TAYLOR SERIES, POWER SERIES

The following represents an (incomplete) collection of things that we covered on the subject of Taylor series and power series.

Warning. Be prepared to prove any of these things during the exam. Things you should memorize:

- the formula of the Taylor series of a given function $f(x)$
- geometric series (i.e. the Taylor expansion of $\frac{1}{1-x}$)
- the Taylor expansions of the functions $e^x, \sin x, \cos x, \ln(1+x)$ and range of validity.
- the relation $f(x) = P_n(x) + R_n(x)$ and Lagrange formula for $R_n(x)$

You should also understand the actual proofs of the Taylor series expansions enumerated above.

1. Taylor Series

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \Leftrightarrow R_n(x) \to 0}$$

In other words, the Taylor expansion takes place only at those values of x for which $R_n(x) \to 0$. If you want to prove from scratch a Taylor series expansion (as we did in the case of e^x, $\cos(x)$, $\sin(x)$ and $\ln(1+x)$) you need to show $R_n(x) \to 0$, and one usually proves this by

- employing Lagrange formula
- estimating $R_n(x)$ (get rid of c)

See the slides of Nov 24 lecture.

Exponential function.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}$$

Understand why this gives, among others, the following formula

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} = \sqrt{e}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$$

Cosine. Know how to estimate the remainder in this case to prove

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}$$

In particular this gives (set $x = 1/2$)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n)!} = \cos 0.5$$

Sine.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \forall x \in \mathbb{R}$$

Hyperbolic Cosine. From

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

one derives

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

If we set $x = 1$ we obtain form example

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{e + 1/e}{2}$$

as opposed to

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \quad \text{and}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1)$$

Geometric series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{holds only for } -1 < x < 1$$

Logarithm. Start with the *fake* geometric series

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

Integrate (apply the nice theorem on power series):

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad x \in (-1, 1)$$

If we want to justify this identity in the range $S = (-1, 1]$, we need to appeal to Abel's theorem. In particular, for $x = 1$ we get

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$$

Equivalently, we have the following power expansion in $x - 1$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \quad \text{valid for } 0 < x \leq 2$$

Approximate Computations. Starting with $f(x) = P_n(x) + R_n(x)$ for a given $f(x)$, one can presumably find n such that $R_n(x)$ is smaller than the desired degree of accuracy (estimate $R_n(x)$!) in order to know that $P_n(x)$ approximates $f(x)$ well enough. Examples: computing $e^{0.2}, 1/e, 1/\sqrt{e}, \sin 0.5$ to three decimal places (i.e. approximate the function by an appropriate Taylor polynomial, etc.)

Example not done in class: compute $\ln(1.4)$ to 2 decimal places by approximating the function $\ln(1+x)$ by Taylor polynomial.

2. Power Series

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, one can determine:

- The radius of convergence $R \geq 0$ with the formula
$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$
- The domain of convergence S which consists of all the numbers x for which the series $\sum a_n x^n$ is convergent: the open interval $(-R, R)$ is for sure included, and then we only have to check the endpoints $x = \pm R$ separately. The power series is divergent outside this range, i.e. for $|x| > R$.

Example. Find the radius of convergence R and the domain of convergence S for each of the following power series:
$$\sum_{n=0}^{\infty} x^n, \sum_{n=1}^{\infty} \frac{x^n}{n}, \sum_{n=0}^{\infty} \frac{x^n}{n^n}, \sum_{n=0}^{\infty} n^n x^n, \sum_{n=0}^{\infty} \frac{x^n}{n!}, \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} x^{2n}$$

Hwk problem: if the series $\sum_{k=0}^{\infty} 4^n a_n$ is convergent, then $\sum_{n=0}^{\infty} a_n(-2)^n$ is also convergent. (the question reduces to understanding the shape of the domain of convergence S of the power series $\sum a_n x^n$)

2.1. **The "Nice Theorem".**

The nice theorem allows us to differentiate/integrate a Taylor series expansion inside the radius of convergence, in order to obtain new identities (Taylor series expansions). If
$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-R, R)$$
then
$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{(differentiate)}$$
$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{(integrate)}$$
The coefficients of the power series obtained through differentiation are na_n.
The coefficients of the power series obtained through integration are $\frac{a_n}{n+1}$.
The above two identities are valid whenever $x \in (-R, R)$.

