Chapter 11

Taylor Series

In Chapter 10 we explored series of constant terms $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$. In this chapter we next analyze series with variable terms, i.e., terms which are functions of a variable such as x. As we will see, perhaps the most naturally arising variable series are the *power series:*

Definition 11.0.1 *A* **power series centered at** $x = a$ *is a series of the form*

$$P(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots, \tag{11.1}$$

where a, a_0, a_1, a_2, *etc., are constants.*

Many familiar functions are in fact equal to infinite series of the form (11.1), at least where the series converge, and so such functions can be approximated to varying degrees by the partial sums of these series. The existence of these *polynomial* partial sums explains, for instance, how calculators and similar devices compute approximate values of these functions for a wide range of inputs. (After all, computing a polynomial's value at a given input requires only a finite number of multiplications and additions, which could even be accomplished with just paper and a pencil.) It also allows for reasonable approximations in applications where the exact equations would be too difficult to solve with the actual functions, but may be simpler with approximations. Moreover, and perhaps surprisingly, there are numerous settings where it is actually easier to deal with such a function as represented by an infinite series than by its usual representation.

We will also see that there are functions which arise in applications and are easily given in the form (11.1), but which are not equal to anything from our usual catalog of known functions. Thus by including for consideration all functions which can be expressed by power series, whether or not they have other conventional representations, we greatly expand our function catalog.

The following are general questions which arise in studying series, along with a preview of the answers.

(1) *If we are given a function* $f(x)$*, how do we produce a power series* (11.1) *which also represents the function* $f(x)$*?*

—In answering this, we will first look at *Taylor Polynomials*,[1] which coincide with the partial sums of power series expansions when a function possesses such an expansion.

[1] Named for English mathematician Brook Taylor (1685–1731). As is often the case with early calculus discoveries, there is some controversy over whom to give credit, since hints of the results were often present earlier, and better statements usually arise later. Nonetheless, apparently after a paper by in 1786 by Swiss mathematician Simon Antoine Jean Lhuilier (1750–1840) referred to "Taylor series," the series and polynomials bear Taylor's name. Lhuilier was also responsible for the "lim" notation in limits, as well as left- and right-hand limits, and many other important aspects of our modern notation.

(2) *Can we always do that?*

—No, and we will eventually look at cases where functions have pathologies which do not allow us to represent them with power series, for example functions which must be defined piece-wise or have discontinuities where we wish to approximate them. However, the functions which do allow for power series representations is vast—too vast to ignore—and it takes some effort to write a function which does not, at least on small intervals, allow for series representations.

(3) *How accurate are the partial sums of the power series as approximations of the function $f(x)$ it represents?*

—This will be approached intuitively, visually and by way of a "Remainder Theorem," which can give bounds on the error, or "remainder," when we use a partial sum to approximate the actual function. (This is Theorem 11.2.1, page 770.)

(4) *Given a power series* (11.1)*, for which values of x does it converge?*

—For this we will rely mostly, but not exclusively, upon a slightly clever application of the Ratio Test. This is perhaps to be expected since power series have a strong resemblance to geometric series.[2]

(5) *Besides approximating given functions through their partial sums, what other computational uses do power series possess?*

—This will be explored in some detail later in the chapter. In short, power series give a new context in which to explore relationships among functions, with some interesting derivative and integration applications, as well as a few "real-world" applications, particularly from physics.

11.1 Taylor Polynomials: Examples and Derivation

Taylor Polynomials are a very important theoretical and practical concept in calculus and higher mathematics. As such, the general form given below should be committed to memory, as often happens naturally as it is revisited repeatedly through examples and exercises. While we will eventually derive these polynomials from reasonable first principles at the end of this section, for now we simply define them.

Definition 11.1.1 *The* N**th order Taylor Polynomial** *for the function $f(x)$ centered at the point a, where $f(a)$, $f'(a)$, $\cdots$, $f^{(N)}(a)$ all exist, is given by*[3]

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \underbrace{\frac{f^{(n)}(a)(x-a)^n}{n!}}_{\text{“}n\text{th-order term”}} + \cdots + \frac{f^{(N)}(a)(x-a)^N}{N!}. \qquad (11.2)$$

[2]This is meant in the sense that, if a_0, a_1, etc., in (11.1) were all the same number, the series would be geometric, with ratio $r = (x - a)$.

[3]We normally do not bother to write the factors $\frac{1}{0!}$ and $\frac{1}{1!}$ in the first two terms, since $0!, 1! = 1$. We also use the convention that $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, etc.

The zeroth, first, second and third order Taylor Polynomials for a function $f(x)$ and centered at $x = a$ would be the following:

$$\begin{aligned}
P_0(x) &= f(a),\\
P_1(x) &= f(a) + f'(a)(x-a),\\
P_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!},\\
P_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}.
\end{aligned}$$

A few notes are appropriate here.

1. The Nth-order Taylor Polynomial with center $x = a$ is the sum of the $(N-1)$st-order Taylor Polynomial with the same center, and the term $\frac{1}{N!}f^{(N)}(a)(x-a)^N$, so we just add a single "term" to a Taylor Polynomial to arrive at the next-order Taylor Polynomial.

2. $P_1(x)$ is the same as the linear approximation of $f(x)$ centered at $x = a$, so it is often called "the first-order approximation of $f(x)$ at (or near) $x = a$." $P_2(x)$ is then called the quadratic, or second-order approximation, $P_3(x)$ the cubic, or third-order approximation, and so on.

Example 11.1.1 *Find $P_0(x), P_2(x), \cdots, P_5(x)$ at $x = 0$ for the function $f(x) = e^x$.*

Solution: *First note that if we construct $P_5(x)$, the first term will be $P_0(x)$, the first two will comprise $P_1(x)$, the first three terms will give $P_2(x)$, and so on.*

Anytime we need to construct a Taylor Polynomial of a function $f(x)$, We first construct the chart of the function and its relevant derivatives at the center. For this example, we construct the following chart with $a = 0$.

$$\begin{aligned}
f(x) = e^x &\implies f(0) = 1\\
f'(x) = e^x &\implies f'(0) = 1\\
f''(x) = e^x &\implies f''(0) = 1\\
f'''(x) = e^x &\implies f'''(0) = 1\\
f^{(4)}(x) = e^x &\implies f^{(4)}(0) = 1\\
f^{(5)}(x) = e^x &\implies f^{(5)}(0) = 1
\end{aligned}$$

Now, according to our definition (11.2),

$$P_5(x) = \underbrace{\underbrace{\underbrace{f(0)}_{P_0(x)} + f'(0)(x-0)}_{P_1(x)} + \frac{f''(0)(x-0)^2}{2!}}_{P_2(x),\text{ etc.}} + \frac{f'''(0)(x-0)^3}{3!} + \frac{f^{(4)}(0)(x-0)^4}{4!} + \frac{f^{(5)}(0)(x-0)^5}{5!}$$

$$= 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5.$$

From the computation above we can also write

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= 1 + x,\\
P_2(x) &= 1 + x + \frac{x^2}{2!},\\
P_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},\\
P_4(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!},\\
P_5(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.
\end{aligned}$$

The polynomials $P_0(x), \cdots, P_5(x)$ are graphed in Figure 11.1, page 752. A couple of observations from that figure are in order:

- Clearly, as we add more terms to get higher-order Taylor Polynomials, the curves tend to more closely follow the behavior of the function, at least near the center ($x = 0$ in the above example). This will be explained as we proceed.

- In all cases, the highest-order nonzero term eventually dominates the polynomial's behavior for large $|x|$. For instance, for large $|x|$, $P_5(x)$ more clearly behaves like the degree-5 polynomial it is, and thus very differently from the original function $f(x) = e^x$:

  1. $|P_5(x)| \to \infty$ as $x \to \pm\infty$, and in particular $x \to -\infty \implies P_5(x) \to -\infty$, though $e^x \to 0^+$ as $x \to -\infty$.
  2. As $x \to \infty$, $e^x - P_5(x) \to \infty$, i.e., the exponential will grow much faster than will any polynomial, including a degree-5 polynomial such as this particular $P_5(x)$.

It is natural to ask why the Taylor Polynomials $P_N(x)$ seem to give us better and better approximations of the function $f(x)$ as we increase N. The following observation gives some hint:

Theorem 11.1.1 *If $f(x)$ is N-times differentiable at $x = a$, then $P_N(x)$, as defined by* (11.2), *satisfies:*

$$\begin{aligned}
P_N(a) &= f(a)\\
P_N'(a) &= f'(a)\\
P_N''(a) &= f''(a)\\
&\vdots\\
P_N^{(N)}(x) &= f^{(N)}(a) \qquad \textit{(i.e., } P_N^{(N)}(x) \textit{ is constant)}\\
P_N^{(m)}(x) &= 0 \qquad \textit{for all } m \in \{N+1, N+2, N+3, \cdots\}
\end{aligned}$$

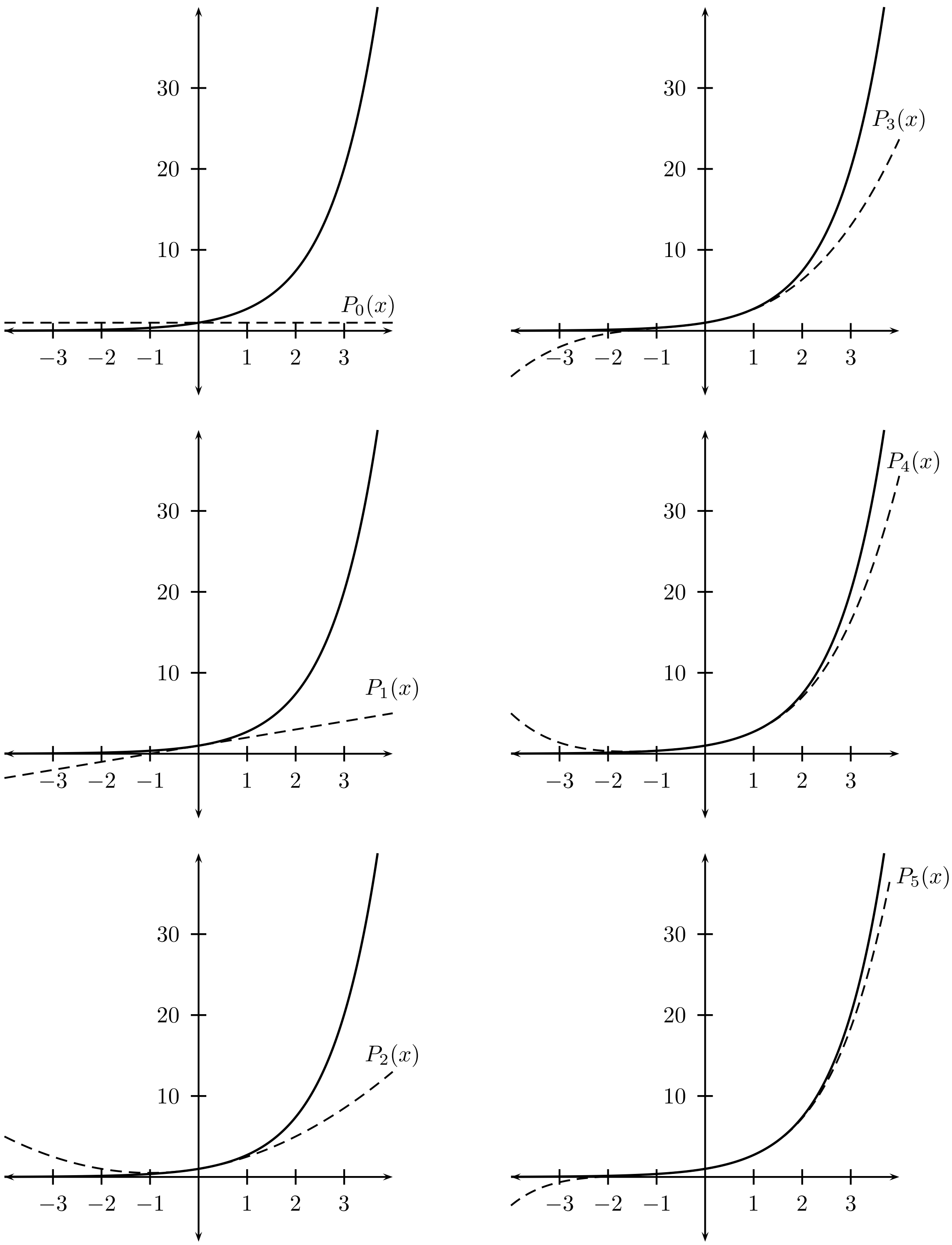


Figure 11.1: Graphs of $y = e^x$ and the Taylor Polynomial approximations $P_0(x)$–$P_5(x)$, plotted with dashed lines.

The upshot of this is that P_N is the simplest polynomial such that $f, f', f'', \cdots, f^{(N)}$ respectively match $P_N, P_N', P_N'', \cdots, P_N^{(N)}$ at the center $x = a$:

P_0 has the same height as f at $x = a$;
P_1 has the same height and slope as f at $x = a$;
P_2 has the same height, slope and second derivative as f at $x = a$;

and so on. While it becomes difficult to visualize how matching higher derivatives with f will continue the trend of better approximation, it should have the ring of truth. For instance, we can claim that the polynomial $P_3(x)$ matches the function $f(x)$ in height, slope, second derivative (concavity?), and the (instantaneous) rate of change in the second derivative at $x = a$. To go to the fourth-order approximation we note that how fast f''' is changing at the center $x = a$ will be "picked up" by $f^{(4)}$, at least in the instantaneous sense, and thus by $P_4(x)$ since it shares the height and first four derivatives with $f(x)$ at $x = a$. This type of reasoning will be addressed again in our error estimates for our approximations $P_N(x) \approx f(x)$, that is, estimates for the size of the errors $f(x) - P_N(x)$ in these approximations. It will also be addressed at the end of this section in our derivation of the Taylor Polynomials from some first principles.

So in our above Example 11.1.1, $P_5(x)$ matches the height and first five derivatives of e^x at $x = 0$, which helps it to "fit" the curve of $y = e^x$ (i.e., approximate the behavior of $f(x)$) better than the lower-order approximations which do not match as many derivatives of $f(x) = e^x$ as does $P_5(x)$. Indeed, $P_5(x)$ is the simplest polynomial which matches the height, slope, "concavity," third derivative, fourth derivative and fifth derivative of e^x at $x = 0$. Higher-order Taylor Polynomials $P_6(x)$, $P_7(x)$ and so on will match all that, and more.

A pattern clearly emerges for $P_N(x)$, centered at $a = 0$ for $f(x) = e^x$. If we desired $P_6(x)$, we would simply add $\frac{1}{6!}x^6$, and if we desired $P_7(x)$ we would then further add $\frac{1}{7!}x^7$, and so on. It would be a simple exercise to generate $P_{20}(x)$ or higher, and to compute its values using any rudimentary programming language.[4]

It should be pointed out that some textbooks use the result of the above Theorem 11.1.1, page 751 as the *definition* of the Taylor Polynomials, meaning that they define the Taylor Polynomial of $f(x)$ centered at $x = a$ as that Nth-degree (or less) polynomial which matches the height and first N derivatives of $f(x)$ at $x = a$. It can be shown that the only such Nth-degree or lower *polynomial* which satisfies the matching of height and all derivatives up to degree N at $x = a$ must in fact be of the form of our definition of the Nth-order Taylor Polynomial (11.2), page 749.

We will derive our formula from a different motivation at the end of this section, not wishing for it to be a distraction here. However, for completeness we include a proof of Theorem 11.1.1 (page 751):

[4]The graphs here and throughout the book are generated with the Postscript language, which is more of a publishing language and far from being a first choice for intense, scientific computations, but is quite adequate here. One technique for making the computations more computer-friendly, and pencil and paper-friendly, is to rewrite the polynomial

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = 1 + x\left(1 + \frac{1}{2}x\left(1 + \frac{1}{3}x\left(1 + \frac{1}{4}x\left(1 + \frac{1}{5}x\right)\right)\right)\right).$$

With the second form, there are fewer multiplications (if we consider, say, x^5 and $5!$ as each comprising four multiplications), and we do not have to rely on the computer to compute powers of large numbers, divided by large factorials, and sum these. It is akin to the process known as *synthetic division* for computing polynomial values.

Proof: First we note how derivative the of a general nth-order term in our polynomial (11.2) simplifies, assuming $n \geq 1$:

$$\frac{d}{dx}\left[\frac{f^{(n)}(a)(x-a)^n}{n!}\right] = \frac{f^{(n)}(a)}{n!}\cdot n(x-a)^{n-1} = \frac{f^{(n)}(a)}{(n-1)!\cdot n}\cdot n(x-a)^{n-1}$$
$$= \frac{f^{(n)}(a)}{(n-1)!}\cdot (x-a)^{n-1}.$$

We made use of the fact that a, $f^{(n)}(a)$ and $n!$ are all constants in the computation above. We will also use the fact that any additive constants, i.e., terms of form "$(x-a)^0$," will have derivative zero. Finally note that any term with $(x-a)^n$, where $n \geq 1$, will be zero at $x = a$.

From these observations it is routine (if not totally transparent) that we can demonstrate the computations in Theorem 11.1.1. To make the pattern clear, we assume here that $N > 3$. In each of what follows, we first take derivatives at each line, and then evaluate at $x = a$.

$$P_N(x) = \sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n \implies P_N(a) = \frac{f^{(0)}(a)}{0!} = f(a)$$
$$P_N'(x) = \sum_{n=1}^{N}\frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} \implies P_N'(a) = \frac{f^{(1)}(a)}{0!} = f'(a)$$
$$P_N''(x) = \sum_{n=2}^{N}\frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2} \implies P_N''(a) = \frac{f^{(2)}(a)}{0!} = f''(a)$$
$$P_N'''(x) = \sum_{n=3}^{N}\frac{f^{(n)}(a)}{(n-3)!}(x-a)^{n-3} \implies P_N'''(a) = \frac{f^{(3)}(a)}{0!} = f'''(a)$$
$$\vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots$$
$$P_N^{(N-1)}(x) = \sum_{n=N-1}^{N}\frac{f^{(n)}(a)}{(n-(N-1))!}(x-a)^{n-(N-1)}$$
$$= \frac{f^{(N-1)}(a)}{0!} + \frac{f^{(N)}(a)}{1!}(x-a) \implies P_N^{(N-a)}(a) = f^{(N-1)}(a)$$
$$P_N^{(N)}(x) = f^{(N)}(a) \implies P_N^{(N)}(a) = f^{(N)}(a)$$
$$P_N^{(m)}(x) = 0, \quad m \in \{N+1, N+2, N+3, \cdots\}, \qquad \text{q.e.d.}$$

Example 11.1.2 *There is a simple real-world motivation for this kind of approach. Suppose a passenger on a train wishes to know approximately where the train is. At some time t_0, he passes the engineer's compartment and sees the mile marker s_0 out the front window. He also sees the speedometer reading v_0. If the train is not accelerating or decelerating noticeably, he can follow his watch and expect the train to move approximately $v_0(t-t_0)$ in the time $[t_0, t]$. In other words,*

$$s \approx s_0 + v_0(t - t_0). \tag{11.3}$$

On the other hand, perhaps he feels some acceleration, as the train leaves an urban area, for instance. If the engineer has an acceleration indicator, and it reads a_0 at time t_0, then the *passenger could assume that the acceleration will be constant for a while (but not too long!), and use*

$$s \approx s_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2. \tag{11.4}$$

If our passenger can even compute how $a = s''$ is changing, then assuming that change is at a constant rate, i.e., that $s'''(t) \approx s'''(t_0)$, we can go another order higher and claim[5]

$$s \approx s_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2 + \frac{1}{3}s'''(t_0)(t - t_0)^3. \tag{11.5}$$

Indeed this will likely be the best estimate thus far when $|t - t_0|$ is small (and s''' is still relatively constant). However, we have to be aware that this latest approximation is a degree-three polynomial, and will therefore act like one as $|t|$ (and therefore $|t - t_0|$) gets large, so we have to always be aware of the range of t for which the approximation is accurate.

Next we look at some more examples.

Example 11.1.3 *Find $P_2(x)$ for $f(x) = \sqrt{1 + x^2}$ centered at $a = 0$.*

<u>Solution</u>: *We compute the first two derivatives, and evaluate them at 0:*

$$\begin{aligned} f(x) &= \sqrt{1+x^2} & &\Longrightarrow f(0) = 1 \\ f'(x) &= \frac{1}{2\sqrt{1+x^2}} \cdot 2x = x(1+x^2)^{-1/2} & &\Longrightarrow f'(0) = 0 \\ f''(x) &= x \cdot \frac{-1}{2}(1+x^2)^{-3/2} + (1+x^2)^{-1/2} \cdot 1 & &\Longrightarrow f''(0) = 1. \end{aligned}$$

Thus

$$\begin{aligned} P_2(x) &= f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2 \\ &= 1 + \frac{1}{2}x^2. \end{aligned}$$

See Figure 11.2, page 756 *for the graphs of $f(x)$ and $P_2(x)$.*

In most applications, one chooses a center $x = a$ so that $a, f(a), f'(a), f''(a)$ and so on are all "nice" numbers, though theoretically we could have found $P_2(x)$ in Example 11.1.3 with $a = \sqrt{3}$. On the other hand, if we can easily enough compute $\sqrt{3}$ (for our $(x - a)^n$ terms), we probably could equally easily compute $\sqrt{x^2 + 1}$.

