

Growth of the number of simple closed geodesics on hyperbolic surfaces

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1. Introduction

In this paper, we study the growth of $s_X(L)$, the number of simple closed geodesics of length $\leq L$ on a complete hyperbolic surface X of finite area. We also study the frequencies of different types of simple closed geodesics on X and their relationship with the Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces.

Simple closed geodesics. Let $c_X(L)$ be the number of primitive closed geodesics of length $\leq L$ on X . The problem of understanding the asymptotics of $c_X(L)$ has been investigated intensively. Due to work of Delsarte, Huber and Selberg, it is known that

$$c_X(L) \sim e^L/L$$

as $L \rightarrow \infty$. By this result the asymptotic growth of $c_X(L)$ is independent of the genus of X . See [Bus] and the references within for more details and related results. Similar statements hold for the growth of the number of closed geodesics on negatively curved compact manifolds [Ma].

However, very few closed geodesics are *simple* [BS2] and it is hard to discern them in $\pi_1(X)$ [BS1].

Counting problems. Let $\mathcal{M}_{g,n}$ be the moduli space of complete hyperbolic Riemann surfaces of genus g with n cusps. Fix $X \in \mathcal{M}_{g,n}$. To understand the

growth of $s_X(L)$, it proves fruitful to study different types of simple closed geodesics on X separately. Let $S_{g,n}$ be a closed surface of genus g with n boundary components. The mapping class group $\text{Mod}_{g,n}$ acts naturally on the set of isotopy classes of simple closed curves on $S_{g,n}$. Every isotopy class of a simple closed curve contains a unique simple closed geodesic on X . Two simple closed geodesics γ_1 and γ_2 are of the same type if and only if there exists $g \in \text{Mod}_{g,n}$ such that $g \cdot \gamma_1 = \gamma_2$. The type of a simple closed geodesic γ is determined by the topology of $S_{g,n}(\gamma)$, the surface that we get by cutting $S_{g,n}$ along γ . We fix a simple closed geodesic γ on X and consider more generally the counting function

$$s_X(L, \gamma) = \#\{\alpha \in \text{Mod}_{g,n} \cdot \gamma \mid \ell_\alpha(X) \leq L\}.$$

Note that there are only finitely many simple closed geodesics on X up to the action of the mapping class group. Therefore,

$$s_X(L) = \sum_{\gamma} s_X(L, \gamma),$$

where the sum is over all types of simple closed geodesics.

We say that $\gamma = \sum_{i=1}^k a_i \gamma_i$ is a *multi-curve* on $S_{g,n}$ if γ_i 's are disjoint, essential, nonperipheral simple closed curves, no two of which are in the same homotopy class, and $a_i > 0$ for $1 \leq i \leq k$. In this case, the length of γ on X is defined by $\ell_\gamma(X) = \sum_{i=1}^k a_i \ell_{\gamma_i}(X)$. We call the multi-curve γ integral if $a_i \in \mathbb{N}$ for $1 \leq i \leq k$ (or rational if $a_i \in \mathbb{Q}$).

In Section 6 we establish the following result:

THEOREM 1.1. *For any rational multi-curve γ ,*

$$(1.1) \quad \lim_{L \rightarrow \infty} \frac{s_X(L, \gamma)}{L^{6g-6+2n}} = n_\gamma(X),$$

where $n_\gamma: \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$ is a continuous proper function.

Measured laminations. A key role in our approach is played by the space $\mathcal{ML}_{g,n}$ of compactly supported measured laminations on $S_{g,n}$: a piecewise linear space of dimension $6g-6+2n$, whose quotient by the scalars $P\mathcal{ML}(S_{g,n})$ can be viewed as a boundary of the Teichmüller space $\mathcal{T}_{g,n}$. The space $\mathcal{ML}_{g,n}$ has a piecewise linear integral structure; the integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with integral multi-curves on $S_{g,n}$. In fact, $\mathcal{ML}_{g,n}$ is the completion of the set of rational multi-curves on $S_{g,n}$.

The mapping class group $\text{Mod}_{g,n}$ of $S_{g,n}$ acts naturally on $\mathcal{ML}_{g,n}$. Moreover, there is a natural $\text{Mod}_{g,n}$ -invariant locally finite measure on $\mathcal{ML}_{g,n}$, the Thurston measure μ_{Th} , given by this piecewise linear integral structure [Th].

For any open subset $U \subset \mathcal{ML}_{g,n}$, we have

$$\mu_{\text{Th}}(t \cdot U) = t^{6g-6+2n} \mu_{\text{Th}}(U).$$

On the other hand, any complete hyperbolic metric X on $S_{g,n}$ induces the length function

$$\begin{aligned} \mathcal{ML}_{g,n} &\rightarrow \mathbb{R}_+, \\ \lambda &\rightarrow \ell_\lambda(X), \end{aligned}$$

satisfying $\ell_{t\lambda}(X) = t \ell_\lambda(X)$.

Let $B_X \subset \mathcal{ML}_{g,n}$ be the unit ball in the space of measured geodesic laminations with respect to the length function at X (see equation (3.1)), and $B(X) = \mu_{\text{Th}}(B_X)$. In Theorem 3.3, we show that the function $B: \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$ is integrable with respect to the Weil-Petersson volume form. The contributions of X and γ to $n_\gamma(X)$ (defined by equation (1.1)) separate as follows:

THEOREM 1.2. *For any rational multi-curve γ , there exists a number $c(\gamma) \in \mathbb{Q}_{>0}$ such that*

$$n_\gamma(X) = \frac{c(\gamma) \cdot B(X)}{b_{g,n}},$$

where $b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X) \cdot dX < \infty$.

Note that $c(\gamma) = c(\delta)$ for all $\delta \in \text{Mod}_{g,n} \cdot \gamma$.

Notes and references. In the case of $g = n = 1$, this result was previously obtained by G. McShane and I. Rivin [MR]. The proof in [MR] relies on counting the integral points in homology of punctured tori with respect to a natural norm. See also [Z] for a different treatment of a related problem.

Polynomial lower and upper bounds for $s_X(L)$ were found by I. Rivin. More precisely, in [Ri] it is proved that for any $X \in \mathcal{T}_{g,n}$, there exists $c_X > 0$ such that

$$\frac{1}{c_X} L^{6g-6+n} \leq s_X(L) \leq c_X \cdot L^{6g-6+2n}.$$

Similar upper and lower bounds for the number of pants decompositions of length $\leq L$ on a hyperbolic surface X were obtained by M. Rees in [Rs].

Idea of the proof of Theorem 1.2. The crux of the matter is to understand the density of $\text{Mod}_{g,n} \cdot \gamma$ in $\mathcal{ML}_{g,n}$. This is similar to the problem of the density of relatively prime pairs (p, q) in \mathbb{Z}^2 . Our approach is to use the moduli space $\mathcal{M}_{g,n}$ to understand the average of these densities. To prove Theorem 1.2, we:

(I): Apply the results of [Mirz2] to show that the integral of $s_X(L, \gamma)$ over the moduli space $\mathcal{M}_{g,n}$

$$P(L, \gamma) = \int_{\mathcal{M}_{g,n}} s_X(L, \gamma) dX$$

is well-behaved. Here the integral on $\mathcal{M}_{g,n}$ is taken with respect to the Weil-Petersson volume form. In fact $P(L, \gamma)$ is a polynomial in L of degree $6g-6+2n$ (§5). Let $c(\gamma)$ be the leading coefficient of $P(L, \gamma)$. So

$$(1.2) \quad c(\gamma) = \lim_{L \rightarrow \infty} \frac{P(L, \gamma)}{L^{6g-6+2n}}.$$

(II): Use the ergodicity of the action of the mapping class group on the space $\mathcal{ML}_{g,n}$ of measured geodesic laminations on $S_{g,n}$ [Mas2] to prove that these densities exist (§6).

Let μ^γ denote the discrete measure on $\mathcal{ML}_{g,n}$ supported on the orbit γ ; that is,

$$\mu^\gamma = \sum_{g \in \text{Mod}_{g,n}} \delta_{g \cdot \gamma}.$$

The space $\mathcal{ML}_{g,n}$ has a natural action of \mathbb{R}_+ by dilation. For $T \in \mathbb{R}_+$, let $T^*(\mu^\gamma)$ denote the rescaling of μ^γ by factor T . Although the action of $\text{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is not linear, it is homogeneous. We define the measure $\mu_{T,\gamma}$ by

$$\mu_{T,\gamma} = \frac{T^*(\mu^\gamma)}{T^{6g-6+2n}}.$$

So given $U \subset \mathcal{ML}_{g,n}$, we have $\mu_{T,\gamma}(U) = \mu^\gamma(T \cdot U) / T^{6g-6+2n}$.

Then, for any $T > 0$:

- the measure $\mu_{T,\gamma}$ is also invariant under the action of $\text{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$, and
- it satisfies

$$(1.3) \quad \mu_{T,\gamma}(B_X) = \frac{s_X(T, \gamma)}{T^{6g-6+2n}}.$$

Therefore, the asymptotic behavior of $s_X(T, \gamma)$ is closely related to the asymptotic behavior of the sequence $\{\mu_{T,\gamma}\}_T$.