2.1.1. *Side Remark.*

Why is this thing called a theorem? To give a simple example, let
$$g(x) = 2x + x^3 + x^4$$
It is easy to differentiate and integrate $g(x)$:
$$g'(x) = 2 + 3x^2 + 4x^3$$
$$\int_0^x g(t) = x^2 + \frac{x^4}{4} + \frac{x^5}{5}$$
Now, the nice theorem says that i can do the same thing even if $g(x)$ was not a polynomial (finite sum of powers), but a power series (infinite sum of powers !). However when dealing with a power series we are facing the issue of convergence, and the process of

differentiation (integration) *term-by-term* needs justification. The nice theorem takes care of that.
Also, think of the nice theorem as allowing as to obtain new identities from old ones.

Example 1. Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, x \in (-1,1)$. Differentiate/integrate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \quad \text{(differentiation)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \int_0^x \frac{dx}{1-x} = -\ln(1-x) \quad \text{(integration)}$$

and these identities are valid for $x \in (-1,1)$.

- We can multiply both sides of the first one to obtain $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, x \in (-1,1)$. For example, taking $x = -\frac{1}{4}$ gives $\sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n} = -\frac{4}{25}$.
- We can take $x = -1$ in the second identity (Abel's theorem) to obtain $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(2)$. In other words, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

Example 2. Start with

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^2 n, \quad x \in (-1,1)$$

Integrate:

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in (-1,1)$$

Extend this identity to $x = 1$ (ok by Abel's theorem):

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Example 3. Put $-x^2$ in the Taylor expansion of the exponential function to obtain the identity

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}, \quad x \in \mathbb{R}$$

Integrate:

$$\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}, \quad x \in \mathbb{R}$$

For $x = 1$ we get

$$\int_0^1 e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}$$

This allows us to compute the integral on the left-hand side (otherwise hard to figure out) to desired accuracy, as in the Example 7 on page 695 of the textbook. **Example 4.** Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+2)}$$

Start with the power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Multiply both sides by x

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Integrate

$$\int_0^x te^t dt = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!(n+2)}$$

Evaluate integral on the left by integration by parts

$$\int_0^x te^t dt = te^t\big|_0^x - \int_0^x e^t = xe^x - e^x + 1$$

Therefore

$$xe^x - e^x + 1 = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!(n+2)}$$

Set $x = 1$ to obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+2)} = 1$$

Set $x = 2$ to obtain

$$\sum_{n=0}^{\infty} \frac{2^n}{n!(n+2)} = \frac{e^2+1}{4}$$

3. Relation between Taylor series and power series

A power series = Taylor series of its sum

In other words, every time you obtain an identity

$$\sum_{n=0}^{\infty} a_n x^n = \text{(something)}$$

then the power series on the left-hand side **must** be the Taylor series of that something on the right-hand side.

Example 1. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

holds for any $x \in \mathbb{R}$ (we proved this statement by means of the remainder formula). Therefore there is no harm in considering x^2 instead of x in the above "formula", only to obtain

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Looking at the boxed principle (above), we can now see that what we have in fact here is the Taylor expansion of the function e^{x^2} which we obtained almost for free. (Convince yourselves that it is not so trivial to construct the Taylor series of the function $f(x) = e^{x^2}$ from scratch. Not to mention that to justify the Taylor series expansion one usually needs to show that $R_n(x) \to 0$, and in the case of $f(x) = e^{x^2}$ Lagrange's formula for the remainder is really complicated.

Example 2. A simpler example is the identity

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

which is valid for $x \in (-1, 1)1$ and obtained from the geometric series (simply by replacing x by $-x^2$). In view of the boxed principle above, this has to be the Taylor expansion of the function $g(x) = \frac{1}{1+x^2}$. Hence, if you need to compute $g^{(10)}(0)$ simply identify the coefficient of x^{10}:

$$\frac{g^{(10)}(0)}{10!} x^{10} = (-1)^n x^{2n} \quad \text{(for some } n) \Rightarrow 2n = 10, n = 5$$

Therefore

$$\frac{g^{(10)}(0)}{10!} = (-1)^5 = -1 \Rightarrow g^{(10)}(0) = -10!$$