In Section 11.5.3 we will see a pattern which will help us compute higher-order Taylor Series for functions such as this. Clearly the derivative computations needed to find $f'''(0), f^{(4)}(0)$ and so on quickly become unwieldy, and so a shortcut will be welcome. For many physics-type problems, however, $P_2(x)$ is a very useful approximation, particularly for $x \in [-1, 1]$.

Example 11.1.4 *Find $P_3(x)$ at $a = 1$ if $f(x) = 2x^3 - 9x^2 + 5x + 11$.*

<u>Solution:</u> *Again we construct a chart.*

$$\begin{aligned} f(x) &= 2x^3 - 9x^2 + 5x + 11 & &\Longrightarrow \quad f(1) = 9 \\ f'(x) &= 6x^2 - 18x + 5 & &\Longrightarrow \quad f'(1) = -7 \\ f''(x) &= 12x - 18 & &\Longrightarrow \quad f''(1) = -6 \\ f'''(x) &= 12 & &\Longrightarrow \quad f'''(1) = 12 \end{aligned}$$

[5]Notice that if f'' were truly constant, then (11.4) would be exact and not an approximation. Similarly, if f''' were truly constant, then (11.5) would be exact.

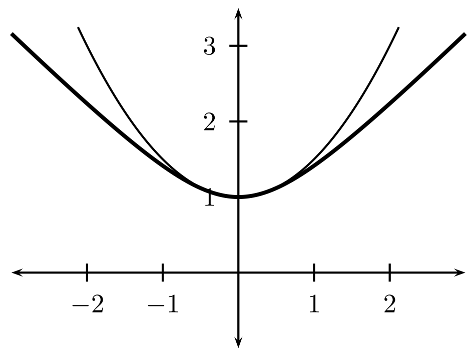


Figure 11.2: Graph of $f(x) = \sqrt{1+x^2}$ (thicker line), along with its second-order Taylor Polynomial $P_2(x) = 1 + \frac{1}{2}x^2$, which is much easier to compute (at least "by hand"), and reasonably accurate if $|x| = |x - 0|$ is small. See Example 11.1.3, page 755.

Now

$$\begin{aligned}P_4(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!}\\ &= 9 - 7(x-1) + \frac{-6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!}\\ &= 9 - 7(x-1) - 3(x-1)^2 + 2(x-1)^3.\end{aligned}$$

This is a trivial, yet important kind of example, for if we expanded out the last line above in powers of x we would get back the original polynomial, which shows that the simplest polynomial matching this function and its first three derivatives at $x = 1$ is the polynomial itself. Furthermore, we can see from our chart, that $f^{(4)}(x) = 0, f^{(5)}(x) = 0$, etc., and so $P_3 = P_4 = P_5 = \cdots$. We will enshrine this result in the following theorem:

Theorem 11.1.2 *Suppose $f(x)$ is an Nth-degree polynomial, i.e.,*

$$f(x) = A_N x^N + A_{N-1}x^{N-1} + \cdots + A_1 x + A_0. \tag{11.6}$$

Then regardless of $a \in \mathbb{R}$, we have $(\forall m \geq N)\,[P_m(x) = f(x)]$.

In other words, a polynomial will be the same as its Taylor Polynomials of all orders which are at least as high as the degree of the polynomial, regardless of the center of the Taylor Polynomial.

The proof is interesting to read through, though the result is more important than the proof. We include the proof here for completeness.

Proof: We will prove this in stages.

(1) An important general observation we will use repeatedly is the following:

$$(\forall x \in \mathbb{R})[g'(x) = h'(x)] \iff (\exists C)[g(x) - h(x) = C]. \tag{11.7}$$

In other words, if two functions have the same derivative functions, then the original two functions differ only by a constant. (This is also true if the functions and derivatives are only considered on single intervals.)

(2) Since f and P_N are both Nth-degree polynomials, we have $f^{(N)}(x)$ and $P^{(N)}(x)$ are constants.

(3) By Theorem 11.1.1, page 751, we have $f^{(N)}(a) = P^{(N)}(a)$.

(4) From (2) and (3), we have

$$P^{(N)}(x) = P^{(N)}(a) = f^{(N)}(a) = f^{(N)}(x). \tag{11.8}$$

Thus $P^{(N)}(x) = f^{(N)}(x)$.

(5) By (1), we can thus conclude that $P^{(N-1)}(x)$ and $f^{(N-1)}(x)$ differ by a constant.

(6) Since $P^{(N-1)}(a) = f^{(N-1)}(a)$, and (5), we must have $P^{(N-1)}(x) = f^{(N-1)}(x)$. In other words, since $P^{(N-1)}(x)$ and $f^{(N-1)}(x)$ differ by a constant, and since $P^{(N-1)}(a) - f^{(N-1)}(a) = 0$, the constant referred to in (5) must be zero.

(7) The argument above can be repeated to get $P^{(N-2)}(x) = f^{(N-2)}(x)$, and so on, until finally we indeed get $P'(x) = f'(x)$.

(8) The last step is the same. From (1), P and f differ by a constant, but since $P(a) = f(a)$, that constant must be zero, so $P(x) - f(x) = 0$, i.e., $P(x) = f(x)$.

It is important that the original function $f(x)$ above was a polynomial, or else the conclusion is false.

The theorem is useful for both analytical and algebraic reasons. If we wish to expand an Nth-degree polynomial (11.6) in powers of $x - a$ (instead of the usual $x = x - 0$), then we can just compute $P_N(x)$ centered at $x = a$. From the theorem, we can easily "re-center" any polynomial, meaning we can write it as a sum of powers of $(x - a)$ instead of x, the original "center" of course being zero.

Example 11.1.5 *Write the following polynomial in powers of x: $f(x) = (x + 5)^4$.*

<u>Solution</u>: We can use the binomial expansion (with Pascal's Triangle, for instance) for this, but we can also use the Taylor Polynomial centered at $a = 0$:

$$\begin{array}{rcl}
f(x) = (x+5)^4 & \implies & f(0) = 625 \\
f'(x) = 4(x+5)^3 & \implies & f'(0) = 4 \cdot 5^3 \\
f''(x) = 4 \cdot 3(x+5)^2 & \implies & f''(0) = 4 \cdot 3 \cdot 5^2 \\
f'''(x) = 4 \cdot 3 \cdot 2(x+5) & \implies & f'''(0) = 4 \cdot 3 \cdot 2 \cdot 5 \\
f^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot 1 & \implies & f^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot 1 \\
f^{(m)}(x) = 0 & & \text{any } m > 4
\end{array}$$

$$\begin{aligned}
P_4(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} \\
&= 5^4 + 4 \cdot 5^3 x + \frac{4 \cdot 3 \cdot 5^2 x^2}{2!} + \frac{4 \cdot 3 \cdot 2 \cdot 5x^3}{3!} + \frac{4 \cdot 3 \cdot 2 \cdot 1x^4}{4!} \\
&= 625 + 500x + 150x^2 + 20x^3 + x^4.
\end{aligned}$$

Because this is $P_4(x)$ for a fourth-degree polynomial function, it equals that polynomial function, i.e.,

$$(x + 5)^4 = 625 + 500x + 150x^2 + 20x^3 + x^4.$$

Of course arguably the more interesting Taylor Polynomials do not involve polynomial approximations of polynomials. The relationship to ordinary polynomials explored above is nonetheless interesting. For the remainder here, we will look at examples where $f(x)$ is not itself a polynomial.

Example 11.1.6 *Consider the function $f(x) = \sqrt[3]{x}$, with $a = 27$.*

a. *Calculate $P_1(x)$, $P_2(x)$, $P_3(x)$.*

b. *Use these to approximate $\sqrt[3]{26}$.*

c. *Compare these to the actual value of $\sqrt[3]{26}$, as determined by calculator.*

<u>Solution:</u> *We take these in turn.*
a. *First we will construct a chart.*

$$\begin{aligned} f(x) &= x^{1/3} & f(27) &= 3 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(27) &= \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(27) &= -\frac{2}{9}\cdot\frac{1}{243} = -\frac{2}{2187} \\ f'''(x) &= \frac{10}{27}x^{-8/3} & f'''(27) &= \frac{10}{27}\cdot\frac{1}{6561} = \frac{10}{177,147} \end{aligned}$$

Thus,

$$\begin{aligned} P_1(x) &= 3 + \frac{1}{27}(x-27) \\ P_2(x) &= 3 + \frac{1}{27}(x-27) + \frac{-\frac{2}{2187}}{2}(x-27)^2 \\ &= 3 + \frac{1}{27}(x-27) - \frac{1}{4374}(x-27)^2 \\ P_3(x) &= 3 + \frac{1}{27}(x-27) - \frac{1}{4374}(x-27)^2 + \frac{\left(\frac{10}{177,147}\right)}{3!}(x-27)^3 \\ &= 3 + \frac{1}{27}(x-27) - \frac{1}{4374}(x-27)^2 + \frac{10}{1,062,882}(x-27)^3. \end{aligned}$$

b. *From these we get*

$$\begin{aligned} P_1(26) &= 3 + \frac{1}{27}(26-27) = 3 + \frac{1}{27}(-1) = 3 - \frac{1}{27} = \frac{80}{27} \approx 2.9629630 \\ P_2(26) &= 3 + \frac{1}{27}(-1) + \frac{1}{4374}(-1)^2 = \frac{12,961}{4374} \approx 2.9627343 \\ P_3(26) &= P_2(26) + \frac{10}{1,062,882}(-1)^3 = \frac{3149513}{1062882} \approx 2.9627249. \end{aligned}$$

c. *The actual value (to 8 digits) is $\sqrt[3]{26} \approx 2.9624961$. The errors $R_1(26), R_2(26)$ and $R_3(26)$, in each of the above approximations are respectively*

$$\begin{aligned} R_1(26) &= \sqrt[3]{26} - P_1(26) \approx 2.9624961 - 2.9629630 = -0.0004669 \\ R_2(26) &= \sqrt[3]{26} - P_2(26) \approx 2.9624961 - 2.9627343 = -0.0002382 \\ R_3(26) &= \sqrt[3]{26} - P_3(26) \approx 2.9624961 - 2.9627249 = -0.0002288. \end{aligned}$$

Thus we see some improvement in these estimates. For other functions it can be more or less dramatic. In Section 11.2 we will state the form of the error, or *remainder* $R_N(x) = f(x) - P_N(x)$, and thus be able to explore the accuracy of $P_N(x)$.

Example 11.1.7 *Find $P_5(x)$ at $a = 0$ for $f(x) = \sin x$.*

Solution: *Again we construct the chart.*

$$\begin{array}{rcl} f(x) = \sin x & \Longrightarrow & f(0) = 0 \\ f'(x) = \cos x & \Longrightarrow & f'(0) = 1 \\ f''(x) = -\sin x & \Longrightarrow & f''(0) = 0 \\ f'''(x) = -\cos x & \Longrightarrow & f'''(0) = -1 \\ f^{(4)}(x) = \sin x & \Longrightarrow & f^{(4)}(0) = 0 \\ f^{(5)}(x) = \cos x & \Longrightarrow & f^{(5)}(0) = 1, \end{array}$$

from which we get

$$\begin{aligned} P_5(x) &= 0 + 1x + \frac{0x^2}{2!} + \frac{-1x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

From this chart we can see an obvious pattern where

$$P_6(x) = P_5(x) + 0 = P_5(x),$$

$$P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + 0 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} = P_7(x),$$

and so on.

This answers the question of how electronic calculators compute $\sin x$: by means of just such a Taylor Polynomial.[6] It also hints at an answer for why physicists often simplify a problem by replacing $\sin x$ with x: that is the simplest polynomial which matches the height, slope and concavity of $\sin x$ at $x = 0$ is a very simple function indeed, namely $P_2(x) = x$.

See Figures 11.3 and 11.4, page 760 to compare $\sin x$ to $P_1(x)$, $P_3(x)$, $\cdots$, $P_{13}(x) = P_{14}(x)$. Clearly the polynomials are increasingly better at approximating $\sin x$ as we add more terms. On the other hand, as $|x|$ gets large these approximations eventually behave like the polynomials they are in the sense that $|P_n(x)| \to \infty$ as $|x| \to \infty$. This is not alarming, since it is the *local* behavior, in this case near $x = 0$ (more generally near $x = a$), that we exploit when we use polynomials to approximate functions. It is worth remembering, however, so that we do not attempt to use a Taylor Polynomial to approximate a function too far from the center, $x = a$, of the Taylor Polynomial.

Example 11.1.8 (Application) *As already mentioned, physicists often take advantage of the second order approximation $\sin x \approx P_2(x) = 0 + x + 0x^2$, that is,*

$$\sin x \approx x \qquad \text{for } |x| \text{ small.} \tag{11.9}$$

[6] Note that when using Taylor Polynomials to compute a trigonometric function such as $\sin x$, the calculus is greatly simplified when we assume x is in radians (which are dimensionless). Therefore a calculator giving its approximation of, say, $\sin 57°$ will convert the angle into radians first.

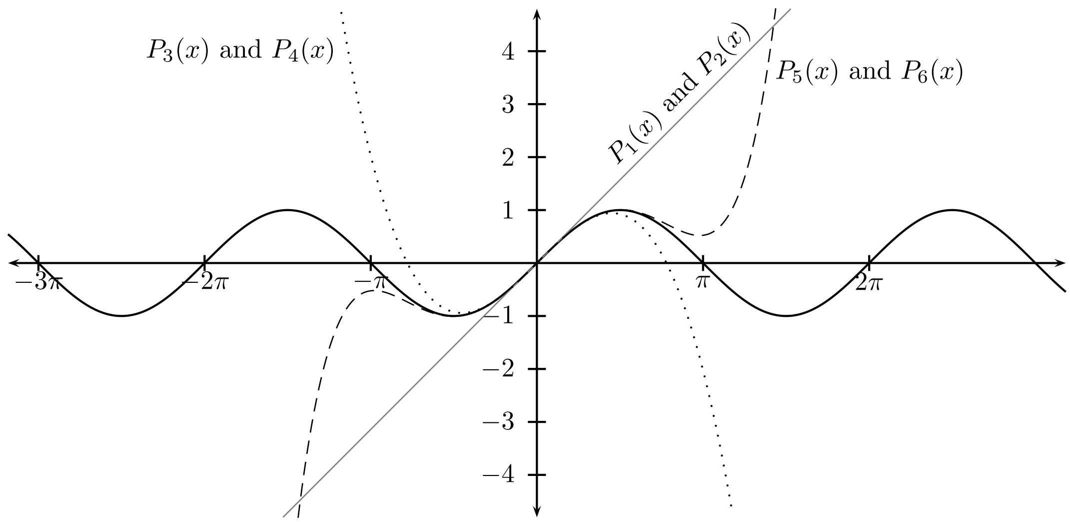


Figure 11.3: $\sin x$, $P_1(x), P_2(x) = x$ (gray), $P_3(x), P_4(x) = x - \frac{x^3}{3!}$ (dots), and $P_5(x), P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (dashed).

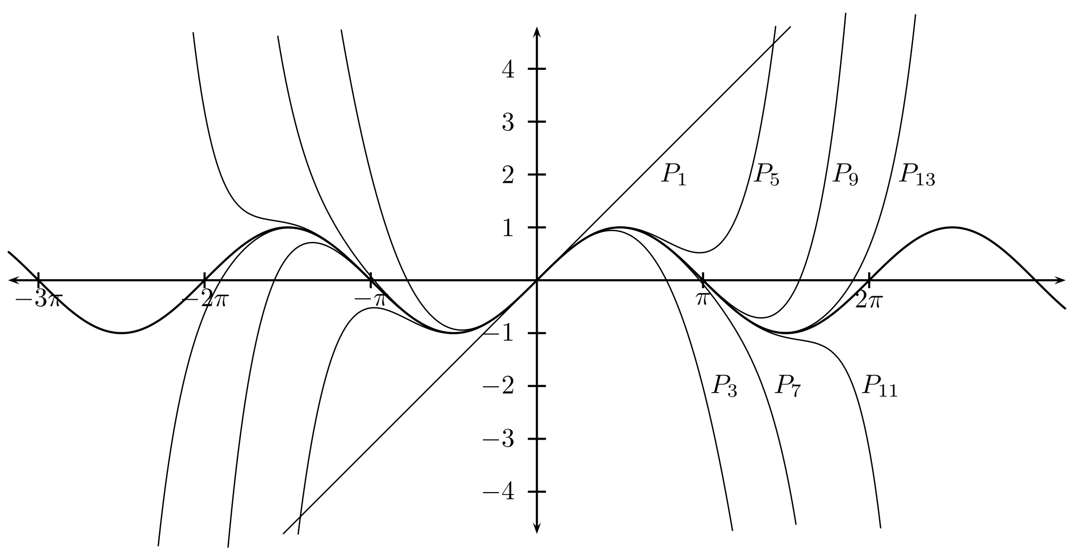
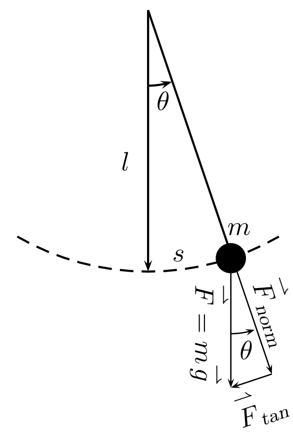


Figure 11.4: $\sin x$ and $P_1(x) = P_2(x)$, $P_3(x) = P_4(x)$, $\cdots$, $P_{13}(x) = P_{14}(x)$.

The classic example is the modeling of the simple pendulum. See the illustration below, which we then model mathematically.

Suppose a pendulum of mass m is hanging from an always taut and straight string of negligible weight. Let θ be the angle the string makes with the downward vertical direction. We will take $\theta > 0$ if θ represents a counterclockwise rotation, as is standard. Also g is the acceleration due to gravity, approximately $32\ \mathrm{ft/sec}^2$ or $9.8\ \mathrm{m/sec}^2$.

The component of velocity which is in the direction of motion of the pendulum is given by $\frac{ds}{dt} = \frac{d(l\theta)}{dt} = l\frac{d\theta}{dt}$, and the acceleration by its derivative, $\frac{d^2s}{dt^2} = \frac{d^2(l\theta)}{dt^2} = l\frac{d^2\theta}{dt^2}$. Now the force in the direction of the motion has magnitude $mg\sin\theta$, but is a restorative force, and is thus in the opposite direction of the angular displacement. It is not too difficult to see that this force is given by $-mg\sin\theta$, for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, by equating the force and the acceleration in the angular direction, we get[7]



$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta \tag{11.10}$$

which simplifies to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{11.11}$$

This is a relatively difficult differential equation[8] *to solve. However, if we assume $|\theta|$ is small, we can use $\sin\theta \approx \theta$ and instead solve the following equation which holds approximately true*[9]*:*

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \tag{11.12}$$

---

[7] For those familiar with moments of inertia, the analog of $F = ma$ is

$$N = I\alpha,$$

where N is torque, I is the moment of inertia, and α is the angular acceleration, in $\mathrm{rad/sec}^2$. Using the fact that, for this example, torque is also defined by $N = F_{\tan}l = -mgl\sin\theta$, we get the equations

$$N = -mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2},$$

giving equation (11.10) after dividing by l.

[8] A *differential equation* is an equation involving the derivatives of a function y (or θ here). The goal in "solving" a differential equation is to find all functions y which satisfy the equation. Courses in differential equations assume the student has learned calculus for two or three semesters, though it is common for simple differential equations to be found in introductory calculus books.

[9] We should point out here that (11.12) is an example of a *simple harmonic oscillator*, which is any physical system governed by an equation of the form

$$Q''(t) = -\kappa Q(t), \qquad \kappa > 0$$

(*kappa*, the lower-case Greek letter kappa being a constant) which has solution

$$Q(t) = A\sin\sqrt{\kappa}\,t + B\cos\sqrt{\kappa}\,t,$$

and period $2\pi/\sqrt{\kappa}$. Examples include springs which are governed by Hooke's Law $F(s) = -ks$, where $k > 0$ and $s = s(t)$. Recall $F = m\frac{d^2s}{dt^2}$, so Hooke's Law becomes $\frac{d^2s}{dt^2} = -\frac{k}{m}\cdot s$, giving a simple harmonic oscillator.

The solution of (11.12) *is*

$$\theta = A\sin\left(\sqrt{\frac{g}{l}}\cdot t\right) + B\cos\left(\sqrt{\frac{g}{l}}\cdot t\right). \tag{11.13}$$

Here A and B are arbitrary constants depending on the initial $(t=0)$ position and velocity of the pendulum. Notice that (11.13) *is periodic, with a period τ where $\tau = 2\pi/\sqrt{g/l}$, i.e.,*

$$\tau = 2\pi\sqrt{\frac{l}{g}}. \tag{11.14}$$

That is the formula found in most physics texts for the period of a pendulum. However, it is based upon an approximation, albeit quite a good one for $|\theta|$ small. Still, the higher we allow the pendulum to swing, the less we can rely on this approximation of the period.

To a novice, it might not be terribly satisfying to resort to approximations when attempting to solve a problem, but "in the lab" and when designing practical applications, understanding how to approximate, *and the limitations of the practice*, are quite valuable, and usually better appreciated with more exposure to the possibilities.

Example 11.1.9 *Let us find $P_6(x)$ where $f(x) = \cos x$ and $a = 0$.*

Solution: *We construct the table again:*

$$\begin{array}{rcr}
f(x) = \cos x & \Longrightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Longrightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Longrightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Longrightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Longrightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Longrightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Longrightarrow & f^{(6)}(0) = -1
\end{array}$$

Since the odd derivatives are zero at $x = 0$, only the even-order terms appear, and we have

$$\begin{aligned}
P_6(x) &= 1 + \frac{-1(x-0)^2}{2!} + \frac{1(x-0)^4}{4!} + \frac{-1(x-0)^6}{6!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.
\end{aligned}$$

From this a pattern clearly emerges, and we could easily calculate

$$P_{14}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}.$$

We might also point out that P_{15} would be the same, since the odd terms were all zero.