In Section 6, we prove the following result:

THEOREM 1.3. *As $T \rightarrow \infty$,*

$$(1.4) \quad \mu_{T,\gamma} \rightarrow \frac{c(\gamma)}{b_{g,n}} \cdot \mu_{\text{Th}},$$

where $c(\gamma)$ is as defined by (1.2)

Note that (1.4) is a statement about the asymptotic behavior of discrete measures on $\mathcal{ML}_{g,n}$, and in some sense it is independent of the geometry of hyperbolic surfaces.

Frequencies of different types of simple closed curves. From Theorem 1.2, it follows that the relative frequencies of different types of simple closed curves on X are universal rational numbers.

COROLLARY 1.4. *Given $X \in \mathcal{M}_{g,n}$ and rational multi-curves γ_1 and γ_2 on $S_{g,n}$, we have*

$$\lim_{L \rightarrow \infty} \frac{s_X(L, \gamma_1)}{s_X(L, \gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)} \in \mathbb{Q}_{>0}.$$

Remark. The same result holds for any compact surface X of variable negative curvature; given a rational multi-curve γ , the rational number $c(\gamma)$ is independent of the metric (§6).

The frequency $c(\gamma) \in \mathbb{Q}$ of a given simple closed curve can be described in a purely topological way as follows ([Mirz1]). For any connected simple closed curve γ , we have

$$\frac{\#(\{\lambda \text{ an integral multi-curve} \mid i(\lambda, \gamma) \leq k\} / \text{Stab}(\gamma))}{k^{6g-6+2n}} \rightarrow c(\gamma)$$

as $k \rightarrow \infty$.

Example. For $i = 1, 2$, Let α_i be a curve on S_2 that cuts the surface into i connected components. Then as $L \rightarrow \infty$

$$\frac{s_X(L, \alpha_1)}{s_X(L, \alpha_2)} \rightarrow 6.$$

In other words, a very long simple closed geodesic on a surface of genus 2 is six times more likely to be nonseparating. For more examples see Section 6.

Connection with intersection numbers of tautological line bundles. In Section 5, we calculate $c(\gamma)$ in terms of the Weil-Petersson volumes of moduli space of bordered hyperbolic surfaces. Hence, $c(\gamma)$ is given in terms of the intersection numbers of tautological line bundles over the moduli space of Riemann surfaces of type $S_{g,n}(\gamma)$, the surface that we get by cutting $S_{g,n}$ along γ [Mirz3]. See equation (5.5).

An alternative proof. In a sequel, we give a different proof of the growth of the number of simple closed geodesics by using the ergodic properties of the earthquake flow on $\mathcal{PM}_{g,n}$, the bundle of measured geodesic laminations of unit length over moduli space.

Acknowledgments. I would like to thank Curt McMullen for his invaluable help and many insightful discussions related to this work. I am also grateful to Igor Rivin, Howard Masur, and Scott Wolpert for helpful comments. The author is supported by a Clay fellowship.

2. Background material

In this section, we present some familiar concepts concerning the moduli space of bordered Riemann surfaces with geodesic boundary components, and the space of measured geodesic laminations.

Teichmüller space. A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f: S \rightarrow X$. The map f provides a *marking* on X by S . Two marked surfaces $f: S \rightarrow X$ and $g: S \rightarrow Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1}: Y \rightarrow X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}(S, L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have

$$\ell_\beta(X) = L_\beta.$$

Let $\text{Mod}(S)$ denote the mapping class group of S , or in other words the group of isotopy classes of orientation-preserving, self-homeomorphisms of S leaving each boundary component set-wise fixed.

Let

$$\mathcal{T}_{g,n}(L_1, \dots, L_n) = \mathcal{T}(S_{g,n}, L_1, \dots, L_n)$$

denote the Teichmüller space of hyperbolic structures on $S_{g,n}$, an oriented connected surface of genus g with n boundary components $(\beta_1, \dots, \beta_n)$, with geodesic boundary components of length L_1, \dots, L_n . The mapping class group $\text{Mod}_{g,n} = \text{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \dots, L_n) / \text{Mod}_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length $\ell_{\beta_i} = L_i$.

By convention, a geodesic of length zero is a cusp and we have

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \dots, 0),$$

and

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \dots, 0).$$

For a disconnected surface $S = \bigcup_{i=1}^k S_i$ such that $A_i = \partial S_i \subset \partial S$, we have

$$\mathcal{M}(S, L) = \prod_{i=1}^k \mathcal{M}(S_i, L_{A_i}),$$

where $L_{A_i} = (L_s)_{s \in A_i}$.

The Weil-Petersson symplectic form. Recall that a *symplectic structure* on a manifold M is a nondegenerate, closed 2-form $\omega \in \Omega^2(M)$. The n -fold wedge product

$$\frac{1}{n!} \omega \wedge \cdots \wedge \omega$$

never vanishes and defines a volume form on M . By work of Goldman [Gol], the space $\mathcal{T}_{g,n}(L_1, \dots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called the *Weil-Petersson symplectic form*, and denoted by ω or ω_{wp} . In this paper, we consider the volume of the moduli space with respect to the volume form induced by the Weil-Petersson symplectic form. Note that when S is disconnected, we have

$$\text{Vol}(\mathcal{M}(S, L)) = \prod_{i=1}^k \text{Vol}(\mathcal{M}(S_i, L_{A_i})).$$

The Fenchel-Nielsen coordinates. A *pants decomposition* of S is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^k$, where $k = 3g - 3 + n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with \mathcal{P} , $\{\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \dots, \tau_{\alpha_k}(X)\}$, consist of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism [Bus]

$$\mathcal{T}_{g,n}(L_1, \dots, L_n) \cong \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \rightarrow (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

By work of Wolpert, the Weil-Petersson symplectic structure has a simple form in the Fenchel-Nielsen coordinates [Wol].

THEOREM 2.1 (Wolpert). *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^k d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

Measured geodesic laminations. Here we briefly sketch some basic properties of the space of measured geodesic laminations. For more details see [FLP], [Th] and [HP].

A *geodesic lamination* on a hyperbolic surface X is a closed subset of X which is a disjoint union of simple geodesics. A *measured geodesic lamination* is a geodesic lamination that carries a transverse invariant measure. Namely, a compactly supported measured geodesic lamination $\lambda \in \mathcal{ML}_{g,n}$ consists of a

compact subset of X foliated by complete simple geodesics and a measure on every arc k transverse to λ ; this measure is invariant under homotopy of arcs transverse to λ . To understand measured geodesic laminations, it is helpful to lift them to the universal cover of X . A directed geodesic is determined by a pair of points $(x_1, x_2) \in (S^\infty \times S^\infty) \setminus \Delta$, where Δ is the diagonal $\{(x, x)\}$. A geodesic without direction is a point on $J = ((S^\infty \times S^\infty) \setminus \Delta) / \mathbb{Z}_2$, where \mathbb{Z}_2 acts by interchanging coordinates. Then geodesic laminations on two homeomorphic hyperbolic surfaces may be compared by passing to the circle at ∞ . As a result, the spaces of measured geodesic laminations on $X, Y \in \mathcal{T}_{g,n}$ are naturally identified via the circle at infinity in their universal covers. The space $\mathcal{ML}_{g,n}$ of compactly supported measured geodesic laminations on $X \in \mathcal{T}_{g,n}$ only depends on the topology of $S_{g,n}$. Moreover, there is a natural topology on $\mathcal{ML}_{g,n}$, which is induced by the weak topology on the set of all $\pi_1(S_{g,n})$ -invariant measures supported on J .

Train tracks. A *train track* on $S = S_{g,n}$ is an embedded 1-complex τ such that:

- Each edge (branch) of τ is a smooth path with well-defined tangent vectors at the end points. That is, all edges at a given vertex (switch) are tangent.
- For each component R of $S \setminus \tau$, the double of R along the interior of edges of ∂R has negative Euler characteristic.

The vertices (or switches) of a train track are the points where three or more smooth arcs come together. The inward pointing tangent of an edge divides the branches that are incident to a vertex into incoming and outgoing branches.

A lamination γ on S is carried by τ if there is a differentiable map $f: S \rightarrow S$ homotopic to the identity taking γ to τ such that the restriction of df to a tangent line of γ is nonsingular. Every geodesic lamination λ is carried by some train track τ . When λ has an invariant measure μ , the carrying map defines a counting measure $\mu(b)$ for each edge b of τ . At a switch, the sum of the entering numbers equals the sum of the exiting numbers.

Let $E(\tau)$ be the set of measures on train track τ ; more precisely, $u \in E(\tau)$ is an assignment of positive real numbers on the edges of the train track satisfying the switch conditions,

$$\sum_{\text{incoming } e_i} u(e_i) = \sum_{\text{outgoing } e_j} u(e_j).$$

By work of Thurston, we have:

- If τ is a birecurrent train track (see [HP, §1.7]), then $E(\tau)$ gives rise to an open set $U(\tau) \subset \mathcal{ML}_{g,n}$.

- The integral points in $E(\tau)$ are in a one-to-one correspondence with the set of integral multi-curves in $U(\tau) \subset \mathcal{ML}_{g,n}$.
- The natural volume form on $E(\tau)$ defines a mapping class group invariant volume form μ_{Th} in the Lebesgue measure class on $\mathcal{ML}_{g,n}$.

