Gell-Mann matrices [2] by the following ways

$$K_{2\otimes 2} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the 2×2 unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Pauli matrices

$$K_{3\otimes 3} \;=\; \frac{1}{3} I_3 \otimes I_3 + \frac{1}{2} \sum_{i=1}^{8} \lambda_i \otimes \lambda_i$$

where

$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$, $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

are the Gell-Mann matrices or 3×3 Gell-Mann matrices. A variant of this last formula with other definition of the Gell-Mann matrices [3] have recently application in optics. We say variant because this paper has expressed the KCM $K_{3\otimes 3}$ in terms of other Gell-Mann matrices, with its definition of the Gell-Mann matrices.

As a generalization [4],

$$K_{n\otimes n} \;=\; \frac{1}{n} I_n \otimes I_n + \frac{1}{2} \sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i$$

where Λ_i's are the $n \times n$ Gell-Mann matrices.

3 KCMs and Charges of Fundamental Fermions

In this section we are going to construct ECOs for fundamental fermions in using KCMs [5]. An ECO we are going to construct here can have charges

of leptons and quarks, together as eigenvalues.
Let us recall at first that fundamental fermions have the quantum numbers J_3, the isospin and Y, the hypercharge. The electric charge Q of a fermion is given by the Gell-Mann-Nishijima formula

$$Q = J_3 + \frac{Y}{2} \tag{1}$$

For the fermions of the standard model (SM) these quantum numbers are given in the following table [6]

		Q	J_3	Y
Neutral Leptons	$\nu_{eL},\, \nu_{\mu L},\, \nu_{\tau L}$	0	$1/2$	-1
Charged Leptons	$e_L,\, \mu_L,\, \tau_L$	-1	$-1/2$	-1
	$e_R,\, \mu_R,\, \tau_R$	-1	0	-2
Quarks u, c, t	$u^r_L,\, u^b_L,\, u^g_L,\, c^r_L,\, c^b_L,\, c^g_L,\, t^r_L,\, t^b_L,\, t^g_L$	$2/3$	$1/2$	$1/3$
	$u^r_R,\, u^b_R,\, u^g_R,\, c^r_R,\, c^b_R,\, c^g_R,\, t^r_R,\, t^b_R,\, t^g_R$	$2/3$	0	$4/3$
Quarks d, s, b	$d^r_L,\, d^b_L,\, d^g_L,\, s^r_L,\, s^b_L,\, s^g_L,\, b^r_L,\, b^b_L,\, b^g_L$	$-1/3$	$-1/2$	$1/3$
	$d^r_R,\, d^b_R,\, d^g_R,\, s^r_R,\, s^b_R,\, s^g_R,\, b^r_R,\, b^b_R,\, b^g_R$	$-1/3$	0	$-2/3$

A matrix relation of the Gell-Mann-Nishijima for eight leptons and quarks of the SM of the same generation, for example e_L, ν_{eL}, u^r_L, u^b_L, u^g_L, d^r_L, d^b_L, et d^g_L has been proposed in [7].

$$\mathbf{Q} = \frac{1}{2}\underbrace{\sigma_0 \otimes \sigma_0 \otimes \sigma_3}_{Y} + \underbrace{\frac{1}{6}\left(\sum_{i=1}^{3} \sigma_i \otimes \sigma_i\right) \otimes \sigma_0}_{J_3} \tag{2}$$

The same ECO can be obtained from a relation between the following ECO of leptons $\mathbf{Q}_L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and ECO of quarks $\mathbf{Q}_Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$

$$\mathbf{Q}_Q - \mathbf{Q}_L = \frac{2}{3} I_2$$

This leads to the following ECO (2), but in terms of the KCM $K_{2\otimes 2}$.

$$\mathbf{Q} = \sigma_0 \otimes \sigma_0 \otimes \mathbf{Q}_L + \frac{1}{3}\left(K_{2\otimes 2} + \sigma_0 \otimes \sigma_0\right) \otimes \sigma_0$$

or

$$\mathbf{Q} = \sigma_0 \otimes \sigma_0 \otimes \mathbf{Q}_Q + \frac{1}{3}\left(K_{2\otimes 2} - \sigma_0 \otimes \sigma_0\right) \otimes \sigma_0$$

We can remark in these formulas that the eigenvalue $+1$ of the KCM $K_{2\otimes 2}$ is associated to the charges of leptons whereas the eigenvalue -1 is associated to the charges of quarks.