Example 11.1.10 *Find P_5 for $f(x) = \ln x$ with center $a = 1$.*

Solution: *First, the table is constructed as usual by computing first $f^{(n)}(x)$ and then $f^{(n)}(a) = f^{(n)}(1)$.*

$$\begin{aligned}
f(x) &= \ln x & &\Longrightarrow & f(1) &= 0\\
f'(x) &= x^{-1} & &\Longrightarrow & f'(1) &= 1\\
f''(x) &= -1x^{-2} & &\Longrightarrow & f''(1) &= -1\\
f'''(x) &= 2x^{-3} & &\Longrightarrow & f'''(1) &= 2\\
f^{(4)}(x) &= -3\cdot 2x^{-4} & &\Longrightarrow & f^{(4)}(1) &= -3\cdot 2\\
f^{(5)}(x) &= 4\cdot 3\cdot 2x^{-5} & &\Longrightarrow & f^{(5)}(1) &= 4\cdot 3\cdot 2
\end{aligned}$$

Now we construct P_5 from (11.2).

$$P_5(x) = 0 + 1(x-1) + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-3\cdot 2(x-1)^4}{4!} + \frac{4\cdot 3\cdot 2(x-1)^5}{5!}.$$

Recalling the definition of factorials, in which $2! = 2\cdot 1$, $3! = 3\cdot 2\cdot 1$, $4! = 4\cdot 3\cdot 2\cdot 1$, and $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$, we see that the above simplifies to

$$P_5(x) = 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.$$

It is not hard to see that $f^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}$, and so $f^{(n)}(1) = (-1)^{n+1}(n-1)!$. The obvious pattern which appears in P_5 should continue for P_6, P_7, etc. Thus we can calculate any $P_N(x)$ for this example:

$$P_N(x) = \sum_{n=1}^{N} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

If we wished to have $N \to \infty$, we get the full **Taylor Series:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

However this might not converge for all x. Indeed, from the ratio test we get

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \Big/ \frac{(-1)^{n+1}(x-1)^n}{n} \right| = \lim_{n\to\infty} \frac{n|x-1|}{n+1} = |x-1|,$$

and so $\rho < 1$ when $|x-1| < 1$, i.e., $x \in (0,2)$, and $\rho > 1$ when $|x-1| > 1$, i.e., $x \in (-\infty,0)\cup(2,\infty)$. When $x = 0$ we have a negative harmonic series $\sum_{n=1}^{\infty} \frac{-1}{n}$ which diverges, and when $x = 2$ we have the conditionally convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Summarizing, thus the series converges for $x \in (0,2]$, and diverges elsewhere.

This use of the Ratio Test is the most common method for computing where such a series—namely one in which we have a formula for the nth-degree term—converges.

From many of the previous examples we see that the table many Taylor Polynomials have patterns which emerge easily from the derivative computations. However, we will see that this is not the case for many important series which we can nonetheless use other methods to derive the pattern. In fact those methods are often easier than attempting what we will later characterize as "brute force," or "from scratch" method of construction here, which is deriving the nth term by computing $f^{(n)}(x)$ to compute $f^{(n)}(a)$ to construct $P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}$.

Motivation for and Derivation of Taylor Polynomials' Forms from First Principles

We end this Section 11.1 with a derivation of the Taylor Polynomials from nearly "first principles." While arguably not crucial to a relative novice student of calculus, it is nonetheless valuable and interesting in its own right because of the insights that can be gleaned from the creative ideas it employs. That said, its placement here is mostly for completeness. It will likely be much better motivated after the subsequent sections in this chapter are mastered, so the reader should feel free to peruse casually here first, and revisit it after studying the rest of the chapter and having a better understanding of the complete context.

One development of $P_N(x)$, left for the exercises, is a derivation based upon the *assumption* that P_N matches f in all derivatives (including the "zeroth") up to order N at the center a, and then the coefficients a_n of the $(x-a)^n$ terms are all found using methods similar to what we used to find coefficients of partial fraction decompositions.

However, the derivation here uses a different motivation, and is based upon reasonable integral approximations. We will continually refer to the special case where $f(x)=e^x$ and the center is $a=0$—for which $P_0(x)$ through $P_5(x)$ are graphed along with $f(x)$ in Figure 11.1, page 752—to illustrate the principles developed here.

In summary, $P_N(x)$ is the (necessarily polynomial) function we arrive at by deduction under the assumptions that

1. we only have the following data for f: $f(a)$, $f'(a)$, $f''(a)$, $\cdots$, $f^{(N)}(a)$, and
2. given no other data for f, we assume that its derivative $f^{(N)}(x)$ is approximately constant, for x near $x=a$. That is,

$$f^{(N)}(x) \approx f^{(N)}(a), \qquad \text{for } x \text{ near } a.$$

If a function did have constant Nth derivative, it would be a polynomial of degree at most N, which we could see by integrating that derivative N times.

The idea is to find simple polynomial approximations for a more complicated function given certain data regarding its behavior. In particular, if we know $f(a)$, $f'(a)$, $f''(a)$, and so on, then we should know something about how the function $f(x)$ behaves near $x=a$, and be able to produce a polynomial which mimics that behavior. In doing so we will make repeated use of the following lemma, which is useful in many other contexts as well:

Lemma 11.1.1 *Given any function g, with derivative g' existing and continuous on the closed interval with endpoints x and a (i.e., $[a,x]$ or $[x,a]$, depending upon whether $x\le a$ or $a\le x$), the following equation holds:*

$$g(x) = g(a) + \int_a^x g'(t)\,dt. \tag{11.15}$$

This is easy enough to verify. Since g is clearly an antiderivative of g', the Fundamental Theorem of Calculus gives

$$g(a) + \int_a^x g'(t)\,dt = g(a) + g(t)\Big|_a^x = g(a) + g(x) - g(a) = g(x),$$

which is the equation (11.15) in reverse, q.e.d.

Two simple observations are worth making here.

1. It is interesting to verify this for the special case of (11.15) when $x=a$: $g(a)=g(a)+\int_a^a g'(t)\,dt = g(a)+0=g(a)$. Recall that such an integral as appears here over any interval of length zero is necessarily zero.

2. This easily lends itself to a simple physics application. If we replace $g(x)$ by $s(t)$, where $s(t)$ is the position at time t and $s_0 = s(t_0)$, we get $s(t) = s_0 + \int_{t_0}^{t_f} s'(t)\,dt$. If $s'(t) = v(t)$ is constant, we get $s(t) = s_0 + v(t_f - t_0)$. If $t_0 = 0$, we get $s = s_0 + vt$. Other cases also follow quickly from (11.15).

Derivation of $P_0(x)$

For a function $f(x)$, if we would like to approximate the value of the function for x near a, the simplest assumption is that the function is approximately constant near $x = a$. The obvious choice for that constant is $f(a)$ itself. Hence we might assume $f(x) \approx f(a)$. (Note that $f(a)$ is itself a constant.) The approximation of $f(x)$ which assumes the function approximately constant is then $P_0(x)$:

$$P_0(x) = f(a). \tag{11.16}$$

This is also called the *zeroth-order approximation* of $f(x)$ centered at $x = a$, and we can write $f(x) \approx P_0(x)$ for x near a, i.e., for $|x-a|$ small. (See again Figure 11.1, page 752.) Summarizing, for x near a,

$$f(x) \approx \underbrace{f(a)}_{P_0(x)}. \tag{11.17}$$

A natural question then arises: how good is the approximation (11.17)? Later we will have a sophisticated estimate on the error in assuming $f(x) \approx P_0(x) = f(a)$. For now we take the opportunity to foreshadow that result by attacking the question intuitively. The answer will depend upon the answers to two related questions, which can be paraphrased as the following.

(i) How good is the assumption that f is *constant* on the interval from a to x?

In other words, how fast is f changing on that interval?

(ii) How far is x from a?

These factors both contribute to the error. For instance if the interval from a to x is short, then a relatively slow change in f means small error $f(x) - P_0(x) = f(x) - f(a)$ over such an interval. Slow change can, however, accumulate to create a large error if the interval from a to x is long. On the other hand, a small interval can still allow for large error if f changes quickly on the interval. The key to estimating how fast the function changes is, as always, the size of its derivative, assuming the derivative exists. Translating (i) and (ii) above into mathematical quantities, we say the bounds of the error will depend upon the following:

(a) the size of $|f'(t)|$ as t ranges from a to x (assuming $f'(t)$ exists for all such t), and

(b) the distance $|x - a|$.

We will see similar factors accounting for error as we look at higher-order approximations $P_1(x)$, $P_2(x)$ and so on in this subsection, and the actual form of the general estimate for the error (also known as the *remainder*) in subsequent sections.

Derivation of $P_1(x)$

It was remarked in the last subsection that P_0 is not likely a good approximation for x very far from a if f' is large. In computing $P_1(x)$, we will not assume f is approximately constant (as we did with P_0), but instead assume that f' is approximately constant. To be clear, here are the assumptions from which P_1 is computed:

- We know $f(a)$ and $f'(a)$;
- $f'(t)$ is approximately constant for t from a to x.

For this derivation we will use the lemma from the beginning of this section (that is Lemma 11.1.1, page 764). Note that the following derivation uses the fact that $f'(a)$ is a constant, and our assumption $f'(t) \approx f'(a)$.

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t)\,dt \\ &\approx f(a) + \int_a^x f'(a)\,dt = f(a) + f'(a)t\Big|_a^x = f(a) + f'(a)x - f'(a)a = \underbrace{f(a) + f'(a)(x-a)}_{P_1(x)}. \end{aligned}$$

Thus we define $P_1(x)$, the *first-order* approximation of $f(x)$ centered at $x = a$ by

$$P_1(x) = f(a) + f'(a)(x-a). \tag{11.18}$$

This was also called the *linear approximation* of $f(x)$ at a in Chapter 5 ((5.13), page 496).

From the graphs in Figure 11.1, page 752 we can see how P_0 and P_1 can differ. Because assuming constant derivative is often less risky, error-wise, than assuming constant height, $P_1(x)$ is usually a better approximation for $f(x)$ near $x = a$, and indeed one can usually stray farther from $x = a$ and have a reasonable approximation for $f(x)$ if $P_1(x)$ is used instead of $P_0(x)$.[10]

Again we ask how good is this newer approximation $P_1(x)$, and again the intuitive response is that it depends upon answers two questions:

(i) How close is $f'(t)$ to constant in the interval between a and x?

(ii) How far are we from $x = a$?

The first question can be translated into, "how fast is f' changing on the interval between a and x?" This can be measured by the size of f'' in that interval, if it exists there. Again translating (i) and (ii) into quantifiables, we get that the accuracy of $P_1(x)$ depends upon

(a) the size of $|f''(t)|$ as t ranges from a to x (assuming $f''(t)$ exists for all such t), and

(b) the distance $|x - a|$.

If f'' is relatively small, then f' is relatively constant, and then the computation we made giving $f(x) \approx f(a) + f'(a)(x-a)$, i.e., $f(x) \approx P_1(x)$, will be fairly accurate as long as $|x - a|$ is not too large. See again Figure 11.1, page 752.

Derivation of $P_2(x)$

To better accommodate the change in f', we next replace the assumption that f' is constant with the assumption that, rather than constant, it is changing at a constant rate. In other words, we assume that f'' is constant. So our assumptions in deriving $P_2(x)$ are:

[10]Note that in an example of motion, this is like choosing between an assumption of constant position, and of constant velocity. Intuitively the constant velocity assumption should yield a better approximation of position, for a while, than would a constant position assumption. However there are functions with very fast oscillations but low magnitude, for which the assumption of a constant height is less problematic than the assumption of a constant derivative, which may be quite large. Indeed a function with a very large derivative may stay surprisingly bounded, while a strictly bounded function can have large values for derivatives, so the value of these assumptions of some kind of constancy must be considered in context. Further consideration of these points is left to the reader.

- $f(a)$, $f'(a)$ and $f''(a)$ are known;
- $f''(t)$ is approximately constant from $t = a$ to $t = x$, i.e., $f''(t) \approx f''(a)$.

Again we use the lemma at the beginning of the section, except this time we use it twice: first, in approximating f'; and then integrating that approximation to approximate f.

$$\begin{aligned} f'(x) &= f'(a) + (f'(x) - f'(a)) \\ &= f'(a) + \int_a^x f''(t)\,dt \\ &\approx f'(a) + \int_a^x f''(a)\,dt \\ &= f'(a) + f''(a)(x-a). \end{aligned}$$

Note that the computation above was the same as from the previous section, except that the part of f' there is played by f'' here, and the part of f there is played by f' here. We integrate again to approximate f. The second line below uses the approximation for f' derived above.

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t)\,dt && \text{(Lemma 11.1.1)} \\ &\approx f(a) + \int_a^x \left[f'(a) + f''(a)(t-a)\right]\,dt && \text{(Approximation for } f' \text{ above)} \\ &= f(a) + f'(a)(x-a) + \left.\left[\frac{f''(a)}{2}(t-a)^2\right]\right|_a^x \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 - \frac{1}{2}f''(a)(a-a)^2 \\ &= \underbrace{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2}_{P_2(x)}. \end{aligned}$$

Thus we define the *second-order* (or *quadratic*) approximation of $f(x)$ centered at $x = a$ by

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2. \tag{11.19}$$

Again, the accuracy depends upon (i) how close $f''(t)$ is to constant from $t = a$ to $t = x$, and (ii) how far we are from $x = a$. These can be quantified by the sizes of (a) $|f'''(t)|$ on the interval from $t = a$ to $t = x$, and (b) how large is $|x - a|$.

It is reasonable to take into account how fast f' changes on the interval from a to x. For P_2 we assume, not that f' is approximately constant as we did with $P_1(x)$, but that the rate of change of f' is constant on the interval, i.e., that f'' is constant (and equal to $f''(a)$) on the interval. In fact this tends to make $P_2(x)$ "hug" the graph of $f(x)$ better, since it accounts for the concavity. Figure 11.1, page 752 shows how $P_0(x)$, $P_1(x)$ and $P_2(x)$ can give progressively better approximations of $f(x)$ near $x = a$ (for the case $f(x) = e^x$ and $a = 0$). The extent to which we err in that assumption is the extent to which f'' (related to concavity) is non-constant, but at least near $x = a$, $P_2(x)$ accommodates concavity, as well as slope and height of the function $f(x)$.

Conclusion

The proof of the final formula for

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

would use an *induction* method, where one proves that

(1) the formula holds true for the first, or first few cases, say $N = 0, 1, 2$ under their respective assumptions (that they match the function and its first N derivatives at $x = a$, and that their degrees are at most N), and

(2) that the formula's truth for the Nth case (regardless of $N \in \mathbb{N} \cup \{0\}$) implies its truth for the $(N+1)$st case. That is,

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!} \implies P_{N+1}(x) = \sum_{n=0}^{N+1} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

Thus the establishment of the formula for P_0, P_1 and particularly P_2 implies it is also established for P_3, which in turn implies it is established for P_4, and so on, so that for instance its truth for P_{1000} is established because it is just a matter of following the implication in (2), also called the *induction step*, 998 times.

While we already proved (1), the proof of (2) is somewhat long and distracting, so we omit it here. However we will include in the exercises the case of computing $P_3(x)$ from scratch, where one assumes knowledge of $f(a), f'(a), f''(a), f'''(a)$ and assumes that $f'''(x) \approx f'''(a)$, i.e., f''' is approximately constant, and integrating back to what that would imply for $P_3(x)$, the function which is an at most degree-3 polynomial and conforms to those assumptions on its derivative, and is thus an approximation of $f(x)$, at least near $x = a$. By the time a student derives the formula for $P_3(x)$ in that manner, it should seem quite reasonable that the pattern will continue for $P_4(x)$, $P_5(x)$ and so on.

Exercises

1. Given $f(x) = \dfrac{1}{1-x}$, and $a = 0$,
   (a) show using (11.2), page 749 that
   $$P_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5.$$
   (b) What do you suppose is the general formula for $P_N(x)$?
   (c) Recalling facts about geometric series, for $|x| < 1$ what is the sum $\sum_{n=0}^{\infty} x^n$?
2. Find $P_5(x)$ if $f(x) = e^{2x}$ and $a = 0$.
3. Find $P_5(x)$ if $f(x) = e^{-3x}$ and $a = 0$.
4. If $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ represents a series for $f(x) = e^x$, then how would we expect to represent the following as series?
   (a) $e^{2x} =$
   (b) $e^{-3x} =$
   (c) $e^{x^2} =$
   (d) $x^2 e^x =$
5. Find $P_5(x)$ where $f(x) = \sin x$ and $a = \pi$.
6. Find $P_3(x)$ if $f(x) = \tan x$ and $a = \dfrac{\pi}{4}$.
7. Find $P_2(x)$ if $f(x) = \tan^{-1} x$ and $a = 0$.
8. Find $P_2(x)$ if $f(x) = \tan^{-1} x$, $a = 1$.

9. Find $P_2(x)$ if $f(x) = \sqrt{1+x^2}$, $a = 0$.

10. Find $P_3(x)$ if $f(x) = x^3$, $a = 1$.

11. Find a formula for $P_N(x)$ if $f(x) = \frac{1}{x}$, $a = 1$.

12. Find a formula for $P_N(x)$ if $f(x) = \frac{1}{x}$, $a = -1$.

13. Find $P_3(x)$ if $f(x) = \sin x$, $a = \frac{\pi}{2}$.

14. Find $P_3(x)$ if $f(x) = \sin x$, $a = -\frac{\pi}{6}$.

15. Show that (11.13) is indeed a solution to (11.12) by taking two time derivatives of each side of (11.12), remembering to employ the chain rule where appropriate.

16. If $\alpha \in \mathbb{R}$, find $P_5(x)$ for $f(x) = (1+x)^\alpha$ and $a = 0$.

17. Suppose at time $t = 1$ we know that $s = 2$, $v = 5$ and $a = -7$. What is likely to be our best approximation for $s(t)$ near time $t = 1$?

18. Assuming $f'''(x) = 6$ for all x, and $f''(2) = 8$, $f'(2) = 7$ and $f(2) = 5$, what is $f(x)$?

19. If we know $f'''(0) = 12$, $f''(2) = 22$, $f'(4) = 92$, and $f(1) = 2$, assuming $f'''(x)$ is constant, what is $f(x)$?

For Exercises 20–22, use $P_4(x)$ centered at $a = 0$ to approximate the given quantity. Compare that to the actual value (given by a calculator or similar device).

20. $f(x) = \sin x$ at $x = \pi/4$.

21. $f(x) = \cos x$ at $x = \pi/4$.

22. $f(x) = e^x$ at $x = 0.5$.

23. Consider $f(x) = \ln x$, and its Taylor Polynomials $P_n(x)$ centered at $a = 1$.

    (a) Compute $P_0(x), P_1(x), \cdots, P_6(x)$. (A pattern should become readily apparent.)

    (b) Using a calculator or similar device find $P_0(2), P_1(2), \cdots, P_6(2)$ as approximations of $\ln 2$. Compare these to $\ln 2$, and comment on the apparent efficiency of the approach $P_n(2) \to \ln 2$ as $n \to \infty$.

    (c) Repeat the above but with $P_0(1/2), P_1(1/2), \cdots, P_6(1/2)$, as approximations of $\ln(1/2)$.

    (d) Note that $\ln(1/2) = -\ln 2$. Does this suggest a more efficient method of approximating $\ln 2$ using Taylor Polynomials? (Note the relative positions of $2, 1/2$ and the center of your polynomials.)

    (e) Repeat (b)–(c) to compute $P_0(1/4), P_1(1/4), \cdots, P_6(1/4)$, compared to $\ln(1/4) = -2\ln 2$.

    (f) Is there any reason why we might not be interested in $P_n(0)$?

24. By applying $\frac{d^2}{dt^2}$ to both sides of (11.13), show that θ satisfies (11.12).

25. Compute $P_3(x)$ in the general case by (1) listing the hypotheses from which $P_3(x)$ arises as an approximation of $f(x)$, and (2) performing the integration steps from those hypotheses. (Read "Conclusion," page 768.)

11.2 Accuracy of $P_N(x)$

All of this makes for lovely graphs, but one usually needs some certainty regarding just how accurate we can expect $P_N(x)$ to be if it is to be used to approximate $f(x)$. Fortunately, there is a way to *estimate*—here meaning to find an upper bound on the size of—the *error* arising from replacing $f(x)$ with $P_N(x)$. This difference $f(x) - P_N(x)$ is also referred to as the *remainder* $R_N(x)$:

$$R_N(x) = f(x) - P_N(x). \tag{11.20}$$

Perhaps the name "remainder" makes more sense if we rewrite (11.20) in the form

$$f(x) = \underbrace{P_N(x)}_{\text{approximation}} + \underbrace{R_N(x)}_{\text{remainder}}. \tag{11.21}$$

Of course if we knew the *exact* value of $R_N(x)$, then by (11.21) we know $f(x)$ since we can always calculate $P_N(x)$ exactly, even with pencil and paper since, after all, it is just a polynomial. Often the best we can expect is to possibly have some estimate on the size of $R_N(x)$. This can often be accomplished by knowing the rough form of R_N, as is given in the following theorem.

Theorem 11.2.1 (Remainder Theorem)[11] *Suppose that f, f', f'', $\cdots$, $f^{(N)}$ and $f^{(N+1)}$ all exist and are continuous on the closed interval with endpoints both x and a. Then*

$$R_N(x) = \frac{f^{(N+1)}(z)(x-a)^{N+1}}{(N+1)!} \tag{11.22}$$

where z is some (unknown) number between a and x.