Moreover, up to scale, μ_{Th} is the unique mapping class group invariant measure in the Lebesgue measure class [Mas2]:

THEOREM 2.2 (Masur). *The action of $\text{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is ergodic with respect to the Lebesgue measure class.*

We remark that the space of measured laminations $\mathcal{ML}_{g,n}$ does not have a natural differentiable structure [Th].

Length functions. The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. The length function can be extended by homogeneity and continuity on $\mathcal{ML}_{g,n}$ [Ker]. More precisely, there is a unique continuous map

$$(2.1) \quad L: \mathcal{ML}_{g,n} \times \mathcal{T}_{g,n} \rightarrow \mathbb{R}_+,$$

such that

- for any simple closed curve α , $L(\alpha, X) = \ell_\alpha(X)$,
- for $t \in \mathbb{R}_+$, $L(t \cdot \lambda, X) = t \cdot L(\lambda, X)$, and
- for any $h \in \text{Mod}_{g,n}$, $L(h \cdot \lambda, h \cdot X) = L(\lambda, X)$.

For $\lambda \in \mathcal{ML}_{g,n}$, $\ell_\lambda(X) = L(\lambda, X)$ is the geodesic length of the measured lamination λ on X . For more details see [Th].

3. Counting integral multi-curves

In this section, we study the growth of the number of integral multi-curves of length $\leq L$ on a hyperbolic Riemann surface X . To simplify notation, let $\mathcal{ML}_{g,n}(\mathbb{Z})$ denote the set of integral multi-curves on $S_{g,n}$.

Counting integral multi-curves. Define $b_X(L)$ by

$$b_X(L) = \#\{\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell_\gamma(X) \leq L\}.$$

In other words, $b_X(L)$ is the number of integral points in $L \cdot B_X \subset \mathcal{ML}_{g,n}$, where

$$(3.1) \quad B_X = \{\lambda \in \mathcal{ML}_{g,n} \mid \ell_\lambda(X) \leq 1\}.$$

In fact the subset $B_X \subset \mathcal{ML}_{g,n}$ is locally convex [Mirz1].

The function $B: \mathcal{T}_{g,n} \rightarrow \mathbb{R}_+$ defined by

$$(3.2) \quad B(X) = \mu_{\text{Th}}(B_X)$$

plays an important role in this section.

PROPOSITION 3.1. *For any $X \in \mathcal{T}_{g,n}$,*

$$\frac{b_X(L)}{L^{6g-6+2n}} \rightarrow B(X)$$

as $L \rightarrow \infty$.

Proof of Proposition 3.1. For any train track τ on $S_{g,n}$, define

$$b_\tau(U, L) = \#(\mathcal{ML}_{g,n}(\mathbb{Z}) \cap (L \cdot U) \cap U_\tau).$$

Recall that U_τ has a linear integral structure (§2), and the points in $U_\tau \cap \mathcal{ML}_{g,n}(\mathbb{Z})$ are in a one-to-one correspondence with the integral points in this chart. Therefore by basic lattice counting estimates, we get

$$(3.3) \quad \frac{b_\tau(U, L)}{L^{6g-6+2n}} \rightarrow \mu_{\text{Th}}(U \cap U_\tau)$$

as $L \rightarrow \infty$. Cover $\mathcal{ML}_{g,n}$ by finitely many train-track charts $U_{\tau_1}, \dots, U_{\tau_k}$. Since the transition functions are volume-preserving, the result follows from equation (3.3) applied to each chart U_{τ_i} . \square

Note that the function B descends to a function over $\mathcal{M}_{g,n}$. On the other hand, given $\lambda \in \mathcal{ML}_{g,n}$, the length function

$$\ell_\lambda: \mathcal{T}_{g,n} \rightarrow \mathbb{R}_+$$

is smooth [Ker]. Hence we have:

PROPOSITION 3.2. *The function $B: \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$, defined by equation (3.2), is continuous.*

In this section, we show that

$$(3.4) \quad b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X) \cdot dX < \infty,$$

where the integral is with respect to the Weil-Peterson volume form.

THEOREM 3.3. *The function B is proper and integrable over $\mathcal{M}_{g,n}$.*

The proof relies on explicit upper and lower bounds for $B(X)$ obtained in Proposition 3.6. For an explicit calculation of $b_{g,n}$, see Theorem 5.3.

Dehn's coordinates for multi-curves. Let

$$\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3+n}\}$$

be a maximal system of simple closed curves on $S_{g,n}$. In order to prove Theorem 3.3, we estimate the hyperbolic length of a multi-curve on X in terms of its combinatorial length with respect to a pants decomposition (see eq. (3.6)).

Consider the Dehn-Thurston parametrization [HP] of the set of multi-curves defined by

$$(3.5) \quad \begin{aligned} DT: \mathcal{ML}_{g,n}(\mathbb{Z}) &\rightarrow (\mathbb{Z}_+ \times \mathbb{Z})^{3g-3+n} \\ \gamma &\rightarrow (m_i, t_i)_{i=1}^{3g-3+n}, \end{aligned}$$

where $m_j = i(\gamma, \alpha_j) \in \mathbb{Z}_+$ is the intersection number of γ and α_j , and $t_j = \text{tw}(\gamma, \alpha_j) \in \mathbb{Z}$ is the twisting number of γ around α_j . Dehn's theorem asserts that these parameters uniquely determine a multi-curve.

Let $Z(\mathcal{P})$ be the set of $(m_i, t_i)_{i=1}^{3g-3+n} \in (\mathbb{Z}_+ \times \mathbb{Z})^{3g-3+n}$ such that the following conditions are satisfied:

Z-1. If $m_i = 0$, then $t_i \geq 0$.

Z-2. If $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}$ bound an embedded pair of pants in $S_{g,n}$, then $m_{i_1} + m_{i_2} + m_{i_3}$ is even.

Then we have:

THEOREM 3.4 (Dehn). *For any pants decomposition \mathcal{P} of $S_{g,n}$, the map*

$$DT: \mathcal{ML}_{g,n}(\mathbb{Z}) \rightarrow Z(\mathcal{P})$$

is a bijection.

See [HP] for more details.

Combinatorial lengths of multi-curves. Let γ be a multi-curve on $X \in \mathcal{T}_{g,n}$. Define the combinatorial length of γ with respect to a pants decomposition $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3+n}\}$ by

$$(3.6) \quad L_{\mathcal{P}}(X, \gamma) = \sum_{i=1}^{3g-3+n} (m_i \cdot S(\ell_{\alpha_i}(X)) + |t_i| \cdot \ell_{\alpha_i}(X)),$$

where $S(x) = \text{arcsinh}\left(\frac{1}{\sinh(x/2)}\right)$. Note that $S(x)$ is equal to the width of the collar neighborhood around a simple closed geodesic of length x on a hyperbolic surface [Bus].

We say a pants decomposition \mathcal{P} is L -bounded on $X \in \mathcal{T}_{g,n}$ if $|\tau_{\alpha}(X)| \leq \ell_{\alpha}(X) \leq L$ for every $\alpha \in \mathcal{P}$.

PROPOSITION 3.5. *Let \mathcal{P} be an L -bounded pants decomposition of $X \in \mathcal{T}_{g,n}$. Then for any multi-curve γ on X ,*

$$(3.7) \quad \frac{1}{c} L_{\mathcal{P}}(X, \gamma) \leq \ell_{\gamma}(X) \leq c L_{\mathcal{P}}(X, \gamma),$$

where the constant c depends only on L, g and n .

Proof. First we prove the lower bound on $\ell_\gamma(X)$ in (3.7). Our proof is inspired by some of the ideas used in [DS] (§5.1).

Without loss of generality, we can assume that γ is a connected simple closed geodesic on X . Fix an orientation on γ . Let p_1, \dots, p_r be the ordered set of intersection points of γ with \mathcal{P} (with respect to the orientation of γ) such that $p_j \in \alpha_{k_j}$. Here $1 \leq k_j \leq 3g - 3 + n$, and

$$r = i(\gamma, \mathcal{P}) = \sum_{j=1}^{3g-3+n} i(\gamma, \alpha_j).$$

Let γ_j denote the segment of γ between p_{j-1} and p_{j+1} , where $p_0 = p_r$, and $p_{r+1} = p_1$. Define \tilde{t}_j to be the twisting number of the arc γ_j around the curve α_{k_j} . Then one can verify easily that

- $2\ell_\gamma(X) = \sum_{i=1}^r \ell(\gamma_j)$,
- $i(\gamma, \alpha_i) = \#\{j \mid k_j = i\}$, and
- $|t_i| = \sum_{\{j: k_j=i\}} |\tilde{t}_j|$.

We show that in terms of the above notation,

$$(3.8) \quad \ell(\gamma_j) \geq c_1 (S(\ell_{\alpha_{k_j}}(X)) + |\tilde{t}_j| \ell_{\alpha_{k_j}}(X)),$$

where c_1 is a constant which only depends on L .

Let $\tilde{\gamma}$ be the lift of γ through p_{j-1} . Let $\tilde{C}_{j-1}, \tilde{C}_j$ and \tilde{C}_{j+1} be the lifts of $\alpha_{k_{j-1}}, \alpha_{k_j}$ and $\alpha_{k_{j+1}}$ which are intersected in order by $\tilde{\gamma}$ such that $\tilde{C}_{j-1} \cap \tilde{\gamma} = p_{j-1}$. Then \tilde{C}_i and \tilde{C}_{i+1} project to two boundary components of a pair of pants in \mathcal{P} .