For including more than eight fundamental fermions, let us construct an ECO in terms of the KCM $K_{3\otimes 3}$. For doing so, let us take the following ECOs respectively of leptons and quarks, $\mathbf{Q}_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ whose diagonal is formed by the electric charge of a neutrino and the electric charges of two charged leptons and $\mathbf{Q}_Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$ whose diagonal is formed by the electric charge of a quark u (a quark c (charm) or a quark t (top)) and the electric charges of quark d, quark s (strange) or quark b (bottom).

$$\mathbf{Q}_Q - \mathbf{Q}_L = \frac{2}{3} I_3 \tag{3}$$

Then we have an ECO for some fermions of the SM

$$\mathbf{Q} = I_3 \otimes I_3 \otimes \mathbf{Q}_L + \frac{1}{3}(K_{3\otimes 3} + I_3 \otimes I_3) \otimes I_3$$

or

$$\mathbf{Q} = I_3 \otimes I_3 \otimes \mathbf{Q}_Q + \frac{1}{3}(K_{3\otimes 3} - I_3 \otimes I_3) \otimes I_3$$

4 Eigenvalues of KCMs in Particle physics

The following relations give the eigenvalues of the KCMs with their multiplicities.

$$K_{2\otimes 2} \equiv diag(-1, \underbrace{+1, +1, +1}_{\text{3 times}})$$

$$K_{3\otimes 3} \equiv diag(\underbrace{-1, -1, -1}_{\text{3 times}}, \underbrace{+1, +1, +1, +1, +1, +1}_{\text{6 times}})$$

$$K_{4\otimes 4} \equiv diag(\underbrace{-1, -1, -1, -1, -1, -1}_{\text{6 times}}, \underbrace{+1, +1, +1, +1, +1, +1, +1, +1, +1, +1}_{\text{10 times}})$$

and so on.

Consider the following n dimensional antisymmetric state and symmetric state
$\Psi_a = \frac{1}{2}\left(|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle\right)$ antisymmetric state
$\Psi_s = \frac{1}{2}\left(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle\right)$ symmetric state

$$K_{n\otimes n}\Psi_a = -\Psi_a$$

$$K_{n\otimes n}\Psi_s = +\Psi_s$$

Thus, these states are eigenfunctions of the KCM $K_{n\otimes n}$.
Now, consider the case of two dimensional states. Then

$$K_{2\otimes 2} \equiv diag(-1, \underbrace{+1, +1, +1}_{\text{3 times}})$$

For looking for the eigenvectors of a KCM we have used the method in [8]. The only antisymmetric and symmetric states from $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ are

$$\left.\begin{array}{l} |1\rangle \otimes |1\rangle \\ \frac{1}{\sqrt{2}}\left[|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle\right] \\ |2\rangle \otimes |2\rangle \end{array}\right\} \text{ 3 Symmetric states} \tag{4}$$

$$\frac{1}{\sqrt{2}}\left[|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle\right] \quad \text{1 Antisymmetric state} \tag{5}$$

The symmetric states are the eigenvectors associated to the eigenvalues +1 whereas the antisymmetric state is the one associated to the eigenvalue -1. We have four different eigenstates of the KCM $K_{2\otimes 2}$. Therefore, they are all we look for.
In particle physics, by using the Clebsch-Gordan coefficients or other methods we can have the following examples.
Example 1
Spin operators and matrices for two electrons [9]

$$\left.\begin{array}{l} |S=1, S_z=+1\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle \\ |S=1, S_z=0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) \\ |S=1, S_z=-1\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle \end{array}\right\} \text{ Symmetrics, Triplet}$$

$$|S=0, S_z=0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) \text{ Antisymmetric, Singlet}$$

where

$$\left|\frac{1}{2}, m_{s_1}\right\rangle\left|\frac{1}{2}, m_{s_2}\right\rangle = \left|\frac{1}{2}, m_{s_1}\right\rangle \otimes \left|\frac{1}{2}, m_{s_2}\right\rangle = \left|\frac{1}{2}, m_{s_1}, \frac{1}{2}, m_{s_2}\right\rangle$$

with $m_{s_1}, m_{s_2} = -\frac{1}{2}$ or $+\frac{1}{2}$.
Example 2: Nucleon wave function (See, for example, [10])
From the rules for addition of angular momenta we know that the combination gives a total isospin of 1 or 0. We obtain a symmetric isotriplet:

$$\left.\begin{array}{ll} I=1, I_3=+1 & |p\rangle\,|p\rangle \\ I=1, I_3=0 & \frac{1}{\sqrt{2}}\left(|p\rangle\,|n\rangle + |n\rangle\,|p\rangle\right) \\ I=1, I_3=-1 & |n\rangle\,|n\rangle \end{array}\right\} \text{ Symmetrics, isotriplet}$$

and an antisymmetric isosinglet:

$$I = 0, I_3 = 0 \qquad \frac{1}{\sqrt{2}}\left(|p\rangle\,|n\rangle - |n\rangle\,|p\rangle\right) \text{ Antisymmetric, isosinglet}$$

where $|p\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and $|n\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$.