With this (11.21) could be rewritten

$$f(x) = \underbrace{f(a) + \frac{f'(a)(x-a)}{1!} + \cdots + \frac{f^{(N)}(a)(x-a)^N}{N!}}_{P_N(x)} + \underbrace{\frac{f^{(N+1)}(z)(x-a)^{N+1}}{(N+1)!}}_{R_N(x)}. \tag{11.23}$$

Thus, the remainder looks just like the next term to be added to construct $P_{N+1}(x)$, except that the term $f^{(N+1)}(a)$ is replaced by the unknown quantity $f^{(N+1)}(z)$.

A few examples of how the form (11.23) plays out are in order.

Example 11.2.1 *Write $f(x) = e^x$ as the sum of $P_4(x)$ and the remainder $R_4(x)$, with center $a = 0$.*

Solution: *Since all derivatives of e^x are not only existing on all of $\mathbb{R}$, but also simply e^x, then of course $f^{(n)}(0) = e^0 = 1$ for all $n = 0, 1, 2, \cdots$, we can write*

$$\begin{aligned} e^x = P_4(x) + R_4(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{f^{(5)}(z)(x-0)^5}{5!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{e^z x^5}{5!}, \end{aligned}$$

for some z between 0 and x.

[11] There are several remainder theorems addressing the size or form of the remainder $R_N(x)$, including one offered by Taylor himself. This form (11.22) is due to Joseph-Louis Lagrange (1736–1813), an Italian-born mathematician and physicist whose importance to both fields—and to the understanding of their interconnectedness—cannot be overstated. However his work tends to deal in advanced topics which are not easily explained without the context of at least upper-division undergraduate mathematics and physics. The remainder theorem above is one exception.

Example 11.2.2 *Write* $\sin x$ *as the sum of* $P_3(x)$ *and the remainder* $R_3(x)$.

<u>Solution</u>: *Note that all derivatives of* $\sin x$ *are of the form* $\pm\sin x$ *or* $\pm\cos x$*, which exist and are continuous on all of* $\mathbb{R}$*. Now we constructed the chart for constructing up to* $P_5(x)$ *for this function in* Example 11.1.7, page 759*, but we will do so again here but in a more summary form:*

n	0	1	2	3	4	5
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$

From this we can write

$$\sin x = P_3(x) + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)x^4}{4!} = x - \frac{x^3}{3!} - \frac{(\cos z)x^4}{4!},$$

for some z *between* 0 *and* x.

In fact we can write $\sin x$ in any of the following ways:

$$\begin{aligned}
\sin x &= x + \frac{(\cos z)x^2}{2!}, && \text{for some } z \text{ between } 0 \text{ and } x,\\
\sin x &= x + \frac{(-\sin z)x^3}{3!}, && \text{for some } z \text{ between } 0 \text{ and } x,\\
\sin x &= x - \frac{x^3}{3!} + \frac{(-\cos z)x^4}{4!}, && \text{for some } z \text{ between } 0 \text{ and } x,\\
\sin x &= x - \frac{x^3}{3!} + \frac{(\sin z)x^5}{5!}, && \text{for some } z \text{ between } 0 \text{ and } x,\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{(\cos z)x^6}{6!}, && \text{for some } z \text{ between } 0 \text{ and } x,
\end{aligned}$$

and so on. The fact that the Taylor Polynomials for $\sin x$, centered at $a = 0$ contain many "zero" terms means that we have a couple of choices for the remainder terms, for instance depending upon whether we wish to consider $x - \frac{1}{3!}x^3$ to be $P_3(x)$ or $P_4(x)$, which are the same for this particular function $\sin x$ with $a = 0$. Note that in each of the cases given above, the z will be between 0 and x, but we should not expect to have the same value for z in each of the above, even if we choose the same value for x.

A general proof of the Remainder Theorem is beyond the scope of this textbook. However, in the exercises the reader is invited to explore how the first case is simply the Mean Value Theorem (Theorem 5.3.1, page 488).

There are several cases where it is useful to find upper bounds (also called *estimates*) on the size of the remainders, which are after all the errors we incur by replacing functions with their Taylor Polynomial approximations.

Example 11.2.3 *Suppose that* $|x| < 0.75$*. In other words,* $-0.75 < x < 0.75$*. Then what is the possible error if we use the approximation* $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$*?*

<u>**Solution:**</u> *Notice that we are asking what is the remainder for the Taylor Polynomial* $P_6(x)$ *(see* Figures 11.3 *and* 11.4, page 760*) where* $f(x) = \sin x$ *and* $a = 0$*, if* $|x| < .75$*. (Recall that, for* $\sin x$*, we have* $P_5 = P_6$ *when* $a = 0$*.) We will use the fact that* $|\sin z| \le 1$ *and* $|\cos z| \le 1$ *no matter what is the value of* z*. Thus*

$$|R_6(x)| = \left|\frac{f^{(7)}(z)(x-0)^7}{7!}\right| = \left|\frac{-\cos z\cdot x^7}{7!}\right| = \frac{1}{7!}|\cos z|\cdot|x|^7 \le \frac{1}{7!}\cdot 1\cdot .75^7 = 0.00002648489.$$

This should be encouraging, since we have nearly five digits of accuracy from a polynomial with only three terms, when our angle is in the range $\pm 0.75 \approx \pm 43°$.

A quick check shows that, to $\sin 0.75 \approx 0.681638760$, $P_6(0.75) \approx 0.6816650391$, and so the difference is $\sin 0.75 - P_6(0.75) \approx -0.000026279$, which is slightly less in absolute value than our error estimate of 0.00002648489.

Example 11.2.4 *Suppose we want to use the approximation* $e^x \approx 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$.

a. *How accurate is this if* $|x| < 5$*?*

b. *How accurate is this if* $|x| < 2$*?*

c. *What if* $|x| < 1$*?*

Solution: *Since the approximating polynomial is* $P_4(x)$ *with* $a = 0$*, we are looking for a bound for*

$$|R_4(x)| = \left|\frac{f^{(5)}(z)x^5}{5!}\right| = \left|\frac{e^z x^5}{5!}\right| = \frac{1}{120}e^z|x|^5.$$

a. $|x| < 5$*: Now* z *is between* 0 *and* x*, and since the exponential function is increasing, the worst possible case scenario is to have the greatest possible value for* z *(which will be* x *or* 0*, which ever is greater). Since the greatest* x *can be is 5, it is safe to use* $e^z < e^5$*. Thus,*

$$|R_4(x)| = \frac{1}{120}e^z|x|^5 < \frac{1}{120}e^5 \cdot 5^5 \approx 3865.$$

Thus we see the exponential is not so well approximated by $P_4(x)$ *for the whole range* $|x| < 5$*.*

b. $|x| < 2$*: Now we have* z *between* 0 *and* x*, and* x *between* -2 *and* 2*, so the the it is only safe to assume* $z < 2$*. Similar to the above, this gives*

$$|R_4(x)| = \frac{1}{120}e^z|x|^5 < \frac{1}{120}e^2 \cdot 2^5 \approx 1.97.$$

We see we have a much better approximation if $|x| < 2$*.*

c. $|x| < 1$ *: Here we can only assume* $z < 1$ *:*

$$|R_4(x)| = \frac{1}{120}e^z|x|^5 < \frac{1}{120}e^1 \cdot 1^5 \approx 0.02265.$$

There are several remarks which should be made about this example.

1. Notice that we "begged the question," since we used calculations of e^5, e^2 and e^1 to approximate the error. This is all correct, but perhaps a strange thing to do since such quantities are exactly what we are trying to approximate with the Taylor Polynomial. But even with this problem, the polynomial is useful because it can be quickly calculated for the whole range $|x| < 5$, 2 or 1 for some application, and the accuracy estimated using only e^5, e^2 or e^1, which are finitely many values.

   One way to avoid this philosophical problem entirely is to use $x > 0 \implies e^x < 3^x$, since 3^x is easier to calculate for the integers we used. For example, $e^5 < 3^5$. However, we need to be somewhat careful, since $x < 0 \implies 3^x < e^x$. Here it would be fine to use 3^x, since we were interested in a larger range of x which included positive numbers. If only interested in $x \in (-5, 0)$, for example, we might use $e^x < 2^x$ there.

2. Note that the error shrinks in a–c, that is as we restrain x so that $|x| < 5, 2, 1$ respectively for two reasons:
   (a) $\left|f^{(5)}(z)\right| = e^z$ shrinks, since z is more constrained.
   (b) $|x|^5$ shrinks, since the maximum possible value of $|x|$ is smaller.

   We benefit from both these factors when we shrink $|x|$.

3. If we truly needed more accuracy for $|x| < 5$, we could take a higher-order Taylor Polynomial, such as $P_{15}(x)$, giving
$$|R_{15}(x)| = \frac{1}{15!}e^z|x|^{15} < \frac{1}{15!}e^5 5^{15} \approx 3.5$$
   This might still seem like a large error, but it is relatively small considering $e^5 \approx 148$. If the error is still too large, consider $P_{20}(x)$, with
$$|R_{20}(x)| = \frac{1}{21!}e^z|x|^{21} < \frac{1}{20!}e^5 5^{20} \approx 0.000277.$$
   When we increase the order of the Taylor Polynomial, we always have the benefit of a growing factorial term $N!$ in the remainder's denominator. As long as the term $\left|f^{N+1}(z)\right|$ does not grow significantly, the factorial will dominate the exponential $|x-a|^{N+1}$.

4. Finally, the exponential will always increase faster as $x \to \infty$ than any polynomial (be it $P_N(x)$ for a fixed N or any other polynomial), and "flatten out" like no polynomial can (excepting the zero polynomial) as $x \to -\infty$, so it is really not a good candidate for approximation very far from zero.

A reasonable question to ask next is how large do we need to have N so that $P_N(x)$ is within a tolerable size. The next examples consider that question.

Example 11.2.5 *Suppose we wish to find a Taylor Polynomial $P_N(x)$ for $f(x) = \cos x$ centered at $x = 0$ so that $P_N(x)$ is within 10^{-7} of $f(x)$ for $|x| < \pi$. What is the range of N which assures this?*

Solution: Here we will use the guaranteed, if seemingly crude, estimate for the size of the error $|R_N(x)|$, in which we again note that $f^{(n)}(z)$ will be of the form $\pm\sin z$ or $\pm\cos z$ regardless of n, and thus $\left|f^{(N)}(z)\right| \le 1$ regardless of z. From this we get
$$|R_N(x)| = \left|\frac{x^{N+1}f^{(N+1)}(z)}{(N+1)!}\right| \le \frac{|x|^{N+1}\cdot 1}{(N+1)!} < \frac{\pi^{N+1}}{(N+1)!}.$$
It is enough that this last term is at most 10^{-7}, but solving such an inequality does not involve elementary algebraic manipulations. Instead we will need experiment with some numerical values, comparing N to $\frac{1}{(N+1)!}\pi^{N+1}$, the latter listed rounded upwards to assure correctness.

$N =$	$\cdots$	15	16	17	18	19	20	$\cdots$
$\lvert R_N\rvert \le$	$\cdots$	5×10^{-6}	8×10^{-7}	2×10^{-7}	3×10^{-8}	4×10^{-9}	6×10^{-10}	$\cdots$

From the chart we see that $N \ge 18$ guarantees that $P_N(x)$ is within 10^{-7} of $\cos x$, for $-\pi < x < \pi$.

We know that the size of the estimate will continue to decrease because with each increment we multiply it by a factor $\pi/(N+1)$, which is less than 1 once $N > 3$.

It is common to use a "worst-case" estimate in computations such as the one above, in that case using $|\pm\sin z|, |\pm\cos z| \le 1$ and $|x| < \pi$. It would be very difficult to find more precise bounds for that range of x.

Example 11.2.6 *Find N so that $P_N(x)$ as an approximation for $f(x) = e^x$ is accurate to within 10^{-5} when $|x| < 2$.*

<u>Solution</u>: *Here we have $f^{(n)}(z) = e^z$ regardless of n, and so for some z between 0 and x (and thus $z \in (-2, 2)$) we have*

$$|R_N(x)| = \frac{e^z|x|^{N+1}}{(N+1)!} \le \frac{e^2 \cdot 2^{N+1}}{(N+1)!}.$$

It is enough that this last quantity be smaller than 10^{-5}. As in the example above, algebraic techniques will not yield an answer directly, and so we will need to perform some numerical experiments. Below we list some values of N and $e^2 \cdot 2^{N+1}/(N+1)!$, the latter rounded upwards and accurate to one significant digit, except for one crucial value, namely $N = 12$.

$N=$	$\cdots$	9	10	11	12	13	14	$\cdots$
$\lvert R_N \rvert \le$	$\cdots$	3×10^{-3}	4×10^{-4}	7×10^{-5}	9.8×10^{-6}	2×10^{-6}	2×10^{-7}	$\cdots$

We see from the chart, and the clear fact that these estimates will continue to decrease, that $N \ge 12$ suffices. Thus $P_{12}(x)$ and higher ordered Taylor Polynomials centered at $a = 0$ will approximate $f(x) = e^x$ within 10^{-5} for $|x| < 2$.

That the estimates on the error in the above example will continue to decrease is again seen by the fact that we can derive the $N = m$ estimate by multiplying the previous estimate and $2/(m + 1)$, which is less than 1 once $m > 1$, and so that next estimate will be smaller.

In the next example we can more directly compute N to give the error bound we desire.

Example 11.2.7 *For $f(x) = \ln x$, assuming $|x - 1| < 0.5$, find N which guarantees that $P_N(x)$ centered at $a = 1$ is within 10^{-5} of $\ln x$.*

<u>Solution</u>: *In* Example 11.1.10, page 763 *we saw that $f^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}$, for $n = 1, 2, 3, \cdots$. (It is a simple enough computation but for space reasons we refer the reader to that example.) We also derived $P_N(x)$ in that example, and can say that for $x > 0$—that is, where all derivatives exist and are continuous (on an interval containing 1), the remainder theorem* (Theorem 11.2.1, page 770) *gives us*

$$\begin{aligned}\ln x &= \underbrace{\sum_{n=1}^{N} \frac{(-1)^{n+1}(n-1)!(x-1)^n}{n!}}_{P_N(x)} + \underbrace{\frac{f^{(N+1)}(z)(x-1)^{N+1}}{(N+1)!}}_{R_N(x)} \\ &= \sum_{n=1}^{N} \frac{(-1)^{n+1}(x-1)^n}{n} + \frac{((N+1)-1)!(-1)^{N+1+1}z^{-(N+1)}(x-1)^{N+1}}{(N+1)!} \\ &= \sum_{n=1}^{N} \frac{(-1)^{n+1}(x-1)^n}{n} + \frac{(-1)^N(x-1)^{N+1}}{(N+1)z^{N+1}}.\end{aligned}$$

So we desire N such that $|x-1| < 0.5 \implies |R_N(x)| < 10^{-5}$. Note that $|x - 1| < 0.5 \iff x \in (0.5, 1.5)$, and since z is between 1 and x we also have $z \in (0.5, 1.5) \implies \frac{1}{z} \in (2/3, 2)$. Thus

$$|R_N(x)| = \left| \frac{(-1)^N(x-1)^{N+1}}{(N+1)} \cdot \left(\frac{1}{z}\right)^{N+1} \right| < \frac{1}{N+1} \cdot (1/2)^{N+1} \cdot 2^{N+1} = \frac{1}{N+1}.$$

A sufficient condition that $|R_N(x)| < 10^{-5}$ is then $\frac{1}{N+1} \le 10^{-5}$, which we can solve easily:

$$\frac{1}{N+1} \le \frac{1}{10^5} \iff 10^5 \le N+1 \iff 99,999 \le N.$$

Thus we can guarantee an error of less than 10^{-5} if $N \ge 99,999$, assuming $|x - 1| < 0.5$.

In the example above we were somewhat lucky that some factors in the remainder estimate canceled. Suppose instead we assume $|x-1|<\frac{3}{4}$. This expands slightly our range of x, so that $-3/4<x-1<3/4$ and so $1/4<x<7/4$, and this has implications regarding our estimate.

If we were to assume $x\in[1,7/4)$, then we have z in the same range (between 1 and x, and therefore in $z\in[1,7/4)$ as well). In such a case $x-1\in[0,3/4)$ and $\frac{1}{z}\in(4/7,1)$, giving our error estimate as

$$|R_N(x)|=\left|\frac{(-1)^N(x-1)^{N+1}}{(N+1)}\cdot\left(\frac{1}{z}\right)^{N+1}\right|\le\frac{\left(\frac{3}{4}\right)^{N+1}}{N+1}\cdot 1^{N+1}=\frac{1}{(N+1)}\cdot\left(\frac{3}{4}\right)^{N+1}.$$

From that estimate we can see clearly that $|R_N(x)|\to 0$ as $N\to\infty$.

Unfortunately, if we have $x\in(1/4,1]$, with z in the same range, we get $x-1\in(-3/4,0]$ and $\frac{1}{z}\in[1,4)$. In this case our most obvious estimate becomes

$$|R_N(x)|=\left|\frac{(-1)^N(x-1)^{N+1}}{(N+1)}\cdot\left(\frac{1}{z}\right)^{N+1}\right|\le\frac{\left(\frac{3}{4}\right)^{N+1}}{N+1}\cdot 4^{N+1}=\frac{3^{N+1}}{N+1},$$

which will grow larger as N grows, and a quick numerical experiment can show this estimate never achieves anything nearly as small as 10^{-5}.

What went wrong in this second case was that our estimate was too crude: we looked at a worst case scenario with x and z separately, when clearly they are coupled. Using completely different techniques, we will see later that, for $x\in(0,2]$, we will have

$$\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n+1}=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots,$$

and so the remainder terms will shrink for a given x, just not "uniformly;" they will tend to shrink faster for x closer to 1, and not in quite the same way for $x\in(0,1)$ as for $x\in(1,2]$.

If we take for granted that the above series expansion is correct for $x\in(0,2]$, then we can use alternating series methods to find the bounds on errors when $x\in[1,2)$. For $x\in(1/4,1]$ we can use a direct comparison test to a geometric series. For instance, if $x=1/4$, the series becomes

$$(-3/4)-\frac{(-3/4)^2}{2}+\frac{(-3/4)^3}{3}-\frac{(-3/4)^4}{4}+\cdots=-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{4}\right)^n.$$

If we call this series $\sum a_n$, then $|a_n|\le(3/4)^n$, from which we can have a geometric series, and from which we have

$$|R_N(1/4)|=\sum_{n=N+1}^{n}\frac{1}{n}(3/4)^n<\sum_{n=N+1}^{n}(3/4)^n=\frac{(3/4)^{N+1}}{1-\frac{3}{4}}=\frac{1}{4}\left(\frac{3}{4}\right)^{N+1}.$$

If we would like to ensure $|R_N(1/4)|<10^{-5}$, we would solve (noting that $\ln(3/4)<0$):

$$\begin{aligned}\frac{1}{4}(3/4)^{N+1}<10^{-5}&\implies\quad (3/4)^{N+1}<4\times 10^{-5}\\&\implies (N+1)\ln(3/4)<\ln\left(4\times 10^{-5}\right)\\&\implies\quad N>\frac{\ln\left(4\times 10^{-5}\right)}{\ln(3/4)}-1\\&\implies\quad N>34.2,\end{aligned}$$

and so we would take $N\ge 35$ to ensure our error is within 10^{-5}, in using $P_N(1/4)$ to approximate $\ln\frac{1}{4}$.

Exercises

For Exercises 1–6, write the function in the form $f(x) = P_N(x) + R_N(x)$, where $P_N(x)$ and $R_N(x)$ are written out explicitly (see Examples 11.2.1–11.2.2).

1. $f(x) = \sin x$, $a = \pi$, $N = 5$
2. $f(x) = \sqrt{x}$, $a = 1$, $N = 3$
3. $f(x) = \frac{1}{x}$, $a = 10$, $N = 4$
4. $f(x) = e^x$, $a = 0$, $N = 9$.
5. $f(x) = \sec x$, $a = \pi$, $N = 2$.
6. $f(x) = \ln x$, $a = e$, $N = 3$.
7. Explain why the series below converges, and to the limit claimed below. (Hint: apply a hierarchy of functions reasoning to $R_N(x)$.)

   $$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x.$$

8. Many physics problems take advantage of the approximation $\tan x \approx x$ for $|x|$ small.
   (a) Conjecture on where this approximation comes from, from a purely mathematical standpoint.
   (b) Estimate the error for each of the three cases $|x| < 1$, 0.1 and 0.01. (Feel free to use a calculator to find upper bounds for the error.)
9. Suppose we wanted to find a Taylor Polynomial for $f(x) = \sin x$, centered at $a = 0$, with accuracy $|R_N(x)| \leq 10^{-10}$ valid for $-2\pi \leq x \leq 2\pi$. Find N for the lowest-order Taylor Polynomial $P_N(x)$ which guarantees that accuracy for that interval, based upon the remainder formula. (This may require some numerical experimentation with the estimates.)
10. Repeat the previous problem, but for $f(x) = e^x$ and the interval $|x| \leq 10$.
11. Show that the Remainder Theorem for $P_0(x)$ is really just the Mean Value Theorem, Theorem 5.3.1, page 488. (Hint: $z = \xi$.)