For $i = j - 1, j$, consider the common perpendicular segment to $\tilde{C}_i, \tilde{C}_{i+1}$, with end-points denoted by Q_i^+, Q_{i+1}^- . Let s_j, d_{j-1} and d_j be respectively the geodesic length of arcs $Q_j^- Q_j^+, Q_{j-1}^+ Q_j^-$ and $Q_j^+ Q_{j+1}^-$. See Figure 1.

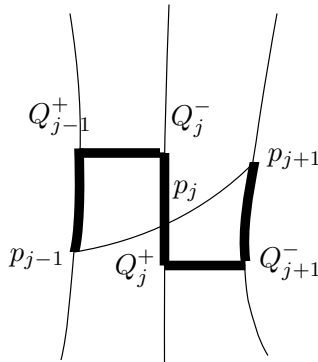


Figure 1

Then we have:

- $d_j \geq S(\ell_{\alpha_{k_j}})$, and
- the shift s_j is given by $s_j = |\tilde{t}_j \ell_{\alpha_{k_j}} + \tau_{\alpha_{k_j}} + e_j|$, where $|e_j| < \ell_{\alpha_{k_j}}$. Since $|\tau_{\alpha_{k_j}}| \leq \ell_{\alpha_{k_j}}$, we get $s_j \geq (|\tilde{t}_j| - 2) \ell_{\alpha_{k_j}}$.

Here both e_j and \tilde{t}_j are independent of the geometry of X . They only depend on the topology of γ relative to the pants decomposition \mathcal{P} . See §5.1 in [DS] for more details.

As a result we have

$$(3.9) \quad d_j + s_j \geq S(\ell_{\alpha_{k_j}}) + (|\tilde{t}_j| - 2) \ell_{\alpha_{k_j}}.$$

Since \mathcal{P} is L -bounded, $\ell_{\alpha_i}(X) \leq L$, and there exists a constant $D_L > 0$ such that $d_j \geq D_L$. Now equation (3.8) follows from the next two basic observations:

(1) Let p_1 and p_2 be two points of distance x and z to a geodesic line. Consider the geodesic quadrangle $p_1 q_1 q_2 p_2$ such that $d(q_1, q_2) = y$, $d(p_1, q_1) = x$, and $d(q_2, p_2) = z$. If $y \geq D(L)$, then

$$(3.10) \quad d = d(p_1, p_2) \geq c(L)(x + y + z),$$

where $c(L)$ is a constant depending only on L . To prove (3.10), note that by basic hyperbolic trigonometry,

$$\cosh(d) = \cosh(x) \cosh(z) \cosh(y) - \sinh(x) \sinh(z).$$

Here x and z are oriented lengths; if p_1 and p_2 lie on opposite sides of $q_1 q_2$, then x and z have opposite signs. See §2.3.2 of [Bus]. This formula implies that $\cosh(d) \geq \cosh(x) \cosh(z) (\cosh(y) - 1)$. Hence there exists $K = K(L) > 0$ such that

$$(3.11) \quad d \geq x + y + z - K(L);$$

see Lemma 5.1 of [DS]. Since $d \geq D(L)$, it is easy to see that

$$d \geq \min\left\{\frac{1}{2}, \frac{D(L)}{2K(L)}\right\}(x + y + z).$$

(2) Note that for $0 \leq x \leq L$, $S(x)/x$ is bounded from below. Hence, by (3.9) there exists a constant c_2 such that

$$(3.12) \quad d_j + s_j \geq c_2(L) (S(\ell_{\alpha_{k_j}}) + |\tilde{t}_j| \ell_{\alpha_{k_j}}).$$

Therefore, by (3.10) for geodesic quadrangles $p_{j-1} Q_{j-1}^+ Q_j^- p_j$ and $p_j Q_j^+ Q_{j+1}^- p_{j+1}$ we have

$$(3.13) \quad \ell(\gamma_j) \geq c(L)(d_j + s_j).$$

Now equation (3.8) follows from (3.12) and (3.13). By adding the inequality (3.8) for $1 \leq j \leq r$, we get

$$\ell_\gamma(X) = \frac{1}{2} \sum_{j=1}^r \ell(\gamma_j) \geq C(L)L_{\mathcal{P}}(X, \gamma).$$

Let $d(\alpha, \beta)$ denote the length of the shortest geodesic path joining two boundaries α and β of a pair of pants in the pants decomposition \mathcal{P} . To obtain the upper bound on $\ell_\gamma(X)$, it is enough to note that $d(\alpha, \beta) \leq c_3 (S(\ell_\alpha(X)) + S(\ell_\beta(X)))$, where the constant c_3 depends only on L . \square

Upper and lower bounds for $B(X)$. Next, we find upper and lower bounds for the function $B(X)$ in terms of the lengths of short geodesics on X . Define $R: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$R(x) = \frac{1}{x |\log(x)|}.$$

PROPOSITION 3.6. *For any $X \in \mathcal{T}_{g,n}$, sufficiently small $\varepsilon > 0$ and $1 \leq L$,*

$$(3.14) \quad C_1 \cdot \prod_{\gamma: \ell_\gamma(X) \leq \varepsilon} R(\ell_\gamma(X)) \leq B(X),$$

and

$$(3.15) \quad \frac{b_X(L)}{L^{6g-6+2n}} \leq C_2 \cdot \prod_{\gamma: \ell_\gamma(X) \leq \varepsilon} \frac{1}{\ell_\gamma(X)},$$

where $C_1, C_2 > 0$ are constants depending only on g, n and ε .

Sketch of the proof. Take ε small enough such that no two closed geodesics of length $\leq \varepsilon$ on a hyperbolic surface meet. For $X \in \mathcal{T}_{g,n}$, let $\alpha_1, \dots, \alpha_s$ be the set of all simple closed geodesics of length $\leq \varepsilon$ on X and

$$\mathcal{P}_X = \{\alpha_1, \dots, \alpha_s, \dots, \alpha_k\}$$

be a maximal set of disjoint simple closed geodesics such that $\ell_{\alpha_i}(X) \leq L_{g,n}$, where $k = 3g - 3 + n$, and $L_{g,n}$ is the Bers' constant for $S_{g,n}$ (see [Bus]). We can assume that \mathcal{P}_X is $L_{g,n}$ -bounded on X .

Given $x, y, L > 0$, consider the set $A_{x,y}(L)$ defined by

$$A_{x,y}(L) = \{(m, n) \mid mx + ny \leq L\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+.$$

Then for $L > 6 \max\{x, y\}$

$$(3.16) \quad \frac{1}{12} \left(\frac{L^2}{x \cdot y} \right) \leq |A_{x,y}(L)|.$$

Also, for $L > 1$

$$(3.17) \quad |A_{x,y}(L)| \leq 3 \left(\frac{L^2}{x \cdot y} + \frac{L}{\min\{x, y\}} + 1 \right) \leq 3L^2 \left(1 + \frac{1}{x \cdot y} + \frac{1}{\min\{x, y\}} \right).$$

Now we can estimate $b_X(L)$, the number of integral multi-curves of length $\leq L$ on X , by applying (3.7). We use the combinatorial length of multi-curve geodesics with respect to the pants decomposition \mathcal{P}_X instead of their geodesic length on X . By setting $x_i = S(\ell_{\alpha_i}(X))$, $y_i = \ell_{\alpha_i}(X)$, Theorem 3.4 implies that

$$(3.18) \quad \frac{1}{c} \prod_{i=1}^k A_{x_i, y_i} \left(\frac{L}{k} \right) \leq b_X(L) \leq c \prod_{i=1}^k A_{x_i, y_i}(L),$$

where $c > 0$ is a constant independent of X and L . Note that

$$\{(m_i, t_i)_{i=1}^{3g-3+n} \mid \forall i \ 1 \leq i \leq 3g-3+n, \ t_i > 0, \ m_i \in 2\mathbb{Z}_+\} \subset Z(\mathcal{P}).$$

On the other hand,

$$S(x)/|\log(x)| \rightarrow 1$$

as $x \rightarrow 0$. It is easy to check that:

- $1/c_0 \leq x \cdot S(x) \leq c_0$, for $\varepsilon \leq x \leq L_{g,n}$,
- $c_1 \leq \max\{x, S(x)\}$, and
- $\min\{x, S(x)\} \leq c_2$.

Here c_0, c_1 and c_2 are constants which only depend on g, n and ε . Therefore, (3.14) follows from (3.16) and (3.18). Similarly, (3.15) follows from (3.17) and (3.18). \square

Properness and integrability of the function B . In this part we show that the upper bound in (3.15) is an integrable proper function.

Proof of Theorem 3.3. Note that $\inf\{\ell_\gamma\}_\gamma \rightarrow 0$ as $X \rightarrow \infty$ in $\mathcal{M}_{g,n}$. Also,

$$R(\varepsilon) \rightarrow \infty$$

as $\varepsilon \rightarrow 0$. So equation (3.14) implies the function B is proper.