Example 2: Combination of quark-quark (See, for example, [11])

One can construct combination of two quarks. Between up quark u and down quark d there are four combinations which may be arranged as

$$2 \otimes 2 = 3 \oplus 1$$

$$\left.\begin{array}{ll} I = 1, I_3 = +1 & |u\rangle\,|u\rangle \\ I = 1, I_3 = 0 & \frac{1}{\sqrt{2}}\left(|u\rangle\,|d\rangle + |d\rangle\,|u\rangle\right) \\ I = 1, I_3 = -1 & |d\rangle\,|d\rangle \end{array}\right\} \text{ Symmetrics, Triplet}$$

$$I = 0, I_3 = 0 \qquad \frac{1}{\sqrt{2}}\left(|u\rangle\,|d\rangle - |d\rangle\,|u\rangle\right) \text{ Antisymmetric, Singlet}$$

Now let us pass to the eigenvalues of the KCM $K_{3\otimes 3}$.

$$K_{3\otimes 3} \equiv diag(\underbrace{-1,-1,-1}_{3 \text{ times}}, \underbrace{+1,+1,+1,+1,+1,+1}_{6 \text{ times}})$$

The only antisymmetric and symmetric states from $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are

$$\left.\begin{array}{l} |1\rangle \otimes |1\rangle \\ \frac{1}{\sqrt{2}}\left[|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle\right] \\ |2\rangle \otimes |2\rangle \\ \frac{1}{\sqrt{2}}\left[|1\rangle \otimes |3\rangle + |3\rangle \otimes |1\rangle\right] \\ \frac{1}{\sqrt{2}}\left[|2\rangle \otimes |3\rangle + |3\rangle \otimes |2\rangle\right] \\ |3\rangle \otimes |3\rangle \end{array}\right\} \text{ 6 Symmetric states} \tag{6}$$

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}\left[|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle\right] \\ \frac{1}{\sqrt{2}}\left[|1\rangle \otimes |3\rangle - |3\rangle \otimes |1\rangle\right] \\ \frac{1}{\sqrt{2}}\left[|2\rangle \otimes |3\rangle - |3\rangle \otimes |2\rangle\right] \end{array}\right\} \text{ 3 Antisymmetric states} \tag{7}$$

Like the case of the KCM $K_{2\otimes 2}$, the symmetric states are the eigenvectors associated to the eigenvalues $+1$ whereas the antisymmetric states are the ones associated to the eigenvalues -1.

Example 1: Combination of quark-quark (See, for example, [11])
We can combine any two of the quarks u, d and s. There are nine such combinations which may be arranged as

$$3 \otimes 3 = 6 \oplus \bar{3}$$

From two quarks we can obtain a sextet (the symmetric combinations);

$$\left.\begin{array}{ll} I=1, I_3=1 & |u\rangle\,|u\rangle \\ I=1, I_3=-1 & |d\rangle\,|d\rangle \\ I=1, I_3=0 & |s\rangle\,|s\rangle \\ I=1, I_3=0 & \frac{1}{\sqrt{2}}\left(|u\rangle\,|d\rangle+|d\rangle\,|u\rangle\right) \\ I=1, I_3=-\frac{1}{2} & \frac{1}{\sqrt{2}}\left(|d\rangle\,|s\rangle+|s\rangle\,|d\rangle\right) \\ I=1, I_3=\frac{1}{2} & \frac{1}{\sqrt{2}}\left(|u\rangle\,|s\rangle+|s\rangle\,|u\rangle\right) \end{array}\right\} \text{Symmetrics, Sixtet}$$

and a triplet (the antisymmetric combinations):

$$\left.\begin{array}{ll} I=0, I_3=0 & \frac{1}{\sqrt{2}}\left(|u\rangle\,|d\rangle-|d\rangle\,|u\rangle\right) \\ I=0, I_3=-\frac{1}{2} & \frac{1}{\sqrt{2}}\left(|d\rangle\,|s\rangle-|s\rangle\,|d\rangle\right) \\ I=0, I_3=\frac{1}{2} & \frac{1}{\sqrt{2}}\left(|u\rangle\,|s\rangle-|s\rangle\,|u\rangle\right) \end{array}\right\} \text{Antisymmetrics, Triplet}$$

Comparing the four examples above respectively with (4), (5), (6) and (7) we can see that the eigenvalues +1 of the KCMs $K_{2\otimes 2}$ and $K_{3\otimes 3}$ are associated to the spin 1 or isospin 1 whereas the eigenvalues -1 are associated to spin 0 or isospin 0 from the combinations of states with spin $\frac{1}{2}$.