11.3 Taylor/Maclaurin Series

Now we come to the heart of the matter. Basically, the *Taylor Series* of a function f which has all derivatives f', f'', $\cdots$ existing at a, is the series we get when we let $N \to \infty$ in the expression for $P_N(x)$. The Taylor Series equals the function if and only if the remainder terms shrink to zero as $N \to \infty$:

11.3.1 Checking Validity of Taylor Series

Theorem 11.3.1 *Recalling the definition of the remainder $R_N(x) = f(x) - P_N(x)$, where $P_N(x)$ is an Nth-order Taylor Polynomial for $f(x)$ centered at some number $a \in \mathbb{R}$, we have*

$$\lim_{N\to\infty} R_N(x) = 0 \iff f(x) = \lim_{N\to\infty} P_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!},$$

that is,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \iff \lim_{N\to\infty} R_N(x) = 0.} \tag{11.24}$$

Proof: First we prove $(\Longleftarrow)$. Assume $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$. Then

$$R_N(x) = f(x) - P_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \longrightarrow 0 \qquad \text{as } N \to \infty.^{12}$$

Next we prove $(\Longrightarrow)$. Assume $R_N(x) \longrightarrow 0$ as $N \to \infty$. Then

$$N \to \infty \implies \underbrace{f(x) - R_N(x)}_{\displaystyle\downarrow \atop \displaystyle f(x)} = \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}}_{\displaystyle\downarrow \atop \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}},$$

which shows $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$, q.e.d.

The series we get from Theorem 11.3.1 above has the following name:

Definition 11.3.1 *Supposing that all derivatives of $f(x)$ exist at $x = a$, the series given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \tag{11.25}$$

is called the **Taylor Series** *of $f(x)$ centered at $x = a$.*

[12] Recall that the "tail end" $\sum_{n=N+1}^{\infty} b_n$ of a convergent series $\sum_{n=0}^{\infty} b_n$ shrinks to zero as $N \to \infty$. This "tail end" is represented by $S - S_N$, where S is the full series and S_N the Nth partial sum. Recall $S_N \to S \implies S - S_N \to 0$.

Thus, Theorem 11.3.1 can be restated that the Taylor Series will equal the function if and only if the remainders R_N from the Taylor Polynomials shrink to zero as $N \to \infty$.

A special case of the Taylor Series is the case $a = 0$. This occurs often enough it is given its own name:

Definition 11.3.2 *If a Taylor Series is centered at $a = 0$, it is called a* **Maclaurin Series**.[13] *In other words, if all derivatives of $f(x)$ exist at $x = 0$, the function's Maclaurin Series is given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}. \tag{11.26}$$

The partial sums are sometimes called *Maclaurin Polynomials.* Note that both Taylor Series, and the special case of the Maclaurin Series, are in fact *power series,* introduced in (11.1), page 748.

In the following propositions, we will consider several Taylor and Maclaurin Series, and show where they converge based on Theorem 11.3.1 (which we restated in (11.24)) and other observations. Showing that $R_N \to 0$ in some cases will require creativity, but once we establish this fact for a series we will assume it from then on, as with those below:

Proposition 11.3.1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x \in \mathbb{R}. \tag{11.27}$$

Proof: Recall that $(\forall x \in \mathbb{R})(\forall n \in \{0, 1, 2, 3, \cdots\}) \left[f^{(n)}(x) = e^x\right]$. Thus, for any fixed $x \in \mathbb{R}$, we have

$$R_N(x) = \frac{f^{(N+1)}(z)x^{N+1}}{(N+1)!} = e^z \frac{x^{N+1}}{(N+1)!}.$$

Now z, while depending upon N and x, is nonetheless between x and 0, and so by the increasing nature of the exponential function we have $e^z < \max\{e^0, e^x\}$, and is thus bounded by $M = \max\{e^0, e^x\}$ (which depends only upon x, and not upon N or z). Thus

$$|R_N(x)| = e^z \frac{|x|^{N+1}}{(N+1)!} \le M \cdot \frac{|x|^{N+1}}{(N+1)!} \longrightarrow M \cdot 0 = 0 \qquad \text{as } N \to \infty,$$

since the numerator grows geometrically (or shrinks geometrically) as N increases, while the denominator grows as a factorial. Recall that the factorial will dominate the exponential regardless of the base (in this case $|x|$) as $N \to \infty$. Since we showed $R_N(x) \to 0$ (be showing the equivalent statement $|R_N(x)| \to 0$) for any x, by Theorem 11.3.1, page 777, (11.27) follows, q.e.d.[14]

It was important to notice that e^z was bounded *once x was chosen,* and that the bound is going to change with each x. The upshot is that for a given x, $R_N(x) \to 0$ but for two different x-values, this convergence of the remainder to zero—and thus the convergence of the Taylor series to the value $f(x)$—can occur at very different rates.

[13] Named for Colin Maclaurin, 1698–1746, a Scottish mathematician. He was apparently aware of Taylor Series, citing them in his work, but made much creative use of those centered at $a = 0$ and so eventually was honored to have the special case named for him.

[14] A clever way to prove more directly that $M \cdot \frac{|x|^{N+1}}{(N+1)!} \longrightarrow 0$ as $N \to \infty$ would be to show that the series $\sum_{n=1}^{\infty} M \cdot \frac{|x|^{n+1}}{(n+1)!}$ converges, which can be proved using a fairly straight-forward Ratio Test. This would show that the "nth term" approaches zero in the limit, since $\sum a_n$ converges $\implies a_n \to 0$, which is the contrapositive of the nth-term test for divergence (NTTFD, Section 10.2).

Also, absolute values were not needed around the e^z-term, since it will always be positive. Finally, to accommodate the case $x = 0$, we substituted the weaker "$\le$" for the "$<$" in our inequality above. For the case $x = 0$, a careful look at the P_N show $R_N(0) \equiv 0$. This is because 0 is where the series is centered. (Recall $P_N(a) = f(a)$.)

Proposition 11.3.2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n+1)!} \qquad \text{for all } x \in \mathbb{R}. \tag{11.28}$$

Proof: Now $f^{(n)}(x)$ is of the form $\pm \sin x$ or $\pm \cos x$, which means it is bounded absolutely by 1, i.e., $\left|f^{(n)}\right| \le 1$. Thus for any given $x \in \mathbb{R}$ we have

$$|R_N(x)| = \left|\frac{f^{(N+1)}(z)x^{N+1}}{(N+1)!}\right| \le 1 \cdot \frac{|x|^{N+1}}{(N+1)!} \to 1 \cdot 0 = 0 \text{ as } N \to \infty.$$

Again this is because the geometric term $|x|^{N+1}$ is a lower order of growth (and may even decay if $x \in (-1, 1)$) than the factorial $(N + 1)!$. Thus, according to Theorem 11.3.1, (11.28) follows, q.e.d.

A nearly identical argument shows that

Proposition 11.3.3

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{for all } x \in \mathbb{R}. \tag{11.29}$$

Not all Taylor series converge to the function for all of $x \in \mathbb{R}$. Furthermore, it is often difficult to prove $R_N(x) \to 0$ when other techniques can give us that the Taylor Series in fact converges. For example, consider the following:

Proposition 11.3.4

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n. \qquad \text{for all } x \in (-1, 1). \tag{11.30}$$

Though we can calculate the series directly (see Exercise 6, page 768), Equation (11.30) is obvious if we read it backwards, realizing that the series is geometric with first term $\alpha = 0$ and ratio x (as in Theorem 10.1.1, page 707). Moreover, the series converges when $|x| < 1$ and diverges otherwise, from what we know of geometric series. From these observations, Proposition 11.3.4 is proved. We will see in Section 11.5 that many of the connections and manipulations we would like to make with Taylor/Maclaurin Series are legitimate. In fact, these methods are often much easier than computations from the Taylor Series definition. Consider Proposition 11.30. The actual remainder is of the form

$$R_N(x) = \frac{(N+1)!\,(1-z)^{-(N+2)}\,x^{N+1}}{(N+1)!} = \frac{x^{N+1}}{(1-z)^{N+2}}. \tag{11.31}$$

We know z is between 0 and x, but without knowing more about where, it is not obvious that the numerator in our simplified R_N will decrease in absolute size faster than the denominator. We will not belabor the point here, but just conclude that resorting to using facts about geometric series is a much simpler approach than attempting to prove $R_N(x) \to 0$ when $|x| < 1$. (See also the discussion after Example 11.2.7, page 774.)

Another interesting Taylor Series is the following:

Proposition 11.3.5 The following is the Taylor Series for $\ln x$ centered at $x = 1$:

$$\begin{aligned}\ln x &= 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \qquad (11.32)\\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \qquad \text{for } |x-1| < 1, \text{ i.e., } x \in (0,2).\end{aligned}$$

We found P_N in Example 11.1.10, page 763. A proof that (11.32) is valid for $(1/2, 2)$ in which one shows $R_N(x) \to 0$ in that interval is left as Exercise **??**. The proof that the series is valid for all of $(0,2)$ is left as an exercise in Section 11.4, and again in Section 11.5 after other methods are available. Finally, it is not difficult to show that the series also converges at $x = 2$ (by the Alternating Series Test) and so the series in fact converges for all of $(0,2]$, so that by Abel's Theorem, introduced later as Theorem 11.4.1, page 783, the series converges to $\ln x$ in all of $(0,2]$.

11.3.2 Techniques for Writing Series using Σ-Notation

There are some standard tricks for writing formulas to achieve the correct terms in the summation. For instance, inserting a factor $(-1)^n$ or $(-1)^{n+1}$ to achieve the alternation of sign, depending upon whether the first term carries a "+" or "−." We also pick up only the odd terms in the $\sin x$ expansion by using the $2n+1$ factors, and get the evens in the $\cos x$ using the $2n$. Perhaps the best way to get comfortable with these manipulations is to write out a few terms of the summations on the right of (11.28), (11.29) and (11.32). For example, we can check the summation notation is consistent in (11.28) as follows (note we define $(-1)^0 = 1$ for simplicity):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &= \underbrace{\frac{(-1)^0 x}{1!}}_{n=0 \text{ term}} + \underbrace{\frac{(-1)^1 x^3}{3!}}_{n=1 \text{ term}} + \underbrace{\frac{(-1)^2 x^5}{5!}}_{n=2 \text{ term}} + \underbrace{\frac{(-1)^3 x^7}{7!}}_{n=3 \text{ term}} + \cdots\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

However it would also be perfectly legitimate to instead write the above series as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} = \underbrace{x}_{n=1} \underbrace{-\frac{x^3}{3!}}_{n=2} + \underbrace{\frac{x^5}{5!}}_{n=3} + \cdots.$$

We see that we get the correct alternation of sign and the correct powers and factorials from our Σ-notation in both cases. Also note that $2n+1$ and $2n-1$ both give odd numbers regardless of $n \in \{0, 1, 2, 3, \cdots\}$ (since $2n$ is even and 1 is odd), and so it becomes simply a matter of whether the series produces the correct terms to be summed.

Example 11.3.1 *Write the following in a compact* Σ*-notation.*

a. $\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots$

b. $x-\frac{x^3}{2}+\frac{x^5}{4}-\frac{x^7}{8}+\cdots$

c. $-\frac{x^2}{1}+\frac{x^4}{1\cdot 3}-\frac{x^6}{1\cdot 3\cdot 5}+\frac{x^8}{1\cdot 3\cdot 5\cdot 7}+\cdots$

Solution:

a. *We see the powers of x are increasing by 1, while the denominators are increasing by 2 with each new term added. The summations will appear different depending upon where the indices begin. Here are two possibilities, though the first is more obvious:*

$$\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots=\sum_{n=1}^{\infty}\frac{x^n}{2n},$$
$$\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots=\sum_{n=0}^{\infty}\frac{x^{n+1}}{2(n+1)}.$$

b. *Here we have only odd powers of x. It is worth noting that therefore the powers of x are increasing by 2. We have alternating factors of ± 1. In the denominator we have powers of 2. This can be written in the following ways (among others):*

$$x-\frac{x^3}{2}+\frac{x^5}{4}-\frac{x^7}{8}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,x^{2n-1}}{2^{n-1}},$$
$$x-\frac{x^3}{2}+\frac{x^5}{4}-\frac{x^7}{8}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n}\,x^{2n+1}}{2^{n}}.$$

c. *The powers of x here are all even, hence increasing by 2 with each step. There is also alternation of signs. Finally the denominators are products of odd numbers, similar to a factorial but skipping the even factors. In a case like this, we allow for a more expanded writing of the pattern in the* Σ*-notation. We write the following:*

$$-\frac{x^2}{1}+\frac{x^4}{1\cdot 3}-\frac{x^6}{1\cdot 3\cdot 5}+\frac{x^8}{1\cdot 3\cdot 5\cdot 7}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^n\,x^{2n}}{1\cdot 3\cdot 5\cdots(2n-1)},$$
$$-\frac{x^2}{1}+\frac{x^4}{1\cdot 3}-\frac{x^6}{1\cdot 3\cdot 5}+\frac{x^8}{1\cdot 3\cdot 5\cdot 7}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,x^{2n+2}}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

If we had some compelling reason, we might even begin at $n=3$, for instance:

$$-\frac{x^2}{1}+\frac{x^4}{1\cdot 3}-\frac{x^6}{1\cdot 3\cdot 5}+\frac{x^8}{1\cdot 3\cdot 5\cdot 7}+\cdots=\sum_{n=3}^{\infty}\frac{(-1)^n\,x^{2n-4}}{1\cdot 3\cdot 5\cdot(2n-5)}.$$

It is understood that the denominator contains all the odd factors up to $(2n-1)$, $(2n+1)$ or $(2n-5)$, depending on the form chosen. Though the first two terms do not contain all of $1\cdot 3\cdot 5$, we put in those three numbers into the Σ*-form to establish the pattern, which is understood to terminate at $(2n-1)$ or $(2n+1)$ even if that means stopping before 3 or 5.*

Whenever there is alternation, expect $(-1)^n$ or $(-1)^{n+1}$ or similar factors to be present. An increase by 2 at each step is achieved by $(2n+k)$, where k is chosen to get the first term correct. An increase by 3 would require a $(3n+k)$. With some practice it is not difficult to translate a series written longhand, but with a clear pattern, into Σ-notation. (For series of constants, we also used $(-1)^n = \cos(n\pi)$.)

Exercises

For Exercises 1–4, show that the Σ-notation for the series below (namely those in (11.27), (11.29), (11.30), and (11.32)) expands to the respective series pattern given on the left.

1. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
2. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.
3. $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$.
4. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$ $= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.
5. Rewrite the power series for $\sin x$ centered at $a = 0$, but in such a way that it starts with $n = 1$.
6. Do the same for $\cos x$.

For Exercises 7–11 write each series using Σ-notation: first starting with $n = 1$, and then starting with $n = 0$.

7. $1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \cdots$
8. $x^2 + \frac{x^4}{4} + \frac{x^6}{9} + \frac{x^8}{16} + \frac{x^{10}}{25} + \cdots$
9. $\frac{x}{2} - \frac{x^2}{2 \cdot 4} + \frac{x^3}{2 \cdot 4 \cdot 6} - \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$
10. $\frac{x}{1 \cdot 1} + \frac{x^3}{3 \cdot 1 \cdot 2} + \frac{x^5}{5 \cdot 1 \cdot 2 \cdot 3}$ $+ \frac{x^7}{7 \cdot 1 \cdot 2 \cdot 3 \cdot 4} + \cdots$
11. $\frac{2}{4} - \frac{4x}{7} + \frac{6x^2}{10} - \frac{8x^3}{13} + \cdots$
12. Prove Proposition 11.29, page 779.
13. Prove that the remainder
$$R_N(x) = \frac{x^{N+1}}{(1-z)^{N+2}}$$
from (11.31) does approach zero as $N \to \infty$ for the case $x \in (-1, 0)$. Note that it is enough to show $|R_N(x)| \to 0$. (Hint: In what interval is $1 - z$ in this case?)

11.4 General Power Series and Interval of Convergence

11.4.1 Definition of General Power Series

While most of our familiar functions can be written as power series, meaning form (11.1), page 748 repeated here as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \tag{11.33}$$

there are many functions which *must* be written as series (for instance, $\int e^{x^2}\,dx$). In some sense, there are more power series than "nice" functions (usual combinations of powers, trigonometric, logarithmic and exponential functions) which also have power series representations. It is therefore interesting to study power series without reference to functions they may or may not have been derived from.

When we are given such a function represented by a power series (11.33), it is clear that $a_0 = f(a)$, but less clear that $a_1 = f'(a)$, or $a_2 = \frac{1}{2!}f''(a)$, etc., which is what happens with Taylor Series where we know the function f and how to compute its derivatives. Even finding $f'(a)$ is somewhat difficult because, as we know from the definition of the derivative,

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x},$$

and it is perhaps an understatement to note that it is not immediately clear how to compute that limit from (11.33). Note how $f(x)$ must be defined—and continuous—in an open interval containing a or this limit which defines the derivative cannot exist. Fortunately, given any function defined by a power series (11.33), we are guaranteed to have only certain cases for its domain. We rely on the following very useful, and eventually intuitive, result.

11.4.2 Abel's Theorem

Theorem 11.4.1 (Abel's Theorem[15]): *A power series of form* (11.33) *will converge at $x = a$ only and diverge elsewhere, or converge absolutely in an open interval $x \in (a-R, a+R)$ and diverge outside the closed interval $[a,b]$ with the same endpoints, i.e., diverge for $x \in (-\infty, a-R) \cup (a+R, \infty)$. If the power series also converges at an endpoint $a-R$ or $a+R$, it will be continuous to the endpoint from the side interior to the interval.*

Definition 11.4.1 *The number R above is called the* **radius of convergence** *of* (11.33)*. We say $R = 0$ if the power series converges at a only. It is quite possible that $R = \infty$. in which case the power series converges on all of $\mathbb{R}$. Otherwise, $R \in (0,\infty)$ is nonzero and finite and the power series*

a. converges for $|x-a| < R$, and

b. diverges for $|x-a| > R$.

[15] Named for Niels Henrik Abel, 1802–1829, a Norwegian mathematician most notable for founding Group Theory, on the way to proving the impossibility of solving the general fifth-degree polynomial equations by a formula with radicals, unlike second-degree (quadratic), third-degree (cubic) or fourth-degree (quartic) equations, which do have formulas for their solutions. While solving the general quadratic equation (using the "quadratic formula") is basic enough, the third-degree and fourth-degree "formulas" are much more involved, and Abel dispelled any hope that such formulas exist for higher-degree polynomial equations. Here we are interested in his (very different) theorem on the convergence of power series.

Because of this result above it makes sense to talk of the *interval of convergence* of a power series. Its form will be one of the following, depending upon the specific series:

$$\{a\}, (-\infty,\infty), (a-R,a+R), [a-R,a+R], [a-R,a+R), (a-R,a+R].$$

For the Taylor Series for e^x, $\sin x$ and $\cos x$, we know this interval of convergence is simply $(-\infty,\infty)=\mathbb{R}$, and so we say $R=\infty$ in those cases. In contrast, the Taylor Series for $\ln x$ centered at $x=1$ converges at least in $|x-1|<1$, as shown in Proposition 11.3.5, page 780.

$$\begin{aligned}\ln x &= 1(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots\\ &=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} \qquad \text{for (at least) } |x-1|<1, \text{ i.e., } x\in(0,2).\end{aligned}$$

While Abel's Theorem does not state whether or not a series converges at the endpoints, it is not difficult to see that the series above converges for $x\in(0,2]$, i.e., converges at the right endpoint $x=2$ (by the Alternating Series test), and diverges at the left endpoint $x=0$ (since there it is the harmonic series). Abel's theorem then *does* say that the series will then be left-continuous at $x=2$, and since so is $\ln x$, they must agree at that point. Thus the series *equals* $\ln x$ on all of $x\in(0,2]$.

11.4.3 Finding the Interval and Radius of Convergence

In most cases, the Ratio and Root Tests are the tools used to find the *interval of convergence* for a given power series. From there we usually observe the actual *radius*, as it is basically half the length of the interval, or equivalently, the distance from the center a to one of the endpoints $a\pm R$. For most cases we will use the Ratio Test.

Example 11.4.1 *Find the interval of convergence for the series* $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

<u>Solution:</u> *Actually we know this series, and that it converges to e^x for all $x\in\mathbb{R}$, so the interval is $(-\infty,\infty)=\mathbb{R}$, and thus $R=\infty$. We deduced this from the form of the remainder $R_N=\frac{1}{(N+1)!}e^z x^{N+1}$.*

But how would we determine where it converges without knowing the form of the remainder? The key here is to use the Ratio Test for an arbitrary x. First we write

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}\equiv\sum_{n=0}^{\infty}u_n.$$

Most textbooks introduce u_n above for convenience in applying the Ratio Test. (The reader should feel free to skip that step where relevant.) Next we calculate

$$\begin{aligned}\rho=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\cdot\frac{n!}{(n+1)!}\\ =\lim_{n\to\infty}|x|\cdot\frac{n!}{n!(n+1)}=\lim_{n\to\infty}|x|\cdot\frac{1}{n+1}=0 \qquad \text{for every } x\in\mathbb{R}.\end{aligned}$$

Recall that the series will converge absolutely if $\rho<1$, and we in fact for this case have $\rho=0$ for every real x. Since $\rho=0<1$ regardless of $x\in\mathbb{R}$, the series converges absolutely on all of $\mathbb{R}=(-\infty,\infty)$, which gives the interval of convergence. (Here we take the radius to be $R=\infty$.)

It is arguably easier to find that the series for e^x converges (absolutely) for all x by using the Ratio Test as above, than using the form of the remainder $R_N(x)$ and showing $R_N(x) \to 0$ as $N \to \infty$. Indeed, the Ratio Test is usually the preferred method for finding where a given power series converges.

