Logical statements: Summary of definitions, notations, and terminology

Logical operations

The basic logical operations are $\wedge$ ("and"), $\vee$ ("or"), $\neg$ (negation), and $\Rightarrow$ ("implies"), and $\iff$ ("equivalent"); they are defined in the following "truth tables":

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Implication, converse, and contrapositive: The most important, but also the most misused and misunderstood, of these logical operations is the logical implication ($P \Rightarrow Q$), and the related operations of the **converse** ($Q \Rightarrow P$), the **contrapositive** ($\neg Q \Rightarrow \neg P$), and the negated implication ($\neg(P \Rightarrow Q)$). The following table (which follows from the above truth table for $P \Rightarrow Q$) shows the differences between these operations:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$	$\neg(P \Rightarrow Q)$
		Implication	Converse	Contrapositive	Negated implication
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	F

Notes:

- The best way to remember the correct logical interpretation of an implication is as follows:

  An implication $P \Rightarrow Q$ is true in all cases except when the assumption ("premise") P is true and the conclusion Q is false; it is false if and only if both P is true and Q is false.

  In particular, this shows that if P is false, then the implication $P \Rightarrow Q$ is true, regardless of the truth value of Q. **In other words, from a false statement one can derive anything.** This is the reason why a "proof" that starts out with the statement to be proved, say P , and derives from this a true statement, say Q, is not valid: The truth of Q and of the implication $P \Rightarrow Q$ does not say anything about the truth value of P; P may be false, while $P \Rightarrow Q$ and Q are both true.

- **The negation of an implication $P \Rightarrow Q$ is not equivalent to another implication involving P and Q.** This is easy to see from the above truth table: $\neg(P \Rightarrow Q)$ is true in exactly one of the four cases and false in three cases. By contrast, an implication with P or $\neg P$ on one side and Q or $\neg Q$ on the other side (e.g., $\neg P \Rightarrow \neg Q$, $\neg Q \Rightarrow \neg P$, $\neg Q \Rightarrow P$, etc.) is always is true in three cases, and false in one case, so it can never match the truth table of $\neg(P \Rightarrow Q)$.

- **Note that the implication $P \Rightarrow Q$ and its contrapositive $\neg Q \Rightarrow \neg P$ have the same truth tables, and hence are logically equivalent.** This is the reason why "proof by contraposition" is a valid method of proof.

Quantifiers

- **Existential quantifier:** $\exists$ ("there exists").
- **Universal quantifier:** $\forall$ ("for all").

Rules for negation

- $\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$ (De Morgan's Law, I)
- $\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$ (De Morgan's Law, II)
- $\neg(\forall x \in S)(P(x)) \iff (\exists x \in S)(\neg P(x))$
- $\neg(\exists x \in S)(P(x)) \iff (\forall x \in S)(\neg P(x))$

English equivalences of logical operations and notations

Here are some English words and phrases that translate into logical implications, equivalences. See p. 33 and p. 29 of the text for similar tables. **Be sure to familiarize yourself with these words and phrases and with their logical interpretation.**

Phrases expressing $P \Rightarrow Q$ ("P implies Q"):

- "**If** P, **then** Q"
- "Q **follows from** P":
- "Q is true **if** P is true"
- "Q is true **whenever** P is true"
- "P is true **only if** Q is true"
- "P is a **sufficient condition** for Q"
- "Q is a **necessary condition** for P":

Phrases expressing $P \Longleftrightarrow Q$ ("P is equvialent to Q"):

- "P holds **if and only if** Q holds"
- "P is a **necessary and sufficient condition** for Q"

Phrases expressing existential quantifiers ($\exists x \dots$)

- "There **exists** $x \in S$ such that ..."
- "... holds **for some** $x \in S$"
- "**For some** $x \in S$ we have ..."

Phrases expressing universal quantifiers ($\forall x \dots$)

- "**For all** $x \in S$, we have ...
- "**For any** $x \in S$, ..."
- "**Every** $x \in S$ satisfies ..."
- "**Given** $x \in S$, we have ..."
- "**If** $x \in S$, then ..."
- "**Whenever** $x \in S$, then ..."

Further resources

This material is covered in the D'Angelo/West text on pp. 27–34; you'll find there many examples illustrating the proper use of logical statements, and some excellent general remarks and advice. Be sure to read this section.

Final Words

The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age. —*Bertrand Russell*

Logic, like whiskey, loses its beneficial effect when taken in too large quantities. —*Lord Dunsany*

Logic is the art of going wrong with confidence. —*Morris Kline*