Next we prove that the function $F: \mathcal{M}_{g,n} \rightarrow \mathbb{R}$ defined by

$$(3.19) \quad F(X) = \prod_{\gamma: \ell_\gamma(X) \leq \varepsilon} \frac{1}{\ell_\gamma(X)},$$

is integrable with respect to the Weil-Petersson volume form on $\mathcal{M}_{g,n}$. Let $\mathcal{M}_{g,n}^{k,\varepsilon} \subset \mathcal{M}_{g,n}$ be the subset consisting of surfaces with k simple closed geodesics of length $\leq \varepsilon$. Note that using the Fenchel-Nielsen coordinates, the set $\mathcal{M}_{g,n}^{k,\varepsilon}$ can be covered by finitely many open sets of the form

$$\pi(\{(x_i, y_i)_1^{3g-3+n} \mid 0 \leq x_1, \dots, x_k \leq \varepsilon, \ x_i \leq L_{g,n}, \ 0 \leq y_i \leq x_i\}).$$

See Section 2. By Theorem 2.1, it is enough to note that for

$$V_{\varepsilon,k} = \{(x_i, y_i)_1^k \mid 0 \leq x_1, \dots, x_k \leq \varepsilon, \ 0 \leq y_i \leq x_i\}$$

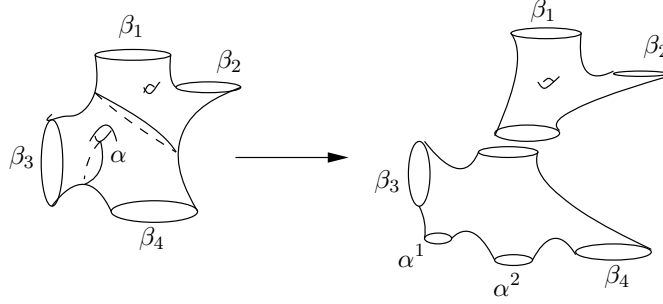


Figure 2. Cutting a surface along a multi-curve

we have

$$\int_{V_{\varepsilon,k}} \frac{1}{x_1 \cdots x_k} dx_1 \cdots dx_k \cdot dy_1 \cdots dy_k < \infty. \quad \square$$

Define $f_L: \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$ by

$$f_L(X) = \frac{b_X(L)}{L^{6g-g+2n}}.$$

By Proposition 3.6, the sequence $\{f_L\}_{L \geq 1}$ satisfies the hypothesis of Lebesgue's dominated convergence theorem. In fact, for every integral multi-curve γ on $S_{g,n}$, we have

$$(3.20) \quad \frac{s_X(L, \gamma)}{L^{6g-6+2n}} \leq f_L(X) \leq C_2 \cdot F(X),$$

where the function F , defined by (3.19), is integrable over $\mathcal{M}_{g,n}$.

4. Integration over the moduli space of hyperbolic surfaces

In this section, we recall the results obtained in [Mirz2] and [Mirz3] for integrating certain geometric functions over the moduli space of hyperbolic surfaces.

Symmetry group of a simple closed curve. For any set A of homotopy classes of simple closed curves on $S_{g,n}$, define $\text{Stab}(A)$ by

$$\text{Stab}(A) = \{g \in \text{Mod}_{g,n} \mid g \cdot A = A\} \subset \text{Mod}_{g,n}.$$

Let $\gamma = \sum_{i=1}^k a_i \gamma_i$, be a multi-curve on $S_{g,n}$. Define the symmetry group of γ , $\text{Sym}(\gamma)$, by

$$\text{Sym}(\gamma) = \text{Stab}(\gamma) / \cap_{i=1}^k \text{Stab}(\gamma_i).$$

Note that for any connected simple closed curve α , $|\text{Sym}(\alpha)| = 1$.

Splitting along a simple closed curve. Consider the surface $S_{g,n} \setminus U_\gamma$, where U_γ is an open set homeomorphic to $\bigcup_1^k ((0,1) \times \gamma_i)$. We denote this surface by $S_{g,n}(\gamma)$, which is a (possibly disconnected) surface with $n + 2k$ boundary components and $s = s(\gamma)$ connected components. Each connected component γ_i of γ , gives rise to two boundary components, γ_i^1 and γ_i^2 on $S_{g,n}(\gamma)$. Namely,

$$\partial(S_{g,n}(\gamma)) = \{\beta_1, \dots, \beta_n\} \cup \{\gamma_1^1, \gamma_1^2, \dots, \gamma_k^1, \gamma_k^2\}.$$

Now for $\Gamma = (\gamma_1, \dots, \gamma_k)$, and $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}_+^k$, let

$$\mathcal{T}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x})$$

be the Teichmüller space of hyperbolic Riemann surfaces homeomorphic to $S_{g,n}(\gamma)$ such that $\ell_{\gamma_i} = x_i$ and $\ell_{\beta_i} = 0$. The group $G(\Gamma) = \prod_{i=1}^k \text{Stab}(\gamma_i)$ naturally acts on $\mathcal{T}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x})$. Now, define

$$\mathcal{M}_{g,n}(\Gamma, \mathbf{x}) = \mathcal{T}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}) / G(\Gamma).$$

Let $\text{Stab}_0(\alpha) \subset \text{Stab}(\alpha)$ denote the subgroup consisting of elements which preserve the orientation of α . Then any $g \in \prod_{i=1}^k \text{Stab}_0(\gamma_i)$ can be thought of as an element in $\text{Mod}(S_{g,n}(\gamma))$. Hence for $(g, n) \neq (1, 1)$, $\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x})$ is a finite cover of $\mathcal{M}_{g,n}(\Gamma, \mathbf{x})$ of order

$$N(\gamma) = \left| \prod_{i=1}^k \text{Stab}(\gamma_i) / \prod_{i=1}^k \text{Stab}_0(\gamma_i) \right|.$$

Therefore,

$$(4.1) \quad \text{Vol}_{wp}(\mathcal{M}_{g,n}(\Gamma, \mathbf{x})) = \frac{1}{N(\gamma)} \prod_{i=1}^s V_{g_i, n_i}(\ell_{A_i}),$$

where

$$S_{g,n}(\gamma) = \bigcup_{i=1}^s S_i,$$

$S_i \cong S_{g_i, n_i}$, and $A_i = \partial S_i$.

There is an exceptional case which arises when $g = n = 1$. In this case, every $X \in \mathcal{M}_{1,1}$ has a symmetry of order 2, $\tau \in \text{Stab}(\gamma)$. As a result, $\text{Vol}(\mathcal{M}_{1,1}(\Gamma, x)) = 1$.

Example. Let α be a connected nonseparating simple closed curve α on $S_{g,n}$. Then there exists an element in $\text{Stab}(\alpha)$ which reverses the orientation of α , and hence $N(\alpha) = 2$.

Simple closed curves on $X \in \mathcal{M}_{g,n}$. Let $[\gamma]$ denote the homotopy class of a simple closed curve γ on $S_{g,n}$. Although there is no canonical simple closed geodesic on $X \in \mathcal{M}_{g,n}$ corresponding to $[\gamma]$, the set

$$\mathcal{O}_\gamma = \{[\alpha] \mid \alpha \in \text{Mod} \cdot \gamma\},$$

of homotopy classes of simple closed curves in the $\text{Mod}_{g,n}$ -orbit of γ on X , is determined by γ . In other words, \mathcal{O}_γ is the set of $[\phi(\gamma)]$ where $\phi: S_{g,n} \rightarrow X$ is a marking of X .

Integration over the moduli space of hyperbolic surfaces. For a multi-curve $\gamma = \sum_{i=1}^k a_i \gamma_i$, we have

$$\ell_\gamma(X) = \sum_{i=1}^k a_i \ell_{\gamma_i}(X).$$

Given a continuous function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$(4.2) \quad f_\gamma(X) = \sum_{[\alpha] \in \text{Mod} \cdot [\gamma]} f(\ell_\alpha(X)),$$

defines a function $f_\gamma: \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$. Then we can calculate the integral of f_γ over $\mathcal{M}_{g,n}$ using the following result [Mirz2]:

THEOREM 4.1. *For any multi-curve $\gamma = \sum_{i=1}^k a_i \gamma_i$, the integral of f_γ over $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson volume form is given by*

$$\int_{\mathcal{M}_{g,n}} f_\gamma(X) dX = \frac{2^{-M(\gamma)}}{|\text{Sym}(\gamma)|} \int_{\mathbf{x} \in \mathbb{R}_+^k} f(|\mathbf{x}|) V_{g,n}(\Gamma, \mathbf{x}) \mathbf{x} \cdot d\mathbf{x},$$

where $\Gamma = (\gamma_1, \dots, \gamma_k)$, $|\mathbf{x}| = \sum_{i=1}^k a_i x_i$, $\mathbf{x} \cdot d\mathbf{x} = x_1 \cdots x_k \cdot dx_1 \wedge \cdots \wedge dx_k$,

$$M(\gamma) = |\{i | \gamma_i \text{ separates off a one-handle from } S_{g,n}\}|,$$

and

$$V_{g,n}(\Gamma, \mathbf{x}) = \text{Vol}_{wp}(\mathcal{M}_{g,n}(\Gamma, \mathbf{x})).$$

By Theorem 4.1, integrating f_γ , even for a compact Riemann surface, reduces to the calculation of volumes of moduli spaces of bordered Riemann surfaces.