Example 11.4.2 *Find the interval and radius of convergence for the series* $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-5)^n}{2n-1}$.

Solution: *Just as above,*

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\dfrac{2^{n+1}(x-5)^{n+1}}{2(n+1)-1}\right)}{\left(\dfrac{2^n(x-5)^n}{2n-1}\right)}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{2n-1}{2(n+1)-1}\cdot\left|\frac{(x-5)^{n+1}}{(x-5)^n}\right|$$
$$= \lim_{n\to\infty} 2\cdot\frac{2n-1}{2n+1}\cdot|x-5| = 2\cdot 1\cdot|x-5|.$$

Remember that the x in the line above is a constant as far as the limit goes (since the limit is in n). To find the region where $\rho < 1$ we simply solve

$$\underbrace{2|x-5|}_{\rho} < 1 \iff |x-5| < \frac{1}{2} \iff -1/2 < x-5 < 1/2$$

$$\iff 9/2 < x < 11/2.$$

Thus we know for a fact that the series converges absolutely for $x \in (9/2, 11/2)$. A similar calculation gives us divergence in $(-\infty, 9/2)\cup(11/2,\infty)$, and we usually do not bother repeating the calculations to see this. The only question left is what happens at the two boundary points.

$\underline{x = 9/2}$:

$$\sum_{n=0}^{\infty}\frac{2^n(9/2-5)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{2^n(-1/2)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{2^n\left(\frac{1}{2}\right)^n(-1)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n-1}.$$

The resultant series converges by the Alternating Series Test (alternates, terms shrink in absolute size monotonically to zero). Thus the series does converge at the left endpoint $x = 9/2$.

$\underline{x = 11/2}$:

$$\sum_{n=0}^{\infty}\frac{2^n(11/2-5)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{2^n(1/2)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{2^n\left(\frac{1}{2}\right)^n}{2n-1} = \sum_{n=0}^{\infty}\frac{1}{2n-1}.$$

This series diverges (limit-comparable to the harmonic series $\sum \frac{1}{n}$). Thus the power series diverges at this endpoint.

The conclusion is that the interval of convergence is $x \in [9/2, 11/2)$.

Note that the center of the interval is $\left(\frac{9}{2}+\frac{11}{2}\right)/2 = 10/2 = 5$, and so the "center" being at $a = 5$ (which we can also read from the original summation notation), we see that the interval extends by $1/2$ to both right and left of the center, so $R = 1/2$. We could also find this by computing half of the length of the interval, i.e., $\left(\frac{11}{2}-\frac{9}{2}\right)/2 = (2/2)/2 = 1/2$.

Example 11.4.3 *Find the radius and interval of convergence for* $\sum_{n=1}^{\infty}(nx)^n$.

Solution: This would be a difficult series to analyze with the Ratio Test (as the reader is invited to attempt), and the Root Test seems more appropriate. Here we use $\rho = \rho_{\text{root}}$*, and get*

$$\rho = \lim_{n\to\infty} \sqrt[n]{|(nx)^n|} = \lim_{n\to\infty} \left[|nx|^n\right]^{1/n} = \lim_{n\to\infty} |nx| = \begin{cases} 0 & \text{if} \quad x = 0, \\ \infty & \text{if} \quad x \neq 0. \end{cases}$$

Thus the series diverges for $x \neq 0$*, and the "interval" of convergence is simply* $[0,0] = \{0\}$*, and the radius is simply* $R = 0$*.*

In the example above, except at $x = 0$ the terms all increased in size rather than shrinking to zero. In effect, $(nx)^n = n^n x^n$ is the product of a very rapidly growing n^n with an exponentially (or " geometrically") growing x^n if $|x| > 1$, and exponentially shrinking x^n if $|x| < 1$. However, even the case of the exponential shrinkage cannot overcome the rapid growth of n^n, which then dominates the behavior of $n^n x^n = (nx)^n$. Cases where $R = 0$ are not the most commonly studied, but they do occur and anyone dealing with series has to be aware of them.

Also notable from this latest example is that there are cases where the Root Test is preferable to the Ratio Test. In fact, as we noted when these two tests were first introduced in Section 10.5, there is even some overlap. Recall that both tests were modeled on comparisons to the Geometric Series $\sum a_0 r^n$.

It should therefore, upon reflection, be no surprise that the Ratio and Root Tests are called upon in many cases to determine where a power series converges. After all, such series $\sum a_n x^n$ can be interpreted to be variations of geometric series.

11.4.4 Taylor/Power Series Connection

There is a nice connection between Taylor and Power Series centered at a given point a. In short, they are the same, assuming there is an interval (of "wiggle room"), around the center of the series, on which the power series converges to the function. We introduce this connection here initially for the reader to note for future reference, and then greatly expand its scope and application in Section 11.5.

To see this connection we first need the following theorem, which we state without proof:

Theorem 11.4.2 Manipulations with Power Series: *Suppose we are given a function defined by a power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{11.34}$$

which converges in some open interval $|x - a| < R$*, where* $R > 0$*.*

$$f^{(n)}(a) = n!\, a_n. \tag{11.35}$$

Note that (11.35) is equivalent to $a_n = \frac{f^{(n)}(a)}{n!}$, so the coefficients of the power series will be exactly the same as those of the Taylor Series, assuming the power series is valid in some open interval $|x - a| < R$, some $R > 0$.

In advanced calculus, functions which can be represented in $|x - a| < R$ by a convergent power series are given a special name:

Definition 11.4.2 *A function* $f(x)$ *which has a power series representation* (11.34) *converging in some open interval* $|x - a| < R$ *(for some* $R > 0$*) is called* **real-analytic** *in that interval.*

Equivalently, a function is real-analytic on an open interval $|x-a|<R$ if and only if its Taylor Series converges to the function in the same interval.

There is a very rich and beautiful theory of real-analytic functions which is beyond the scope of this text. It is a theory which has a remarkably simple extension to functions of a complex variable

$$z \in \mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}, \qquad i = \sqrt{-1}.$$

This may seem a complication, but the theory is often simplified by this generality, after which the real-analytic results follow from the complex theory. In fact the term *radius of convergence* comes from the complex-analytic theory, where the complex values z for which $\sum a_n(z-a)^n$ converges lie in a disk of radius R inside the complex plane $\mathbb{C}$. Such are topics for advanced calculus or complex analysis courses, usually at the senior or graduate levels. However, we will explore some aspects of the theory suitable for this level of textbook in Section 11.6.

Exercises

For Exercises 1–13, find

(a) the interval of convergence, including endpoints where applicable, and

(b) the radius of convergence.

1. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n\sqrt{n}}$.

2. $f(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$. See Example 11.5.5, page 792.

3. $\ln x = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$. See Proposition 11.3.5, page 780.

4. $f(x) = \displaystyle\sum_{n=0}^{\infty} nx^n$.

5. $f(x) = \displaystyle\sum_{n=0}^{\infty} n!\,x^n$.

6. $e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. See Proposition 11.27, page 778.

7. $f(x) = \displaystyle\sum_{n=2}^{\infty} \frac{(x+1)^n}{(\ln n)^n}$.

8. $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^n}$.

9. $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2(x-5)^n}{(2n)!}$.

10. $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n^2 \cdot 10^n}$.

11. $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

12. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!\,(x-3)^n}{n^n}$.

13. $f(x) = \displaystyle\sum_{n=2}^{\infty} \frac{3^n(x+2)^n}{\ln n}$.

14. Assume for a moment that all our work with Taylor Series can be generalized to the complex plane $\mathbb{C}$. Note that $i=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, etc. Use all this and known Maclaurin Series to prove *Euler's Identity:*

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (11.36)$$

Note that this implies that $e^{i\pi}=-1$, or more interestingly $e^{i\pi}+1=0$ an often-cited, beautifully compact equation relating four of the most important numbers in mathematics.

15. Use (11.36) and the facts that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, to show the following relationship between trigonometric and hyperbolic functions (see Exercise 6, page 797):

    (a) $\cos x = \cosh(ix)$;

    (b) $\sin x = \dfrac{1}{i}\sinh(ix)$.

16. Use Exercises 14 and 15 to prove the following trigonometric identities:

    (a) $\sin^2 x + \cos^2 x = 1$;

    (b) $\sin 2x = 2\sin x\cos x$;

    (c) $\cos 2x = \cos^2 x - \sin^2 x$;

    (d) $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

11.5 Valid Manipulations with Taylor/Power Series

Taylor Series are very robust in the sense that most algebra and calculus-based methods for constructing functions from other functions translate to series. Some care must be taken to ensure a proper interval of convergence results, but even that consideration follows fairly easily from the process.

Here we will look at both algebraic and calculus-based manipulations of Taylor Series. In so doing, it should become clear that such methods are often preferable to brute-force computations from the definition of Taylor Series. Furthermore, some functions *require* us to use series representations rather than previous types of formulas, and such manipulations are sometimes quite helpful in finding representations from known functions.

11.5.1 Algebraic Manipulations

We begin this subsection with a bit of theory which is mostly straightforward, and somewhat interesting, but we will be somewhat brief with it here so it will not become a distraction. The main theorem is the following:

Theorem 11.5.1 *If there are two power series representations of a function $f(x)$ which are valid within an open interval surrounding the center a, i.e., if there exists $\delta > 0$ such that $x \in (a-\delta, a+\delta)$ implies*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k = \sum_{k=0}^{\infty} b_k (x-a)^k,$$

then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, and so on.

The theorem is stating that any two power series representations (including a Taylor Series) of the same function with the same center must really be the same series. In other words, any power series representation for a function is unique at each point where it is valid. From (11.35) of Theorem 11.4.2, page 786 we then also get that *any valid power series representation of a function within an open interval is also its Taylor Series with the same center.*

Example 11.5.1 *Use the Maclaurin Series for e^x to calculate the Maclaurin Series for e^{x^2}.*

<u>**Solution:**</u> *We simply replace x with x^2 in the series for e^x.*

$$\begin{aligned} & e^x &=& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ \implies & e^{x^2} &=& 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \\ \iff & e^{x^2} &=& 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}. \end{aligned}$$

valid for all $x \in \mathbb{R}$, since the original series was valid everywhere (and $x \in \mathbb{R} \implies x^2 \in \mathbb{R}$, and can therefore be inputted to the original series for e^x).

A few comments are in order regarding how the theory implies the validity of the series representation for e^{x^2} above. Because the series $e^x = \sum \frac{x^n}{n!}$ is true for any $x \in \mathbb{R}$, we could also use our abstract function notation to write

$$e^{(\ )}=\sum_{n=0}^{\infty}\frac{(\ )^n}{n!}=1+(\ )+\frac{(\ )^2}{2!}+\frac{(\ )^3}{3!}+\frac{(\ )^4}{4!}+\cdots,$$

and any input $(\ )\in\mathbb{R}$ on the left can be equivalently input on the right, and the values of the outputs will be the same. That should also be true of the value of the output if the input is x^2:

$$e^{\left(x^2\right)}=\sum_{n=0}^{\infty}\frac{\left(x^2\right)^n}{n!}=1+\left(x^2\right)+\frac{\left(x^2\right)^2}{2!}+\frac{\left(x^2\right)^3}{3!}+\frac{\left(x^2\right)^4}{4!}+\cdots,$$

but then we can simplify each of the terms within the summation on the right-hand side, as we did in Example 11.5.1 above. Asking if a series representation makes sense at actual *values*, and observing the affirmative answer, helps us to see the validity of the new series. (We will argue similarly in subsequent examples.)

Furthermore, we can also dispel any doubt that this is superior to calculating such a Taylor Series from the original definition of Taylor Series. Recall that we would need formulas for $f^{(n)}(x)$ to compute $f^{(n)}(0)$ to compute the Taylor Coefficients. The first two are easy enough: $f(x)=e^{x^2}$; $f'(x)=2xe^{x^2}$. For f'', we need a product rule and another chain rule: $f''(x)=2x\left(2xe^{x^2}\right)+2e^{x^2}=2e^{x^2}(4x+1)$. Next we would need another product rule and a chain rule to find f''', for which simplifying would be even more difficult. By then, we would likely conclude the *algebraic* method above is superior. Similarly it is not difficult to compute the following:

Example 11.5.2 *Find the Maclaurin Series for $f(x)=x^3\sin 2x$.*

Solution: *We will construct this series in stages, beginning with the series for $\sin x$.*

$$\begin{aligned}
&\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!},\\
\Longrightarrow\quad &\sin 2x=(2x)-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}\\
&\qquad=2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}x^{2n+1}}{(2n+1)!},\\
\Longrightarrow\ &x^3\sin 2x=x^3\left(2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\cdots\right)=x^3\left(\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}x^{2n+1}}{(2n+1)!}\right)\\
&\qquad=2x^4-\frac{8x^6}{3!}+\frac{32x^8}{5!}-\frac{128x^{10}}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}x^{2n+4}}{(2n+1)!}.
\end{aligned}$$

valid for all $x\in\mathbb{R}$.

Again it is not difficult to see that the series should be valid at any given value for $x\in\mathbb{R}$, since we can place any value into the series for $\sin(\ \ )$, including the value $2x$ (which is defined regardless of our choice of $x\in\mathbb{R}$), simplify each term, multiply the series by another "constant" such as x^3 (only n has a range of values within given sum), and get the correct value for $x^3\sin 2x$. Since the correct power series centered at zero should be unique, it must be the one computed above.

This example also shows how Σ-notation can make shorter work of some series constructions.

11.5.2 Derivatives and Integrals

As has already been mentioned, many of the manipulations we would hope we can do with Taylor Series are in fact possible. For instance, we can take derivatives and integrals as expected:

Theorem 11.5.2 *Suppose that $f(x)$ is given by some Taylor Series*

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} a_n(x-a)^n. \tag{11.37}$$

1. *(Also a theorem of Abel.) If the series converges in an open interval containing x, then inside that interval, we can differentiate "term by term" to get*

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}. \tag{11.38}$$

2. *Furthermore, integrating (11.37) term by term we get*

$$\begin{aligned}\int f(x)\,dx &= a_0(x-a) + a_1\frac{(x-a)^2}{2\cdot 1!} + a_3\frac{(x-a)^3}{3\cdot 2!} + \cdots + C\\ &= \sum_{n=0}^{\infty} a_n\frac{(x-a)^{n+1}}{(n+1)!} + C, \end{aligned} \tag{11.39}$$

with the special case that, if the series converges on the closed interval with endpoints a and x, we have

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} a_n\frac{(t-a)^{n+1}}{(n+1)!}\Bigg|_a^x = \sum_{n=0}^{\infty} a_n\frac{(x-a)^{n+1}}{(n+1)!}. \tag{11.40}$$

A very simple demonstration of the derivative part of this theorem is the following:

Example 11.5.3 *We do the following calculation $\frac{d}{dx}e^x = e^x$, but using series to show the reasonableness of the theorem above.*

$$\begin{aligned}\frac{de^x}{dx} = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) &= 0 + 1 + \frac{2x}{2\cdot 1} + \frac{3x^2}{3\cdot 2\cdot 1} + \frac{4x^3}{4\cdot 3\cdot 2\cdot 1} + \cdots\\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\\ &= e^x,\end{aligned}$$

as expected. Using Σ-notation, keeping in mind that the first ($n=0$) term differentiates to zero, we get

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot nx^{n-1} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^{n-1} = \sum_{n=0}^{\infty}\frac{1}{n!}x^n = e^x.$$

The step where we rewrite the new series to begin at $n=0$ is clear if a few terms are written out in the expansions of each series.

The series for $\dfrac{1}{1-x}$ was given in (11.30), page 779, but was shown easily remembered due to its relationship with a simple geometric series (see also (10.9), page 706):

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x},$$

valid for $|x| < 1$. We will use this in some examples below.

Example 11.5.4 *Compute the series for* $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right]$ *centered at* $a = 0$.

Solution: This is a straightforward computation, either with the term-by-term expansion or with the Σ*-notation, and an optional rewriting of the final summation. (Note how the first term vanishes in the derivative.)*

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\right] = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots,$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty}\left(nx^{n-1}\right) = \sum_{n=0}^{\infty}(n+1)x^n.$$

Since the original series was valid for $|x| < 1$*, so will be the new series. (The reader is welcome to perform a ratio test to confirm this.)*

Example 11.5.5 *Use the series for* $\dfrac{1}{1-x}$ *to derive a series for* $\dfrac{1}{1+x^2}$*. Then use that series to find a series for* $\tan^{-1} x$.

Solution: *We first replace* x *with* $-x^2$ *in that series, since* $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{(valid for } |x| < 1\text{)} \\
\implies \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots = \sum_{n=0}^{\infty}(-x^2)^n \\
&= 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n}.
\end{aligned}$$

This is valid wherever $|x^2| < 1$*, which it is not too difficult to see is again wherever* $|x| < 1$.[16] *Next we use the fact that* $\tan^{-1} 0 = 0$*, so that*

$$\tan^{-1} x = \tan^{-1} x - \tan^{-1} 0 = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}(-1)^n \left.\frac{t^{2n+1}}{2n+1}\right|_0^x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} - 0.$$

[16] Recall that $|x^2| = |x|^2$. Also recall that the square root function is increasing on $[0, \infty)$, and so (by definition) preserves inequalities. Thus

$$|x|^2 < 1 \iff \sqrt{|x^2|} < \sqrt{1} \iff |x| < 1.$$

Thus

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{11.41}$$

Alternatively, the final form in (11.41) *can be had by the more expanded form:*

$$\begin{aligned}\tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left[1 - t^2 + t^4 - t^6 + t^8 - \cdots\right] dt \\ &= \left[t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \cdots\right]\Bigg|_0^x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots .\end{aligned}$$

Once again, this is valid where $|x^2| < 1$*, i.e., where* $|x| < 1$*. However, we see that the series converges by the Alternating Series Test at* $x = 1$*, and so the interval of convergence is in fact* $x \in (-1, 1]$*. We know that the series equals* $\tan^{-1} x$ *even at* $x = 1$ *because both* $\tan^{-1} x$ *and the series are left-continuous as* $x \to 1^-$*, the former due to the fact* $\tan^{-1} x$ *is continuous for* $x \in \mathbb{R}$*, and the series is continuous where it converges by Abel's Theorem.*

In fact one valid, if not terribly efficient, method of computing π is from using

$$\pi = 4 \cdot \frac{\pi}{4} = 4\tan^{-1}(1) = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right). \tag{11.42}$$

Example 11.5.6 *Find* $\displaystyle\int_0^x e^{t^2}\,dt$.

<u>**Solution:**</u> *It is an interesting but futile exercise to try to find the antiderivatives of* e^{x^2} *using the usual tricks: substitution, integration by parts, etc. It is well-known that there is no "closed form" for this antiderivative, i.e., using the usual functions in the usual manners. It is also true that, since* e^{x^2} *is continuous on* $\mathbb{R}$*, there must exist continuous antiderivatives.*[17] *Our discussion here presents a strategy for calculating this integral: writing the integrand as a series, and integrating term by term. As before, we will write the steps and the solution in two ways: one method is to write out several terms of the series and declare a pattern; the other, done simultaneously, is to use the* Σ*-notation. Hopefully by now they are equally simple to deal with.*

$$\begin{aligned}
& & e^t &= 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots &&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
&\implies & e^{t^2} &= 1 + \frac{t^2}{1!} + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots &&= \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} \\
&\implies & e^{t^2} &= 1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots &&= \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \\
&\implies & \int_0^x e^{t^2}\,dt &= \int_0^x \left(1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots\right) dt &&= \int_0^x \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\right) dt \\
& & &= \left(t + \frac{t^3}{3 \cdot 1!} + \frac{t^5}{5 \cdot 2!} + \frac{t^7}{7 \cdot 3!} + \cdots\right)\Bigg|_0^x &&= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)n!}\Bigg|_0^x \\
& & &= \left(x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots\right) &&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}.
\end{aligned}$$

[17] This comes from one of the statements of the Fundamental Theorem of Calculus.

Thus

$$\int_0^x e^{t^2}\,dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}.$$

We could also write the general antiderivative

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C.$$

Other antiderivatives which must be found this way are $\int \sin x^2\,dx$, $\int \cos x^2\,dx$.

11.5.3 The Binomial Series and an Application

The following series comes up in enough applications that it is worth some focus. It is the following:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots \tag{11.43}$$

This series (11.43) is known as the *Binomial Series*. It can also be written

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)x^n}{n!}.$$

This series is valid for $|x| < 1$, and sometimes also valid at one or both endpoints $x = \pm 1$. It is not difficult to prove, and is a worthwhile exercise. In fact, for $\alpha \in \{0, 1, 2, 3, \cdots\}$, the function is a polynomial and the series terminates (in the sense that all but finitely many terms are zero), simply giving an expansion of the polynomial, valid for all x.

