The Beautiful Mathematical Explorations of Maryam Mirzakhani

After her untimely death, Maryam Mirzakhani's life is best remembered through her work.

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Maryam Mirzakhani in 2014.

The meanderings of sadness that news of Maryam Mirzakhani's death brought compelled me to read every article I could find about her, and when I could not find more articles, I started to read the comments of the readers. Many of them wrote that they saw her as "unrelatable and incomprehensible," given that they were "math challenged." However, Maryam was the opposite of unrelatable. She reminded us, in words and actions, that mathematical ideas can be understood if one puts enough persistence into the task.

She was not the star of her lectures; the only stars were the mathematical ideas. She spoke calmly and clearly and radiated deep enjoyment of the process.

Maryam had the ability to quickly perceive the appropriate wavelength of her interlocutor and speak accordingly, a rare quality in a mathematician. She listened with attention and was very generous with her time. Behind her kind serenity one could perceive a steely tenacity and a deep well of ideas and, of course, a passion for math and an incessant quest for the fantastic “aha” moment. This moment often took years for her, because she worked on profound questions.

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After one of her lectures, we walked together chatting. Suddenly, the voice of a child came from an adjacent room and Maryam exclaimed, “Anahita!” The voice belonged to her daughter. Maryam’s exclamation lit up the room. She sounded totally different than she had during the lecture. Her entire humanity was in the exclamation.

Maryam’s work connected ideas from different areas of mathematics. Part of this work consisted of counting closed curves on surfaces. A mathematical surface is, roughly speaking, the outer layer of a solid object. In topology, surfaces are studied up to the point of deformation — they’re allowed to bend and stretch but not to tear — thus the old joke that a topologist can’t tell a coffee cup from a doughnut. Surfaces can also have holes and edges. In this way, a disk and a cylinder can both be thought of as surfaces.

Closed curves on a surface are like extremely thin rubber bands wrapping around it. Curves are also studied up to deformation on the surface. For instance, in a cylinder, every closed curve can be deformed into another curve that goes around the cylinder once, twice, three times or more — or no times around at all, which means it can be deformed to a point.

Surfaces can also be studied from a geometric point of view. In this case, a stretchable surface can be “enriched” with a metric, which is a way to measure distances and angles.

Mathematicians call the surface determined by the outer layer of a doughnut a torus. One way to define a metric on a torus consists of imagining that we are minuscule beings living on the variegated surface of the doughnut, where we measure distances and angles. The landscape we see may well change when we move from place to place.

Jan Vondrák

There is a different way to conceive of a metric on a torus: Imagine that we are still minuscule beings, but now we live on something like a square. The “something like” refers to the following special property: If we walk in a straight line and reach one of the sides, say the top one, if we want to continue walking straight we must “re-enter” from the bottom side, in the same direction and at the point directly below the point where we exited. Some computer games, such as Pac-Man, take place in this something-like-a-square planet.

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It takes some thought to convince ourselves that this planet is also a torus. The metric in this case is not the one that we "see" when we live on the surface of any doughnut, but the something-like-a-square idea shows how we can define it. An interesting property of the Pac-Man metric is that at every point, the landscape around us looks exactly the same. It is perfectly flat.

If the topology of a surface is "complicated enough" (that is, if it is not a sphere, a torus, a disk or a cylinder), the surface admits so-called hyperbolic metrics. In these metrics, the landscape at each point looks like a mountain pass or a Pringles potato chip. Like the Pac-Man metric on the doughnut, these metrics cannot be visualized as distances measured on the outer layer of a solid. But as we learned from the late William Thurston, after some workouts of the mind we can start seeing them with our mathematical eyes.

The set of all rubber-band-like curves on a surface that can be deformed into a given curve is called a deformation class. A remarkable property of these hyperbolic metrics is that in every deformation class, only one closed curve has the shortest possible length in the class. This shortest curve is called a geodesic.

Part of Maryam's work involved counting these geodesic curves on surfaces with a hyperbolic metric. Before her, mathematicians knew that a surface has a finite number of geodesics under a certain length. Moreover, mathematicians knew that the number of geodesics shorter than a given length grows exponentially with the bound on this length. This exponential growth limits our ability to make exhaustive computations.

One of the questions Maryam answered is about the growth in the number of geodesics with no crossings. She divided the set of non-crossing geodesics into types. Two non-crossing geodesics are the same type if, in some sense, they “sit” on the surface in an equivalent way.

She proved that on a hyperbolic surface, the number of non-crossing geodesics of a given type grows like a polynomial (not as an exponential) as the bound on the length increases. This helps with exhaustive computations. Maryam provided explicit and meaningful formulas for the coefficients of this polynomial. (The reader might draw closed curves on a piece of paper with three holes in it to begin to see the possible complexity of these curves.) Furthermore, this work also led to an original proof of a celebrated conjecture of Edward Witten from string theory.

A bit more than a decade ago when the mathematical world started hearing about Maryam Mirzakhani, it was hard not to mispronounce her then-unfamiliar name. The strength and beauty of her work made us learn it. It is heartbreaking not to have Maryam among us any longer. It is also hard to believe: The intensity of her mind made me feel that she would be shielded from death.

Maybe the best way for mathematicians to honor her memory is for us to continue to develop her fine results.