Idea of the proof of Theorem 4.1. Here we briefly sketch the main idea of how to calculate the integral of f_γ over $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson volume form when γ is a connected simple closed curve. See [Mirz2] for more details.

First, consider the covering space of $\mathcal{M}_{g,n}$

$\pi^\gamma: \mathcal{M}_{g,n}^\gamma = \{(X, \alpha) \mid X \in \mathcal{M}_{g,n}, \text{ and } \alpha \in \mathcal{O}_\gamma \text{ is a geodesic on } X\} \rightarrow \mathcal{M}_{g,n}$,
where $\pi^\gamma(X, \alpha) = X$. The hyperbolic length function descends to the function

$$\ell: \mathcal{M}_{g,n}^\gamma \rightarrow \mathbb{R}_+$$

defined by $\ell(X, \eta) = \ell_\eta(X)$. Therefore,

$$\int_{\mathcal{M}_{g,n}} f_\gamma(X) dX = \int_{\mathcal{M}_{g,n}^\gamma} f \circ \ell(Y) dY.$$

On the other hand, the function f is constant on each level set of ℓ and we have

$$\int_{\mathcal{M}_{g,n}^\gamma} f \circ \ell(Y) dY = \int_0^\infty f(t) \text{Vol}(\ell^{-1}(t)) dt,$$

where the volume is taken with respect to the volume form induced on $\ell^{-1}(t)$.

The decomposition of the surface along the simple closed curve γ gives rise to a description of $\mathcal{M}_{g,n}^\gamma$ in terms of moduli spaces corresponding to simpler surfaces. This observation leads to formulas for the integral of f_γ in terms of the Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces and the function f as follows.

As before, let $S_{g,n}(\gamma)$ denote the surface obtained by cutting the surface $S_{g,n}$ along γ . Also, $\mathcal{T}_{g,n}(\gamma, \ell_\gamma = t)$ denotes the Teichmüller space of Riemann surfaces homeomorphic to $S_{g,n}(\gamma)$ such that the lengths of the two boundary components corresponding to γ are equal to t . We have a natural circle bundle

$$\begin{array}{c} \ell^{-1}(t) \subset \mathcal{M}_{g,n}^\gamma \\ \downarrow \\ \mathcal{M}_{g,n}(\gamma, \ell_\gamma = t) = \mathcal{T}_{g,n}(\gamma, \ell_\gamma = t) / \text{Stab}(\gamma). \end{array}$$

We consider the S^1 -action on the level set $\ell^{-1}(t) \subset \mathcal{M}_{g,n}^\gamma$ induced by twisting the surface along γ . The quotient space $\ell^{-1}(t)/S^1$ inherits a symplectic form from the Weil-Petersson symplectic form. On the other hand, $\mathcal{M}_{g,n}(\gamma, \ell_\gamma = t)$ is equipped with the Weil-Petersson symplectic form. Also,

$$\ell^{-1}(t)/S^1 \cong \mathcal{M}_{g,n}(\gamma, \ell_\gamma = t)$$

as symplectic manifolds. So we expect to have

$$\text{Vol}(\ell^{-1}(t)) = t \text{Vol}(\mathcal{M}_{g,n}(\gamma, \ell_\gamma = t)).$$

But the situation is different when γ separates off a one-handle in which case the length of the fiber of the S^1 -action at a point is in fact $t/2$ instead of t [Mirz2]. Hence, for any connected simple closed curve γ on $S_{g,n}$,

$$(4.3) \quad \int_{\mathcal{M}_{g,n}} f_\gamma(X) dX = 2^{-M(\gamma)} \int_0^\infty f(t) t \text{Vol}(\mathcal{M}_{g,n}(\gamma, \ell_\gamma = t)) dt,$$

where $M(\gamma) = 1$ if γ separates off a one-handle, and $M(\gamma) = 0$ otherwise.

The Weil-Petersson volumes of the moduli spaces of hyperbolic surfaces. In [Mirz2], by using an identity for the lengths of simple closed geodesics on hyperbolic surfaces and using Theorem 4.1, we obtain a recursive method for calculating volume polynomials.

THEOREM 4.2. *The volume $V_{g,n}(b_1, \dots, b_n) = \text{Vol}_{wp}(\mathcal{M}_{g,n}(b))$ is a polynomial in b_1^2, \dots, b_n^2 ; that is,*

$$V_{g,n}(b) = \sum_{|\alpha| \leq 3g-3+n} C_\alpha \cdot b^{2\alpha},$$

where $C_\alpha > 0$ lies in $\pi^{6g-6+2n-2|\alpha|} \cdot \mathbb{Q}$.

THEOREM 4.3. *The coefficient C_α in Theorem 4.2 is given by*

$$(4.4) \quad C_\alpha = \frac{1}{2^{|\alpha|} |\alpha|! (3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \cdot \omega^{3g-3+n-|\alpha|},$$

where ψ_i is the first Chern class of the i^{th} tautological line bundle, ω is the Weil-Petersson symplectic form, $\alpha! = \prod_{i=1}^n \alpha_i!$, and $|\alpha| = \sum_{i=1}^n \alpha_i$.

See [Mirz2] and [Mirz3] for more details.

Examples. One can use the recursive formula obtained in [Mirz2], or other similar recursive formulas to calculate the coefficients of the volume polynomials.

(1) By [Mirz2],

$$(4.5) \quad V_{1,1}(b) = \frac{1}{24}(b^2 + 4\pi^2),$$

$$(4.6) \quad V_{1,2}(b_1, b_2) = \frac{1}{192}(4\pi^2 + b_1^2 + b_2^2)(12\pi^2 + b_1^2 + b_2^2).$$

(2) In general, for $g = 0$,

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} = \binom{n-3}{\alpha_1 \dots \alpha_n}.$$

(3) For $n = 1$ and $g > 1$, we have [FP], [IZ]:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{6g-4} = \frac{1}{24^g g!}.$$

Therefore, by Theorem 4.3 the leading coefficient of the polynomial $V_{g,1}(L)$ is equal to

$$(4.7) \quad \frac{L^{6g-4}}{24^g g! (3g-2)! 2^{3g-2}}.$$

For more on calculating intersection pairings over $\overline{\mathcal{M}}_{g,n}$ see [ArC].

5. Counting curves and Weil-Petersson volumes

In this section we establish a relationship between $s_X(L, \gamma)$ and the Weil-Petersson volume of moduli spaces of bordered Riemann surfaces. We use this relationship to calculate $b_{g,n}$ in terms of the leading coefficients of volume polynomials.

Let $P(L, \gamma)$ be the integral of $s_X(L, \gamma)$ over $\mathcal{M}_{g,n}$, given by

$$P(L, \gamma) = \int_{\mathcal{M}_{g,n}} s_X(L, \gamma) dX.$$

Now by using Theorem 4.1 for $f = \chi([0, L])$, we obtain the following result:

PROPOSITION 5.1. *For any multi-curve $\gamma = \sum_{i=1}^k a_i \gamma_i$, the integral of $s_X(L, \gamma)$ is given by*

$$(5.1) \quad P(L, \gamma) = \frac{2^{-M(\gamma)}}{|\text{Sym}(\gamma)|} \int_0^L \int_{\sum_{i=1}^k a_i x_i = T} V_{g,n}(\Gamma, \mathbf{x}) \mathbf{x} dx dT,$$

where $\mathbf{x} = (x_1, \dots, x_k)$, and $\Gamma = (\gamma_1, \dots, \gamma_k)$.

Note that even though $V_{g,n}(\Gamma, \mathbf{x})$ depends on the choice of $\Gamma = (\gamma_1, \dots, \gamma_k)$, the right-hand side of (5.1) only depends on γ . Using Theorem 4.2, we get:

COROLLARY 5.2. *For any multi-curve γ , $P(L, \gamma)$ is a polynomial of degree $6g-6+2n$ in L . If γ is a rational multi-curve, then $c(\gamma)$, the leading coefficient of this polynomial, is a positive rational number.*

Notation. Define $c(\gamma)$ by

$$(5.2) \quad c(\gamma) = \lim_{L \rightarrow \infty} \frac{P(L, \gamma)}{L^{6g-6+2n}}.$$

By Corollary 5.2, $c(\gamma)$ is the coefficient of $L^{6g-6+2n}$ in $P(L, \gamma)$. Moreover, if γ is a rational multi-curve, then by Theorem 4.2, $c(\gamma) \in \mathbb{Q}_{>0}$.

Let $\Gamma = (\gamma_1, \dots, \gamma_k)$. Recall that by Theorem 4.2,

$$\text{Vol}_{wp}(\mathcal{M}_{g,n}(\Gamma, \mathbf{x}))$$

is a polynomial of degree $6g - 6 + 2n - 2k$ in x_1, \dots, x_k (see equation (4.1)). Let $(2s_1, \dots, 2s_k)_\Gamma \in \mathbb{Q}_{>0}$ denote the coefficient of $x_1^{2s_1} \cdots x_k^{2s_k}$ in this polynomial, and

$$(5.3) \quad b_\Gamma(2s_1, \dots, 2s_k) = (2s_1, \dots, 2s_k)_\Gamma \frac{\prod_{i=1}^k (2s_i + 1)!}{(6g - 6 + 2n)!}.$$

Also, as before

$$M(\gamma) = |\{i | \gamma_i \text{ separates off a one-handle from } S_{g,n}\}|.$$

Let

$$(5.4) \quad \mathcal{S}_{g,n} = \{\eta | \eta \text{ is a union of simple closed curves on } S_{g,n}\} / \text{Mod}_{g,n}.$$

Note that $|\mathcal{S}_{g,n}| < \infty$. An element $\eta \in \mathcal{S}_{g,n}$ can be written as $\eta = \eta_1 \cup \cdots \cup \eta_k$ where η_i 's are disjoint nonhomotopic, nonperipheral simple closed curves on $S_{g,n}$. Then $\hat{\eta} = \sum_{i=1}^k \eta_i$ defines an integral multi-curve.