The derivation of (11.43) is straightforward. See Exercise 23. Here are some quick examples:

$$\begin{aligned}
\frac{1}{\sqrt{1+x}} &= 1 - \tfrac{1}{2}x + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2}{2!} + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3}{3!} + \cdots && \left(\alpha = -\tfrac{1}{2}\right)\\
\frac{1}{1+x^2} &= 1 - (x^2) + \tfrac{(-1)(-2)(x^2)^2}{2!} + \tfrac{(-1)(-2)(-3)(x^2)^3}{3!} + \cdots && (\alpha = -1)\\
&= 1 - x^2 + x^4 - x^6 + \cdots && (\text{“}x\text{”} = x^2)\\
(1+x)^3 &= 1 + 3x + \tfrac{3\cdot 2x^2}{2!} + \tfrac{3\cdot 2\cdot 1x^3}{3!} + \tfrac{3\cdot 2\cdot 1\cdot 0x^4}{4!} + \tfrac{3\cdot 2\cdot 1\cdot 0\cdot(-1)x^5}{5!} + \cdots\\
&= 1 + 3x + 3x^2 + x^3 && (\alpha = 3)
\end{aligned}$$

Actually, the last one is valid for all x, and the one above it was found as a step in Example 11.5.5, page 792. Other algebraic manipulations can also sometimes put a function into a form suitable for applying the Binomial Series. Consider the following, with $\alpha = 1/3$, we complete the square under the radical, and the natural center of the resulting series is $a = -1$. (Note also that $\sqrt[3]{-A} = \sqrt[3]{-1}\sqrt[3]{A} = -\sqrt[3]{A}$, since 3 is odd.)

$$\begin{aligned}
\sqrt[3]{x^2+2x} &= \sqrt[3]{x^2+2x+1-1} = \sqrt[3]{(x+1)^2-1} = -\sqrt[3]{1-(x+1)^2}\\
&= -\left[1 + \frac{\frac{1}{3}(x+1)^2}{1!} + \frac{\frac{1}{3}\cdot\frac{-2}{3}(x+1)^4}{2!} + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}(x+1)^6}{3!} + \cdots\right]
\end{aligned}$$

Similarly, each of the following manipulations are valid though they yield different intervals of convergence:

$$
\begin{aligned}
(3-8x)^{1/4} = (1-(8x-2))^{1/4} &= 1 + \frac{\frac{1}{4}[-8(x-\frac{1}{4})]}{1!} + \frac{\frac{1}{4}\cdot\frac{-3}{4}[-8(x-\frac{1}{4})]^2}{2!} \\
&\quad + \frac{\frac{1}{4}\cdot\frac{-3}{4}\cdot\frac{-7}{4}[-8(x-\frac{1}{4})]^3}{3!} + \cdots, \\
(3-8x)^{1/4} = 3^{1/4}\left(1-\frac{8}{3}x\right)^{1/4} &= 3^{1/4}\left[1 + \frac{\frac{1}{4}\left(\frac{-8x}{3}\right)}{1!} + \frac{\frac{1}{4}\cdot\frac{-3}{4}\left(\frac{-8x}{3}\right)^2}{2!}\right. \\
&\quad \left. + \frac{\frac{1}{4}\cdot\frac{-3}{4}\cdot\frac{-7}{4}\left(\frac{-8x}{3}\right)^3}{3!} + \cdots\right].
\end{aligned}
$$

The first representation is centered at $x = 1/4$, and definitely valid where $|8x-2| \le 1$, i.e., where $x \in (1/8, 3/8)$, while the second is centered at $x = 0$ and definitely valid where $|8x/3| < 1$, i.e., where $x \in (-3/8, 3/8)$.

While binomial series tend to be complicated to write, there are elegant applications. One particularly beautiful application relates Albert Einstein's Special Relativity to Newtonian Mechanics. This application is given in the following example.

Example 11.5.7 (Application) *According to Einstein, kinetic energy is that energy which is due to the motion of an object, and can be defined as $E_k = E_{\text{total}} - E_{\text{rest}}$, this being a function of velocity for a given mass m:*

$$E_k(v) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} - mc^2.$$

Contained in the above is the very famous equation $E_{\text{rest}} = mc^2$. Also notice that the total energy E_{total} blows up as $v \to c^-$ or $v \to -c^+$, i.e., as velocity approaches the speed of light. At $v = \pm c$, we are dividing by zero in the total energy, and thus the theory that ordinary objects cannot achieve the speed of light (for it would take infinite energy to achieve it).

Now let us expand this expression of $E_k(v)$ by applying the Binomial Series to $\left(1-\frac{v^2}{c^2}\right)^{-1/2}$, with $\alpha = -1/2$ and replacing x with $-v^2/c^2$. Thus $E_k = mc^2\left[(1-v^2/c^2)^{-1/2} - 1\right]$ becomes

$$E_k(v) = mc^2\left(1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{v^2}{c^2}\right)^2}{2!} + \cdots\right) - mc^2 \tag{11.44}$$

$$\approx mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - mc^2 \qquad \text{when } \frac{v^2}{c^2} \text{ is small.} \tag{11.45}$$

Multiplying this out, we see that

$$E_k \approx mc^2 + mc^2 \cdot \frac{1}{2}\frac{v^2}{c^2} - mc^2 = \frac{1}{2}mv^2. \tag{11.46}$$

Summarizing,

$$E_k(v) \approx \frac{1}{2}mv^2 \qquad \text{when } |v| << c. \tag{11.47}$$

Here the notation $|v| << c$ means that $|v|$ is much smaller than c, giving us that v^2/c^2 is very small. So we see that Newton's kinetic energy formula $E_k = \frac{1}{2}mv^2$ is just an approximation of Einstein's, which is to be expected since Newton was not considering objects at such high speeds. In effect, Newton could not see the whole kinetic energy curve, where Einstein's theories could detect more phenomena which governed the behavior of the curve of E_k versus v through a larger range of velocities v.

Exercises

The following are very useful exercises for students to attempt themselves. One should first attempt these using the written out expansion

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

and then using the Σ-notation if possible, comparing the results.

1. Use the series for e^x to find a series expansion for the general antiderivative of e^{x^2}. (You can try to find the antiderivative using non-series methods, but it cannot be written using the usual functions. It is interesting to attempt to use the old methods, to see why they fail.)

2. Use the Maclaurin series for $\sin x$ to do the following:

   (a) Write a series for $\sin 2x$.

   (b) Use the series above to prove that $\frac{d}{dx}\sin 2x = 2\cos 2x$. (It may help to also write the series for $2\cos 2x$ separately.)

   (c) Write a series for $\cos x^2$.

   (d) Use the series above, and the series for $-2x\sin x^2$, to prove that $\frac{d}{dx}\cos x^2 = -2x\sin x^2$.

3. Use the Maclaurin Series for $\sin x$ and $\cos x$ to show that

$$\sin(-x) = -\sin x,$$
$$\cos(-x) = \cos x.$$

In each of the following, unless otherwise stated, leave your final answers in Σ-notation.

4. Find the Maclaurin series for $f(x) = \ln(x+1)$ using (11.32). Where is this series valid?

5. Approximate $\displaystyle\int_0^{\sqrt{\pi}} \cos x^2\,dx$ by computing the first five nonzero terms of the Maclaurin series for $\int \cos x^2\,dx$.

6. The Hyperbolic Functions: The three most important hyperbolic functions are

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad (11.48)$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad (11.49)$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \qquad (11.50)$$

Though not immediately obvious, it is true that $\tanh x$ is invertible, and that its inverse has the property that

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}. \qquad (11.51)$$

Find the Maclaurin series for $f(x) = \tanh^{-1} x$ given that

$$\tanh^{-1} x = \int_0^x \frac{1}{1-t^2}\,dt. \qquad (11.52)$$

(See Example 11.5.5, page 792.) Where is this series valid? (Actually the integral in (11.52) can also be computed with partial fractions, and the final answer written without resorting to series.)

7. (Proof of Proposition 11.3.5) Derive the Taylor Series for $\ln x$ with $a = 1$ using the fact that
$$\ln x = \int_1^x \frac{1}{t}\,dt$$
for $x > 0$, and
$$\frac{1}{t} = \frac{1}{1-(1-t)}.$$
Where is this series guaranteed valid?

8. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$ using a Taylor Series centered at $a = 0$.

9. Evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{x}$ using a Taylor Series centered at $a = 0$.

10. Do the same for $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

11. Evaluate the integral $\displaystyle\int_0^{0.4} \cos x^2\,dx$ by using the Taylor Polynomial $P_3(x)$ for $\cos x$ centered at $a = 0$ (and therefore $P_6(x)$ for $\cos x^2$). This is called a Fresnel integral, which appears in studies of optics.

12. In (11.42), page 793 we see that
$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$
How many terms should we add in the series above to be assured that our sum is
   (a) within 0.01 of π?
   (b) within 0.00001 of π?

13. Use Maclaurin series for $\sin x$ and $\cos x$ to demonstrate the following:
   (a) $\frac{d}{dx}\sin x = \cos x$.
   (b) $\frac{d}{dx}\cos x = -\sin x$.

14. Use the fact that $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$, and that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ to find the Maclaurin Series expansion for
$$f(x) = \frac{1}{(1-x)^2}.$$

15. Use the facts that $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$, and that $\frac{1}{1+t^2} = \frac{1}{1-[-t^2]}$ to compute the Maclaurin Series for $\tan^{-1} x$.

16. Show that $\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$ by using the series for e^x centered at $x = 0$.

17. Starting from the series for e^x, compute the Taylor Series for
   (a) $\sinh x = \dfrac{e^x - e^{-x}}{2}$
   (b) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

For Exercises 18–22, approximate the definite integrals $\int_a^b f(x)\,dx$ by replacing $f(x)$ with an appropriate Taylor Polynomial $P_4(x)$, centered at $a = 0$ **if possible**, and centered elsewhere if necessary. Also, compare your approximation to the exact value for each integral.

18. $\displaystyle\int_0^{\pi/4} \sin x\,dx$

19. $\displaystyle\int_1^2 e^x\,dx$

20. $\displaystyle\int_1^3 \ln x\,dx$

21. $\displaystyle\int_1^3 \sqrt{1+x}\,dx$

22. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

23. Derive the series (11.43) using the formula for Taylor/Maclaurin Series where $f(x) = (1+x)^\alpha$ and $a = 0$.

24. Find a series representation for the following functions using the binomial series (11.43). Do not attempt to use Σ-notation, but rather write out the first five terms of the series to establish the pattern.
    (a) $f(x) = (1+x)^{3/2}$
    (b) $f(x) = (1-x)^{3/2}$
    (c) $f(x) = \dfrac{1}{\sqrt[3]{1+x}}$
    (d) $f(x) = \dfrac{1}{\sqrt[3]{1+x^3}}$
    (e) $f(x) = \dfrac{x^3}{\sqrt{1+x}}$
    (f) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$.

25. find the series expansion for $f(x) = \ln\left(1+x^2\right)$ by using the fact that $\ln\left(x^2+1\right) = \displaystyle\int_0^x \frac{2t}{1+t^2}\,dt$.

26. Find a more general form of the binomial series by using (11.43) to derive a series for
$$f(x) = (b+x)^\alpha \qquad (11.53)$$
and determine for what values of x is it valid. (Hint: Use (11.43) after factoring out b^α from f.)

27. Complete the square and use the binomial series to write a series expansion for the following. Also determine an interval $|x-a|<R$ where the series is guaranteed to be valid.
    (a) $f(x) = \dfrac{1}{\sqrt{x^2-6x+10}}$
    (b) $f(x) = \sqrt{4x^2+12x+13}$
    (c) $f(x) = (-2x^2+3x+5)^{-2/3}$

28. Using (11.44), page 795 to show that $E_k(v) \ge \frac{1}{2}mv^2$ for $|v|<c$, with equality only occurring when $v=0$. Thus (11.47) is always an underestimation unless $v=0$. (Hint: Look at the signs of all the terms we ignore in the approximation.)

29. Approximate $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx$ by using the the Binomial Series expansion for $\sqrt{1+x^3} = \left(1+x^3\right)^{\frac{1}{2}}$, and using the first three nonzero terms of this expansion in your integral.

30. Consider $f(x) = e^{x^2}$.
    (a) Write the Maclaurin series for $f(x)$.
    (b) Find $f^{(9)}(0)$.
    (c) Find $f^{(10)}(0)$.

31. It can be shown that
$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right).$$
Use this fact to approximate π by using the Taylor Series for $\tan^{-1} x$ centered at $a=0$ and the approximation $P_5(x)$.

32. Use the fact that
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$
to prove the assertion at the beginning of the previous exercise.

33. As in Exercise 31, estimate π by using the fact that
$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

34. As in Exercise 31, estimate π by using the fact that
$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right).$$

35. Write the Maclaurin series for $f(x) = \frac{1}{2}\sin 2x$ by
    (a) using the series for $\sin x$.
    (b) using instead the series for $\sin x$ and $\cos x$ and the fact (from the double angle formula) that
    $$f(x) = \sin x \cos x.$$
    (Just write out the first several terms of the product, being careful to distribute correctly, to verify the answer is the same as in part (a).)

11.6 Complications and the Role of Complex Numbers

A common engineering and science research technique is to assume there is a function which describes some relationship between two variables, and that the function has a Taylor Series representation. Then the researcher might look at data and attempt to find the best fitting polynomial of some specified degree that fits the data. Some limit on the degree has to be specified, since anytime we have N data of the form $\{(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)\}$, where none of the x_i are repeated with different y_i, we can fit an $(N-1)$-degree polynomial to that data perfectly, but its predictive abilities may be little or nonexistent, since high-degree polynomials tend to have rather violent behavior, particularly as $|x| \to \infty$ or even between the data points.

Moreover, Taylor Series not only assume that all derivatives exist at the center, but by Abel's Theorem and our ability to differentiate series, we expect the function in question to have all of its derivatives inside the interval of convergence. The contrapositive of that fact gives us that once we run into a problem with the function or one of its derivatives as we move from the center of the series, we cannot move any farther from the center and expect the series to be valid for the function. The upshot is that the researcher who assumes a Taylor Series expansion of a function must be careful to only use that assumption within intervals where the function and its derivatives should all be defined. Attempting to "fit" data to polynomials beyond that will likely have little or no predictive value.

We will first look at some cases where we can expect problems with our Taylor Series, in the sense that we cannot expect the given function to be equal to a Taylor Series. Most of those cases will upon reflection become pretty obvious, but some are more subtle.

As we have seen already, a Ratio Test can often give us the open part of an interval of convergence (with the endpoints usually checked separately), though we were able to avoid the Ratio Test for some of our series derived from, say, the geometric series. From the discussion above (further developed below) we can also see problems with assuming a valid Taylor Series when functions run into other difficulties, which a Ratio Test will not necessarily detect (the series may converge but not to the function). We will explore this in Subsection 11.6.1.

It turns out that the most natural place for series to "live" and be observed is not so much the real line $\mathbb{R}$ and its intervals, but the complex plane

$$\mathbb{C} = \left\{ x + iy \,\middle|\, x, y \in \mathbb{R},\, i = \sqrt{-1} \right\} \tag{11.54}$$

and its open discs, an open disc meaning the interior of a circle (not including the circle itself). This allows "wiggle room" in all directions from the center, which allows for things such as derivatives, where in $\mathbb{R}$ we only require "wiggle room" to the left and right. In fact it is from this complex context that the term *radius of convergence* comes to us. We will look into this further in Subsection 11.6.3. That discussion usually waits until students finish a 2–3 semester calculus sequence and proceed to a Differential Equations course, but it is included here to give some more context to Taylor Series.

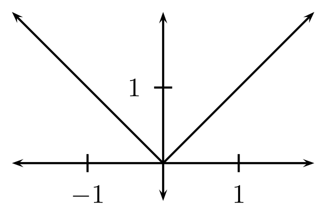
11.6.1 Troubles stemming from continuity problems

Most of our familiar functions are analytic where they are defined, and so can be represented by Taylor Series for usefully large intervals. These functions include all polynomials, rational functions, exponentials, roots, logarithms and trigonometric functions, as well as combinations of these through addition, subtraction, multiplication, division and composition. We already mentioned that there are power series which are perfectly respectable functions, but which cannot be written as combinations of familiar functions. This may leave the student with the incorrect impression that we can always find and manipulate Taylor Series for all functions with impunity.

However, there are many functions we encountered in Chapter 3 which had more pathological behaviors and will not always be analytic where defined. Therefore Taylor Series are often useless and inappropriate in dealing with such functions, at least if we wish to center a series at a problematic point of the function, or assume a series will be valid as we allow the input variable, say x, have its value move "through" a problematic value.

The purpose of this section is to alert the student to situations in which Taylor Series—or even Taylor Polynomials—are not appropriate for approximation except possibly with careful modifications.

Example 11.6.1 *Consider $f(x) = |x|$, which we graph below:*



The following problems arise with attempting to use a Taylor Series representation for $f(x) = |x|$:

- *If we attempt to construct Taylor Polynomials at $a = 0$, we would have to stop at $P_0(x) = 0$ because there are no derivatives to compute at $a = 0$. Furthermore, $P_0(x)$ is clearly a terrible approximation of $f(x)$ as we stray from its center $x = 0$.*

- *If we were to construct a Taylor Series for $f(x)$ at, say, $a = 1$ we would find that the series would terminate after the first-order term, because except at $x = 0$, locally this function is a line. Consider for instance the Taylor Series centered at $a = 1$, where we have, for $N \geq 2$,*

$$\begin{aligned} P_N(x) &= f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \frac{1}{3!}f'''(1)(x-1)^3 + \cdots + \frac{1}{N!}(x-1)^N \\ &= 1 + 1(x-1) + 0 + 0 + 0 + \cdots + 0 \\ &= x. \end{aligned}$$

We would get the same series (letting $N \to \infty$) for any other center $a > 0$, which a direct computation would show. Furthermore, such a series would **not** *be the same as the function for $x < 0$, since $f(x) \neq x$ when $x < 0$.*

- *Similarly for $a = -1$ we would have $N \geq 2 \implies$*

$$\begin{aligned} P_N(x) &= f(-1) + f'(-1)(x+1) + \frac{1}{2!}f''(-1)(x+1)^2 + \frac{1}{3!}f'''(1)(x+1)^3 + \cdots + \frac{1}{N!}(x+1)^N \\ &= -1 - 1(x+1) + 0 + 0 + 0 + \cdots + 0 \\ &= -x. \end{aligned}$$

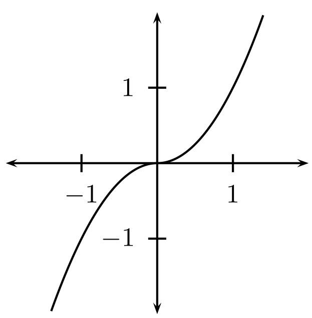
This is equal to $f(x)$ for $x \leq 0$ but is incorrect for $x > 0$.

What ruins the above series' chances of being the same as the function on all of $\mathbb{R}$ is the fact that the absolute value function is not differentiable at $x = 0$. Anywhere else we can have a Taylor Series equal to the function locally, but not globally.

The coefficients of the Taylor Series follow from the local behavior of the function, not its global behavior. On $x \in (0, \infty)$ we have $f(x) = x$, with its obvious Taylor Series (which simplifies to just x), while on $x \in (-\infty, 0)$ with its obvious Taylor Series (which simplifies to just $-x$). However, neither of these Taylor Series can equal the function on the other side of $x = 0$: a Taylor Series centered at $a > 0$ will be incorrect for $x < 0$, and a Taylor Series centered at $a < 0$ will be incorrect for $x > 0$.

While we can see the "kink" at $x = 0$ in the graph for $f(x) = |x|$, which causes a major discontinuity in the derivatives there, sometimes the problem is more subtle, from the graphical perspective. It might not be so subtle from the functional definition perspective: piece-wise defined functions are often suspect. Recall that $|x|$ is *defined* to be x on $[0, \infty)$ and $-x$ on $(-\infty, 0)$.

Example 11.6.2 *Consider the function* $f(x) = \begin{cases} x^2, & \text{if} \quad x \geq 0 \\ -x^2 & \text{if} \quad x < 0 \end{cases}$ *with graph and derivatives*



$$f'(x) = \begin{cases} 2x, & \text{if} \quad x \geq 0 \\ -2x & \text{if} \quad x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2, & \text{if} \quad x > 0 \\ -2 & \text{if} \quad x < 0. \end{cases}$$

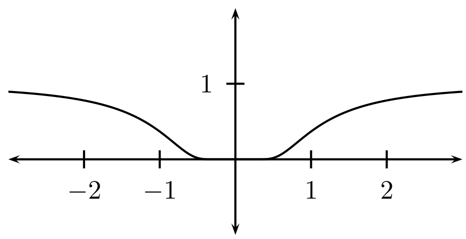
This function has similar complications as the previous, except they emerge in the next-higher-order Taylor Polynomials:

- *At $a = 0$ we can construct $P_1(x) = 0$ (zero height and slope) but we cannot construct $P_2(x)$ or higher because $f''(0)$ does not exist.*
- *For a (positive) center $a > 0$ we can construct even the full Taylor Series, which will simplify to x^2, but not be equal to the function for $x < 0$.*
- *For a (negative) center $a < 0$ we can construct the full Taylor Series, which will simplify to $-x^2$, but will not equal the function for $x > 0$.*

Example 11.6.3 *Consider the function*

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0. \end{cases} \tag{11.55}$$

Clearly, as $x \to 0$ we have form $e^{-\infty}$ and so $x \to 0 \implies f(x) \to 0$, and since $f(0) = 0$ we have continuity at $x = 0$. The function is also symmetric with respect to the y-axis. It is notable that $f(x) \to 0$ somewhat quickly as $x \to 0$ because of the growth in $1/x^2$, and thus negative growth in $-1/x^2$. Indeed the function is graphed below. Though it appears "flat" it is only zero at $x = 0$, which would take a much higher resolution graphic to verify.



It is an interesting exercise to show that all derivatives of $f(x)$ exist everywhere, including at $x = 0$. Furthermore, after computing a few derivatives and some of the ensuing limits one can show that $(\forall n \in \{0, 1, 2, 3, \cdots\}) f^{(n)}(0) = 0$. Thus the Maclaurin Series for $f(x)$ would be itself zero, as in

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0) \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{0x^n}{n!} = 0,$$

and so while this series would be equal to $f(x)$ at $x = 0$, it would not be elsewhere, as $f(x) = 0 \iff x = 0$. So even though all derivatives exist for f at $x = 0$, and the function and its derivatives are all continuous on $x \in \mathbb{R}$, the Taylor Series centered at $a = 0$ does not converge to the function except at the center of the series.