We would like to make rremark the following combinations of color states [12].

From the color states $|R\rangle$, $|B\rangle$ and $|G\rangle$ we can combine two any colors. Then we obtain a sextet (the symmetric combinations);

$$\left.\begin{array}{l} |R\rangle\,|R\rangle \\ |G\rangle\,|G\rangle \\ |B\rangle\,|B\rangle \\ \frac{1}{\sqrt{2}}\left(|R\rangle\,|G\rangle+|G\rangle\,|R\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|G\rangle\,|B\rangle+|B\rangle\,|G\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|B\rangle\,|R\rangle+|R\rangle\,|B\rangle\right) \end{array}\right\} \text{Symmetrics, Sixtet}$$

which are eigenvectors of the KCM $K_{3\otimes 3}$ associated to the eigenvalue +1 and a triplet (the antisymmetric combinations):

$$\left.\begin{array}{l} \frac{1}{\sqrt{2}}\left(|R\rangle\,|G\rangle-|G\rangle\,|R\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|G\rangle\,|B\rangle-|B\rangle\,|G\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|B\rangle\,|R\rangle-|R\rangle\,|B\rangle\right) \end{array}\right\} \text{Antisymmetrics, Triplet}$$

which are eigenvectors of the KCM $K_{3\otimes 3}$ associated to the eigenvalue -1.

5 Conclusion

Expression of a KCM in terms of the generalized Gell-Mann matrices have been reviewed. We have also reviewed that an ECO in terms of a KCM for leptons and quarks together has been constructed. For such ECO the eigenvalues +1 are associated to charges of leptons whereas the eigenvalues -1 are associated to the charges of quarks. For two electrons, nucleon wave function, combination of quarks u and d and combination of any two quarks from the quarks u, d, and s the spin or isospin +1 is the physical meaning of the eigenvalues +1 of the KCMs $K_{2\otimes 2}$ and $K_{3\otimes 3}$, the spin or isospin 0 is the physical meaning of the eigenvalue -1 of these KCMs. Then the symmetric states and antisymmetric states are respectively the associated eigenvectors.

Acknowledgement 1 *We would like to thank the Doctors Ratsimbarison Mahasedra of iHEPMAD, Hanitriarivo Rakotoson of University of Antananarivo and Ravo Tokiniaina Ranaivoson of Madagascar-INSTN for discussions, the Professor M Knecht of CPT Marseille for his remark after the talk and the Professor Stephan Narison of the University of Montpellier 2, the International Committee and Local organisation of the HEPMAD16 15TH Anniversary.*

References

[1] L.D. Faddev, *Algebraic Aspects of the Bethe ansatz*, Int. J. Mod. Phys.A, 10, No 13, May, 1848(1995).

[2] C. Rakotonirina, *Tensor Commutation matrices in Finite Dimension*, arXiv: math.GM/0508053, (2005).

[3] C. J. R. Sheppard, M. Castello, and A. Diaspro, *Three-dimensional polarization algebra*, J. Opt. Soc. Am. A, **33**, 1938-1947 (2016).

[4] C. Rakotonirina, *Expression of a tensor commutation matrix in terms of the generalized Gell-Mann matrices*, Int. J. Math. Math. Sci. 2007, 20672 (2007).

[5] C. Rakotonirina, A.A. Ratiarison, *Swap Operators and Electric Charges of Fermions*, International Journal of Theoretical and Applied Physics (IJTAP), Vol.3, No.1 (2013), pp. 15-24.

[6] J. C. Baez, J. Huerta, *The Algebra of Grand Unified Theories*, Bull. Am. Math. Soc. 47: 483-552, 2010.

[7] P. Zenczykowski, *Space, Phase Space and Quantum Numbers of Elementary Particles*, Acta Phys. Pol. B 38, 2053 (2007).

[8] N. Wheeler, *Comments concerning Julian Schwinger's "on angular momentum"*, Reed College Physics Department, (2000).

[9] F. M. Fernández, *The Kronecker product and some of its physical applications*, Eur. J. Phys. 37 (2016) 065403.

[10] D. Griffiths, *Introduction to Elementary Particles*, , Wiley-VCH, p.131 (2008).

[11] M. V. N. Murthy, *Notes on Elementary Particle Physics*, The Institute of Mathematical Sciences (2012).

[12] P.Z. Skands, *Introduction to QCD*, arXiv:1207.2389v4 [hep-ph] 2015.