Calculation of $c(\gamma)$ and $b_{g,n}$. Now we can explicitly calculate the value of the integral of the function B over $\mathcal{M}_{g,n}$.

THEOREM 5.3. *In terms of the above notation, we have:*

- (1) *The frequency $c(\gamma)$ of a multi-curve $\gamma = \sum_{i=1}^k a_i \gamma_i$ is equal to*

$$(5.5) \quad c(\gamma) = \frac{2^{-M(\gamma)}}{|\text{Sym}(\gamma)|} \times \sum_{|\mathbf{s}|=3g-3+n-k} \frac{b_\Gamma(2s_1, \dots, 2s_k)}{a_1^{2s_1+2} \cdots a_k^{2s_k+2}}.$$

Here $\Gamma = (\gamma_1, \dots, \gamma_k)$, $\mathbf{s} = (s_1, \dots, s_k) \in \mathbb{Z}_+$, and $|\mathbf{s}| = \sum_{i=1}^k s_i$.

- (2) *We have*

$$b_{g,n} = \sum_{\eta \in \mathcal{S}_{g,n}} B_\eta,$$

where for $\eta = \bigcup_{i=1}^k \eta_i$,

$$B_\eta = \frac{2^{-M(\hat{\eta})}}{|\text{Sym}(\hat{\eta})|} \times \sum_{|\mathbf{s}|=3g-3+n-k} b_{\hat{\eta}}(2s_1, \dots, 2s_k) \prod_{i=1}^k \zeta(2s_i + 2),$$

and $\tilde{\eta} = (\eta_1, \dots, \eta_k)$.

Proof. Part 1. To prove equation (5.5), note that given $a_1, \dots, a_k \in \mathbb{R}_+$, and $s_1, \dots, s_k \in \mathbb{Z}_+$, we have

$$\begin{aligned} \int_{a_1 x_1 + \dots + a_k x_k = T} x_1^{2s_1+1} \dots x_k^{2s_k+1} dx_1 \dots dx_k \\ = \frac{(2s_1+1)! \dots (2s_k+1)! \cdot T^{2|\mathbf{s}|+2k-1}}{a_1^{2s_1+2} \dots a_k^{2s_k+2} \cdot (2|\mathbf{s}|+2k-1)!}. \end{aligned}$$

Now the result follows from Theorem 4.2, (5.3), and Proposition 5.1.

Part 2. As a result of Proposition 3.1,

$$b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X) dX = \int_{\mathcal{M}_{g,n}} \lim_{L \rightarrow \infty} \frac{b_X(L)}{L^{6g-6+2n}} dX.$$

On the other hand, for any $X \in \mathcal{T}_{g,n}$,

$$b_X(L) = \sum_{\eta \in \mathcal{S}_{g,n}} \tilde{s}_X(L, \eta),$$

where for $\eta = \eta_1 \cup \dots \cup \eta_k \in \mathcal{S}_{g,n}$

$$\tilde{s}_X(L, \eta) = \sum_{\gamma = \sum a_i \eta_i \in \mathcal{ML}_{g,n}(\mathbb{Z})} s_X(L, \gamma).$$

Given $\tilde{\eta} = (\eta_1, \dots, \eta_k)$, and $\mathbf{a} \in \mathbb{N}^k$, let $\mathbf{a} \cdot \tilde{\eta} = \sum_{i=1}^k a_i \cdot \eta_i \in \mathcal{ML}_{g,n}(\mathbb{Z})$. It is easy to check that $\text{Sym}(\mathbf{a} \cdot \tilde{\eta}) \subset \text{Sym}(\hat{\eta})$, and

$$|\{\mathbf{a}_1 \in \mathbb{N}^k \mid \exists g \in \text{Mod}_{g,n} \mathbf{a}_1 \cdot \tilde{\eta} = g(\mathbf{a} \cdot \tilde{\eta})\}| = \frac{|\text{Sym}(\hat{\eta})|}{|\text{Sym}(\mathbf{a} \cdot \tilde{\eta})|}.$$

Therefore, we have

$$\tilde{s}_X(L, \eta) = \sum_{\mathbf{a} \in \mathbb{N}^k} \frac{|\text{Sym}(\mathbf{a} \cdot \tilde{\eta})|}{|\text{Sym}(\hat{\eta})|} s_X(L, \mathbf{a} \cdot \tilde{\eta}).$$

Hence,

$$b_{g,n} = \sum_{\eta \in \mathcal{S}_{g,n}} \int_{\mathcal{M}_{g,n}} \lim_{L \rightarrow \infty} \frac{\tilde{s}_X(L, \eta)}{L^{6g-6+2n}} dX.$$

Now (3.20) allows us to use Lebesgue's dominated convergence theorem. As a result, we get

$$\begin{aligned} \int_{\mathcal{M}_{g,n}} \lim_{L \rightarrow \infty} \frac{\tilde{s}_X(L, \eta)}{L^{6g-6+2n}} dX &= \sum_{\mathbf{a} \in \mathbb{N}^k} \frac{|\text{Sym}(\mathbf{a} \cdot \tilde{\eta})|}{|\text{Sym}(\hat{\eta})|} \lim_{L \rightarrow \infty} \frac{P(L, \mathbf{a} \cdot \tilde{\eta})}{L^{6g-6+2n}} \\ &= \sum_{\mathbf{a} \in \mathbb{N}^k} \frac{|\text{Sym}(\mathbf{a} \cdot \tilde{\eta})|}{|\text{Sym}(\hat{\eta})|} c(\mathbf{a} \cdot \tilde{\eta}). \end{aligned}$$

Now the result follows from (5.5). See [Mirz1] for more details. \square

Note that by Theorem 4.2, for $|\mathbf{s}| = 3g - 3 - k$, $b_{\bar{\eta}}(2s_1, \dots, 2s_k) \in \mathbb{Q}_{>0}$. On the other hand, $\zeta(2i) \in \pi^{2i} \cdot \mathbb{Q}$. Hence we get:

COROLLARY 5.4. *For any g, n , with $2g - 2 + n > 0$, $b_{g,n}$ is a rational multiple of $\pi^{6g-6+2n}$.*

In the simplest case when $g = n = 1$, $|\mathcal{S}_{1,1}| = 1$, and

$$b_{1,1} = \zeta(2) = \frac{\pi^2}{6}.$$

6. Counting different types of simple closed curves

In this section we use the ergodicity of the action of the mapping class group on the space of measured laminations to obtain the following results:

THEOREM 6.1. *For any rational multi-curve γ and $X \in \mathcal{T}_{g,n}$,*

$$s_X(L, \gamma) \sim \frac{B(X)}{b_{g,n}} c(\gamma) L^{6g-6+2n},$$

as $L \rightarrow \infty$.

Note that $b_{g,n}$ and $c(\gamma)$ (defined by equations (3.4) and (5.2)) are both constants independent of X and L ; see Theorem 5.3. Therefore, we get:

COROLLARY 6.2. *For any $X \in \mathcal{T}_{g,n}$, as $L \rightarrow \infty$*

$$\frac{s_X(L, \gamma_1)}{s_X(L, \gamma_2)} \rightarrow \frac{c(\gamma_1)}{c(\gamma_2)}.$$

Since there are only finitely many isotopy classes of simple closed curves on $S_{g,n}$ up to the action of the mapping class group, the following result is immediate:

COROLLARY 6.3. *The number of simple closed geodesics of length $\leq L$ on $X \in \mathcal{M}_{g,n}$ has the asymptotic behavior*

$$s_X(L) \sim n(X) L^{6g-6+2n}$$

as $L \rightarrow \infty$, where $n : \mathcal{M}_{g,n} \rightarrow \mathbb{R}_+$ is proper and continuous.

Discrete measures on $\mathcal{ML}_{g,n}$. Any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, defines a sequence of discrete measures on $\mathcal{ML}_{g,n}$, $\{\mu_{T,\gamma}\}$, such that for any open set $U \subset \mathcal{ML}_{g,n}$

$$\mu_{T,\gamma}(U) = \frac{\#(T \cdot U \cap \text{Mod}_{g,n} \cdot \gamma)}{T^{6g-6+2n}}.$$

There is a close relation between the asymptotic behavior of this sequence of measures and counting different types of simple closed geodesics. First, we prove the following result on the asymptotic behavior of $\mu_{T,\gamma}$ as $T \rightarrow \infty$:

THEOREM 6.4. *For any rational multi-curve γ , as $T \rightarrow \infty$*

$$(6.1) \quad \mu_{T,\gamma} \xrightarrow{w^*} \frac{c(\gamma)}{b_{g,n}} \cdot \mu_{\text{Th}},$$

where μ_{Th} is the Thurston volume form on $\mathcal{ML}_{g,n}$.

Remarks. 1. Let Y be a closed orientable surface of genus g with bounded negative curvature. Then each homotopy class of closed curves contains a unique closed geodesic. Consider the space $\mathcal{ML}(Y)$ of measured geodesic laminations on Y and let $\ell_\gamma(Y)$ be the geodesic length of γ on Y . The length function extends to a continuous function on $\mathcal{ML}(Y)$. Moreover, $\mathcal{ML}(Y) \cong \mathcal{ML}_g$. Since Theorem 6.4 is independent of the Riemannian metric on the surface, both Theorem 6.1 and Corollary 6.2 hold for Y .

2. Using the same method, one can check that the results of Theorem 6.1 and Corollary 6.2 also hold for any hyperbolic surface $X \in \mathcal{M}_{g,n}(L_1, \dots, L_n)$ with geodesic boundary components.

Proof of Theorem 6.4. It is enough to prove the result for integral multi-curves. The argument has three main steps:

Step 1. Given $X_0 \in \mathcal{T}_{g,n}$, by Proposition 3.6, and (3.20) we have

$$\mu_{T,\gamma}(L \cdot B_{X_0}) = \frac{s_{X_0}(L, T, \gamma)}{T^{6g-6+2n}} \leq C(X_0, L),$$

where $C(X_0, L)$ is a constant depending only on X_0 and L . In particular it is independent of T . On the other hand, given a compact subset $K \subset \mathcal{ML}_{g,n}$, there exists L such that $K \subset L \cdot B_{X_0}$. As a result, we have

$$\limsup_{T \rightarrow \infty} \mu_{T,\gamma}(K) < \infty.$$

Therefore, any subsequence of $\{\mu_{T,\gamma}\}$ contains a weakly-convergent subsequence.

Step 2. We show that any weak limit of the sequence $\{\mu_{T,\gamma}\}$ is a multiple of the measure μ_{Th} . Assume that

$$(6.2) \quad \mu_{T_i,\gamma} \rightarrow \nu_J$$

as $T_i \in J \rightarrow \infty$. We show that ν_J belongs to the Lebesgue measure class; that is for any $V \subset \mathcal{ML}_{g,n}$ with $\mu_{\text{Th}}(V) = 0$, we have $\nu_J(V) = 0$. Let $U \subset \mathcal{ML}_{g,n}$ be a convex open set in a train track chart. Using Proposition 3.1, we have:

$$(6.3) \quad \nu_J(U) \leq \liminf_{i \rightarrow \infty} \mu_{T_i,\gamma}(U) \leq \lim_{i \rightarrow \infty} \frac{b(T_i, U)}{T_i^{6g-6+2n}} = \mu_{\text{Th}}(U).$$

Since we can approximate V with open subsets of $\mathcal{ML}_{g,n}$ satisfying (6.3), the measure ν_J belongs to the Lebesgue measure class. Then the ergodicity of the

action of the mapping class group on $\mathcal{ML}_{g,n}$ (Theorem 2.2) implies that

$$\nu_J = k_J \mu_{\text{Th}}.$$

Step 3. Finally, we show $\nu_J = k \cdot \mu_{\text{Th}}$, where k is independent of the subsequence J . Note that for any $X \in \mathcal{T}_{g,n}$, $s_X(T, \gamma) = \mu_{T,\gamma}(B_X)$. Equation (6.2) implies:

$$(6.4) \quad \frac{s_X(T_i, \gamma)}{T_i^{6g-6+2n}} \rightarrow k_J \cdot B(X)$$

as $i \rightarrow \infty$. Now we integrate both sides of (6.4) over $\mathcal{M}_{g,n}$. By using (3.20), Corollary 5.2, and (5.2), we get

$$\begin{aligned} k_J \cdot b_{g,n} &= k_J \cdot \int_{\mathcal{M}_{g,n}} B(X) dX = \int_{\mathcal{M}_{g,n}} \lim_{i \rightarrow \infty} \frac{s_X(T_i, \gamma)}{T_i^{6g-6+2n}} dX \\ &= \lim_{i \rightarrow \infty} \int_{\mathcal{M}_{g,n}} \frac{s_X(T_i, \gamma)}{T_i^{6g-6+2n}} dX = \lim_{i \rightarrow \infty} \frac{P(T_i, \gamma)}{T_i^{6g-6+2n}} = c(\gamma). \end{aligned}$$

On the other hand, by Theorem 3.3, $b_{g,n} < \infty$. Therefore,

$$k_J = \frac{c(\gamma)}{b_{g,n}}$$

is independent of J , and hence

$$\mu_{T,\gamma} \rightarrow \frac{c(\gamma)}{b_{g,n}} \cdot \mu_{\text{Th}}. \quad \square$$

Proof of Theorem 6.1. Since ∂B_X has measure zero, equation (6.1) implies that

$$\mu_{T,\gamma}(B_X) \rightarrow \frac{c(\gamma)}{b_{g,n}} \cdot \mu_{\text{Th}}(B_X).$$

Now the result is immediate since

$$\mu_{T,\gamma}(B_X) = \frac{\#(L \cdot B_X \cap \text{Mod}_{g,n} \cdot \gamma)}{L^{6g-6+2n}} = \frac{s_X(L, \gamma)}{L^{6g-6+2n}}. \quad \square$$

Examples. Here we explicitly calculate the frequencies of different types of simple closed curves in some simple cases.

- (1) First, we consider the case of $g = 2$. Then by equation (4.6), for any nonseparating simple closed curve α_1

$$\text{Vol}(\mathcal{M}(S_2(\alpha_1), \ell_{\alpha_1} = x)) = V_{1,2}(x, x) = \frac{1}{48}(2\pi^2 + x^2)(6\pi^2 + x^2).$$

Since $N(\alpha_1) = 2$ and $M(\alpha_1) = 0$, the leading coefficient of $P(L, \alpha_1)$ is equal to $\frac{1}{2 \times 48 \times 6}$. Equation (5.5) implies that

$$c(\alpha_1) = \frac{1}{2 \times 48 \times 6}.$$

Similarly, by equation (4.5), for any separating simple closed curve α_2 ,

$$\text{Vol}(\mathcal{M}(S_2(\alpha_2), \ell_{\alpha_2} = x)) = V_{1,1}(x) \times V_{1,1}(x) = \left(\frac{x^2}{24} + \frac{\pi^2}{6}\right)^2.$$

In this case, $M(\alpha_2) = 1$, and $N(\alpha_2) = 2$. By equation (5.5)

$$c(\alpha_2) = \frac{1}{24 \times 24 \times 6}.$$

Hence, Corollary 6.2 implies that

$$\lim_{L \rightarrow \infty} \frac{s_X(L, \alpha_1)}{s_X(L, \alpha_2)} = \frac{c(\alpha_1)}{c(\alpha_2)} = 6.$$

Roughly speaking, on a surface of genus 2, a long, random connected, simple, closed geodesic is separating with probability $\frac{1}{7}$.

- (2) Let β_i be a connected simple closed curve on $S_{0,n}$ satisfying

$$S_{0,n}(\beta_i) \cong S_{0,i+1} \cup S_{0,n-i+1}.$$

Then as in Section 4, the coefficient of L_1^{2n-4} in $V_{0,n+1}(L_1, \dots, L_n, L_{n+1})$ equals $\frac{1}{2^{n-2}(n-2)!}$. In this case, $N(\beta_i) = |\text{Sym}(\beta_i)| = 1$. Hence, by (5.5), we have

$$c(\beta_i) = \frac{1}{2^{n-4}(i-2)! (n-i-2)! (2n-6)}.$$

Hence, given $X \in \mathcal{T}_{0,n}$

$$\frac{s_X(L, \beta_i)}{s_X(L, \beta_j)} \rightarrow \frac{\binom{n-4}{i-2}}{\binom{n-4}{j-2}}$$

as $L \rightarrow \infty$.

- (3) Let γ_i be a separating connected simple closed curve on a surface of genus g that cuts the surface into two parts of genus i and $g-i$. For simplicity, we assume that $g > 2i > 2$. In this case, $N(\gamma_i) = 1$ and $M(\gamma_i) = 0$. Also,

$$\text{Vol}(\mathcal{M}(S_g(\gamma_i), \ell_{\gamma_i} = x)) = V_{i,1}(x) \times V_{g-i,1}(x).$$

On the other hand, by (4.7) the leading term of the polynomial $V_{g,1}(L)$ is equal to

$$\frac{L^{6g-4}}{(3g-2)! g! 24g 2^{3g-2}}.$$

Now since $|\text{Sym}(\gamma_i)| = 1$, by (5.5) the frequency of a simple closed curve of type γ_i is equal to

$$c(\gamma_i) = \frac{1}{2^{3g-2} 24^g i!(g-i)! (3g-2)!(3(g-i)-2)!(6g-6)}.$$

For $X \in \mathcal{T}_g$, we have

$$\lim_{L \rightarrow \infty} \frac{s_X(L, \gamma_i)}{b_X(L)} = \frac{c(\gamma_i)}{b_g}.$$

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(Received April 11, 2004)