Such a function as in Example 11.6.3 above is certainly smooth at $x = 0$ and indeed all of $\mathbb{R}$, as are all of its derivatives, but it is not *real-analytic* at $x = 0$ because it cannot be represented as a power series in an open interval containing $x = 0$. (Neither were the functions in the previous Examples 11.6.1, 11.6.2; see Definition 11.4.2, page 786.)

In fact the Taylor Series centered at any $a \in \mathbb{R}$ would only be guaranteed to converge to the function at on $|x - a| < |a|$, because it could not extend to "the other side of zero," and we know that it must converge within a certain "radius" of the center, and diverge once past that radius from the center. In the next subsection we will see that what is crucial is what happens inside the *complex plane*, where the term "radius of convergence" makes more sense.

11.6.2 The Complex Plane

Here we will look very briefly at the *complex plane*, which is the geometric interpretation of complex numbers $z = x + iy$, where $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$. We would call x the *real part* of z, and iy to be the *imaginary part* of z.

At first this seems preposterous because clearly $\sqrt{-1} \notin \mathbb{R}$, since the square of any real number will not be negative. While it may seem easy to dismiss any number with an "imaginary" part iy as being a figment of the imagination and of no actual consequence, there nonetheless are many important physical phenomena best described using complex numbers, as their geometric properties (which we develop below) have many real-world analogs. Furthermore, complicated "real-number" phenomena are often most easily analyzed by lifting them into the complex plane, making observations there, and bringing these observations back into the real line.[18]

So if we take as given that there is a number system which includes all the real numbers, but also a quantity $i = \sqrt{-1}$, we get the following multiplication facts:

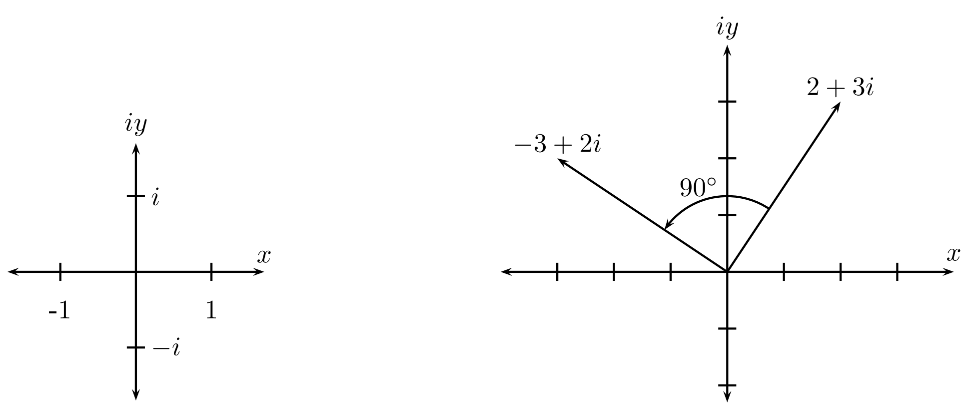
$$\left.\begin{array}{rclcrcl} i &=& i & \qquad & i^5 &=& i \\ i^2 &=& -1 & & i^6 &=& -1 \\ i^3 &=& -i & & i^7 &=& -i \\ i^4 &=& 1 & & i^8 &=& 1 \end{array}\right\} \qquad (\forall n \in \mathbb{N} \cup \{0\}) \qquad \left[\begin{array}{rcr} i^{4n+1} &=& i \\ i^{4n+2} &=& -1 \\ i^{4n+3} &=& -i \\ i^{4n+4} &=& 1 \end{array}\right]$$

[18]It is akin to giving someone lost in a wilderness an aerial map, or a brief lift in a helicopter, so that they can glimpse their predicament from above. This could indeed be useful in finding a path out of the wilderness, even if the actual solution is still to be taken at ground level.

In fact the above pattern follows for $n \in \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, but that requires a discussion of division for the negative exponents. Before discussing division, one has to first discuss multiplication, which has its own complications. Assuming in the discussion below that $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} z_1 &= x_1 + iy_1 \\ z_2 &= x_2 + iy_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

We see a kind of intermingling of the real and imaginary parts of z_1 and z_2 to form the real and imaginary parts of the product z_1z_2. While that may appear quite complicated and esoteric, in fact there is a geometric interpretation which is not all that difficult. For instance, multiplying by the *imaginary unit* i has the same effect as a $\pi/2$ ($90°$) rotation in the *complex plane*, where we graph $z = x_1 + iy_1$ the same way we graph (x_1, y_1) in what looks like the regular xy-plane, though here the horizontal axis is referred to as the *real axis*, and the vertical axis is referred to as the *imaginary axis*. In the diagrams below we do see how multiplying by i is indeed the same as rotating the point around the origin $0 = 0 + i \cdot 0$ by $\pi/2$.



Note how in the first graph, each time we multiply by i we "travel" from $i^0 = 1$, to $i^1 = i$, $i^2 = -1$, $i^3 = -i$, back to $i^4 = 1$ and so on. In the second graph note the relative positions of $2 + 3i$ and $i(2 + 3i) = -3 + 2i$: the latter is a $\pi/2$ rotation from the former.

This already hints at why complex numbers can be useful in the physical sciences: rotations in a plane can be modeled as multiplications by powers of i.

The scope of this text would have to be greatly expanded to prove the validity of the following, but the reader should be assured by the presence of dozens of textbooks on the subject, that we are allowed to perform calculus in complex variables (properly understood), which allows us to accept, for instance, the following identity of Euler:[19]

$$e^{i\theta} = \cos\theta + i \sin\theta. \tag{11.56}$$

[19]Leonhard Euler (sounds like "oiler"), 1707–1783 was an extremely prolific Swiss mathematician and physicist. A student studying graduate level mathematics will read his name often, perhaps more often than that of any other historical figure. He had a particular talent for discovering facts ahead of the time in which they could actually be proved rigorously, such as his identity (11.56).

This follows from the Maclaurin series for e^θ, $\cos\theta$ and $\sin\theta$, where (of course) θ is in radians:

$$\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\
&= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta} \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

This equation (11.56) is useful in many contexts. For instance, it can be used to find the most basic trigonometric identities that involve more than one angle, if we consider two expansions for $\exp[i(\alpha+\beta)]$:

$$\begin{aligned}
e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta), \quad \text{and} \\
e^{i(\alpha+\beta)} &= e^{i\alpha+i\beta} \\
&= e^{i\alpha}e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta), \\
\Longrightarrow \qquad \cos(\alpha+\beta) + i\sin(\alpha+\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).
\end{aligned}$$

Now anytime we have $x_1 + iy_1 = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we must have $x_1 = x_2$ and $y_1 = y_2$; that is, the real parts x_1, x_2 must be the same and the imaginary parts iy_1, iy_2 must be the same. Setting the two different forms above for the real part equal, and doing the same for the imaginary parts (divided by i), we get

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}$$

The pair of trigonometric identities above are proved geometrically in most trigonometric textbooks, but the proof using complex numbers and Euler's identity as above is routine once one is comfortable with complex numbers. Many more trigonometric identities follow from these, and the facts that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ (both of which can be proved using their own Maclaurin Series). For instance, if we set $\alpha, \beta = \theta$ we have $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and $\sin 2\theta = 2\sin\theta\cos\theta$ from these.

This gives rise to further geometric aspects of complex numbers. Consider Figure 11.5. It is customary to define, for $z = x + iy$, the "absolute value" of z, given by[20]

$$|z| = \sqrt{x^2 + y^2},$$

which is the distance from z to the origin $0 = 0 + i \cdot 0$. (Similarly $|x|$ is the distance from x to zero but on the real line.) We can also define an angle θ which the ray from 0 to z makes with the positive real axis, measured counterclockwise. If we do so, it is not hard to see that $x = |z|\cos\theta$ and $y = |z|\sin\theta$. It is common to see $|z|$ replaced by the real variable r, so $r = \sqrt{x^2+y^2}$ and

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

[20]This quantity $|z|$ has many other names such as the *modulus, norm, magnitude,* and *length* of z.

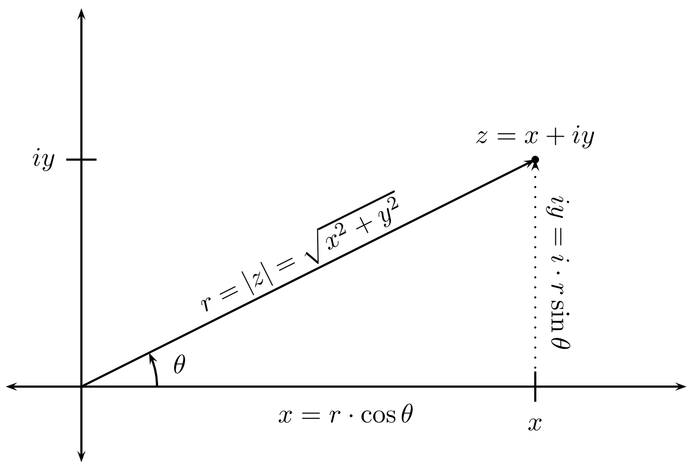


Figure 11.5: A complex number $z = x+iy$ written in polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

This is called the *polar form* of the complex number z. (A similar theme is developed with the usual Cartesian Plane, $\mathbb{R}^2$, in Chapter 12.)

This gives us some interesting aspects of complex multiplication. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

so that when we multiply two complex numbers, in the product their lengths (absolute values) are multiplied, and their angles are added.

Besides giving further illumination on the idea that multiplying by i is the same as revolving the complex number 90° around the origin, this also lets us "work backwards" to solve some other interesting problems. For instance, what should be the square root of i? One problem with answering this is that there are actually two square roots of i, as there are two square roots of -1, namely i and $-i$, and there are two square roots of 9, namely 3 and -3. We usually choose one to be "the square root," and so with the complex plane we might choose only those whose angles θ are within $[0, \pi)$, though that is only one convention. In fact in most applications we are interested in all roots, so in the computations below we use quotation marks around the expressions for the roots. We also exploit the ambiguity regarding what exactly should be θ, since once we have a workable θ we also have $\theta + 2n\pi$ also legitimate, for $n \in \mathbb{Z}$.

Example 11.6.4 *Find all fourth roots of 16.*

<u>Solution</u>: Here we write 16 in the form $|z|e^{i\theta}$ using four consecutive legitimate values for θ, and then formally (or "naively") apply the 1/4 power:

$$\begin{aligned}
16 = 16e^{i\cdot 0} &\implies \text{“}16^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{0}{4}} = 2\left(\cos 0 + i\sin 0\right) &&= 2\left(1 + i\cdot 0\right) &&= 2,\\
16 = 16e^{i\cdot 2\pi} &\implies \text{“}16^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{2\pi}{4}} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) &&= 2\left(0 + i\cdot 1\right) &&= 2i,\\
16 = 16e^{i\cdot 4\pi} &\implies \text{“}16^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{4\pi}{4}} = 2\left(\cos\pi + i\sin\pi\right) &&= 2\left(-1 + i\cdot 0\right) &&= -2,\\
16 = 16e^{i\cdot 6\pi} &\implies \text{“}16^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{6\pi}{4}} = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) &&= 2\left(0 + i\cdot(-1)\right) &&= -2i.
\end{aligned}$$

If we were to write $16 = 16e^{8\pi i}$, *we would get the same root as we got in the first case above, namely* "$16^{1/4}$" $= 2e^{i\cdot 2\pi} = 2$ *as before. Similarly with the other possible values of* θ: *we would again get only the four previous fourth roots, namely* $\pm 2, \pm 2i$.

These fourth roots of 16 can also be found by solving $x^4 = 16$ using high school algebra, but the technique above also allows us to find any roots of any number which we can write in the form $z = re^{i\theta}$.

Example 11.6.5 *Find the square roots of* i.

Solution: We proceed as above, noting that i *makes an angle of* 90° *with the positive real axis. We will use* $\theta = \frac{\pi}{2}$ *and* $\theta = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}$ *to find our two second roots.*

$$i = 1e^{i\cdot\pi/2} \implies \text{“}i^{1/2}\text{”} = 1^{1/2}e^{i\cdot\frac{\pi}{4}} = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{1+i}{\sqrt{2}},$$

$$i = 1e^{i\cdot 5\pi/2} \implies \text{“}i^{1/2}\text{”} = 1^{1/2}e^{i\cdot\frac{5\pi}{4}} = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 1\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -\frac{1+i}{\sqrt{2}}.$$

Thus the square roots of i *are* $\pm(1+i)/\sqrt{2}$. *Note that these make* 45° *and* 225° *angles with the positive real axis, so when we square these—and thus double the angles—we arrive at angles of* 90° *and* 450°, *which are where we will find* i. *The lengths of either root are* $\sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$, *so when we square these roots we get a complex number with length* $1^2 = 1$. *So our computed roots have the correct angle and the correct length when squared.*

The reader can verify that adding another multiple of 2π *to the original angle for* i *will yield one of the same two square roots of* i *in the process above.*

Anytime we graph the nth roots of a number, on the complex plane these roots will always have the same absolute value (distance from the origin), and successive ones will make angles of $2\pi/n$ between them, because we write the original number with successive angles in increments of 2π, so when we take the "$1/n$" power we get angles differing by $2\pi/n$. This also explains why there will be exactly n such roots, after which the process's outcomes are repeated.

11.6.3 The Complex Plane's Role

While very useful and interesting in their own right, the main purpose of introducing complex numbers here is to show their importance in the theory of power series. In particular, Abel's Theorem is actually a theorem about power series for complex numbers:

Theorem 11.6.1 *Any power series* $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, *where* $z_0, a_0, a_1, a_2, \dots \in \mathbb{C}$, *will converge absolutely either*

(i) at z_0 *only, or*

(ii) on all of $\mathbb{C}$, *or*

(iii) within a circle where $|z - z_0| < R$ *for some* $R > 0$, *and diverge where* $|z - z_0| > R$

In each of these cases, the convergence will be absolute, meaning that $\sum \left|a_k (z - z_0)^k\right|$ *will converge.*

This theorem then applies to $\mathbb{R} \subseteq \mathbb{C}$, and we see that when we intersect the "open circles" of convergence in $\mathbb{C}$ for a series centered at some $a \in \mathbb{R}$, with the real line $\mathbb{R}$, we get open intervals in $\mathbb{R}$ of convergence centered at $a \in \mathbb{R}$. Like the previous statement of Abel's Theorem, there is no mention of the boundary, which is the actual circle $|z - z_0| = R$ in $\mathbb{C}$.

The theorem can shed some light on why the Taylor Series for certain "well-behaved" functions—unlike those in Subsection 11.6.1—fail to converge on all of $\mathbb{R}$: they might not be so well behaved in $\mathbb{C}$.

Example 11.6.6 *Consider the function $f(x) = 1/(x^2+1)$. This function and all of its derivatives exist on all of $\mathbb{R}$, as the reader can verify. Its Maclaurin Series is given by*

$$\frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-1)^k x^{2k},$$

which we get from geometric series methods. The interval of convergence is $x \in [-1, 1]$.

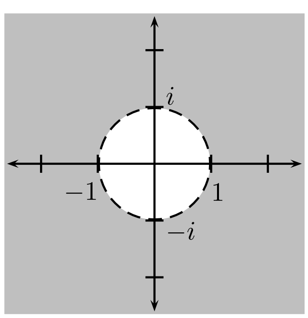
If instead we look at f as a function of a complex variable z with the same formula, we have

$$f(z) = \frac{1}{z^2+1}$$

which is undefined at $z = \pm i$, where the denominator would be zero. With Abel's Theorem stating that outside of a circle of some radius R the series representation

$$f(z) = \frac{1}{z^2+1} = \sum_{k=0}^{\infty}(-1)^k z^{2k}$$

will diverge, and converge inside the open disc bounded by the circle, it is reasonable that the series for $f(z)$ should converge for $|z| < 1$, and diverge for $|z| > 1$ since $|z| = 1$ is where the function first encounters any discontinuities (in this case, in the function and all its derivatives).



In the diagram at the left, the white region is $|z| < 1$, where the series converges (absolutely), and the gray region is $|z| > 1$, where it diverges. Note that $\pm i$ are on the boundaries of the dashed circle. These points $\pm i$ are precisely where $f(z) = \frac{1}{z^2+1}$ has a discontinuity (dividing by zero), and so we should expect the Maclaurin series to be valid at most up to the circle, as per Abel's Theorem, and this would imply absolute convergence of the real-variable series for $f(x)$ within $x \in (-1, 1)$. (One must test the boundary points separately.)

So when we attempt to determine the region in which a series expansion for a function is valid, the more complete context is $\mathbb{C}$. For instance, if we are only looking in $\mathbb{R}$ then the function $f(x) = \frac{1}{x^2+1}$ has no problems in the function itself or its derivatives anywhere in $\mathbb{R}$, but when we consider the context of $\mathbb{C}$, perhaps rewriting it as $f(z) = 1/(z^2+1)$, we can immediately detect a problem at $z = \pm i$.

Example 11.6.7 *Find the largest open interval of convergence for the Taylor Series representation of $f(x) = \frac{1}{x^2+1}$ centered at $x = 5$. Do not write the actual series.*

Solution: Again, in $\mathbb{C}$ the only discontinuities in the function or its derivatives are at $z = \pm i$, which are a distance $\sqrt{5^2 + 1^2} = \sqrt{26}$ from the center $a = 5$, and so the largest open interval of convergence would be $x \in \left(5 - \sqrt{26}, 5 + \sqrt{26}\right)$. The reader is encouraged to draw the open disc in $\mathbb{C}$ as above, though it would be centered at $z = 5$ and would extend to its boundary which would contain $z = \pm i$. (Note that the left endpoint of the real interval of convergence would be negative.) The actual series would be of the form $\sum a_k(x-5)^k$.

The technique above would be much easier than first finding the actual form of the series, and then using a Ratio Test technique to find the actual interval of convergence.

11.6.4 Summary

The reader might at this point be wondering how we know the series referred to in the above example would converge to the function in that interval, while the Maclaurin Series for the function in Example 11.6.3, page 801 does not, even though there are no troubles on all of $\mathbb{R}$ with the function or derivative. The explanation is that the function e^{-1/z^2} has some very violent behavior near $z = 0$ in the complex plane, behavior which does not occur anywhere along the real line.[21]

The correct explanation, which again is not proved here due to the scope of this textbook, is that our usual functions found in this textbook, *with the exception of those defined piecewise* (including $|x|$), will have Taylor Series which converge in any open disc $|z - z_0| < R$, where z_0 is the center and where R is the distance from z_0 to the nearest discontinuity. This was the analysis in Example 11.6.6, page 807 and the subsequent Example 11.6.7. This applies to all combinations of polynomial, root, trigonometric, arc-trigonometric, exponential and logarithmic functions using addition, subtraction, multiplication, division and functional composition (meaning the output of one function is fed as an input into another). It is also helpful to know (by a contrapositive-type argument using Abel's Theorem) that any function with a Taylor Series which converges to that function on all of $\mathbb{R}$ must have that series converge on all of $\mathbb{C}$: if it did not converge on all of $\mathbb{C}$, it could not on all of $\mathbb{R}$ either, as a problem in $\mathbb{C}$ would limit the size of a disc of convergence there, which could therefore not include all of $\mathbb{R}$.

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[21] The point $z = 0$ is called, in complex function theory, an *essential singularity.* In fact, as we can see from the series for e^z, we could write

$$e^{-1/z^2} = 1 - \frac{1}{z^2} + \frac{1}{2! \cdot z^4} - \frac{1}{3! \cdot z^6} + \cdots,$$

we can expect more and more "singular" behavior as $z \to 0$ in $\mathbb{C}$, meaning as $0 < |z| < \varepsilon$ for smaller and smaller $\varepsilon > 0$. Recall how $z^{-n} = (1/z)^n$ will make angle $n\theta$ from the positive real axis, where θ is the angle made by $1/z$, and so these terms in the above series, until the factorials take over, can have some dramatic behavior in the partial sums. (That is not so much the case when $\theta \in \{0, \pi\}$, i.e., when $z \in \mathbb{R}$.)

A surprising and beautiful theorem of complex analysis says that any open disc containing an essential singularity z_0 will "map to" all of $\mathbb{C}$ excepting perhaps a single value, so for such a function f we have the output from the function, with input from the disc, is all of $\mathbb{C}$ or could possibly miss a single value in $\mathbb{C}$. Thus

$$\left\{ f(z) \,\middle|\, 0 < |z - z_0| < \varepsilon \right\} = \mathbb{C}, \text{ or } \mathbb{C} - \{w_0\},$$

where $w_0 \in \mathbb{C}$ depending upon the function. Once a student of complex variables is aware of the nature of an essential singularity (having a series representation with infinitely many negative powers of $(z - z_0)$ being the signature of such functions and their singularities at z_0), detecting them is routine, and that student could use that knowledge to again help detect where a function can be represented by a convergent Taylor Series, and where that is impossible. In fact $f(z) = e^{-1/z^2}$ can have a series in any disc that avoids the singularity, namely the origin. In fact the only value not in the range of the function is zero, though that value is approached as $z \to 0^{\pm}$, that is, along the real axis. That is why we defined $f(x)$ to be zero at $x = 0$ in Example 11.6.3, page 801.

We can conclude that we can find Taylor Series representations for most of the functions we encounter in this textbook, and that these series will be valid on intervals the limits of which might be easier to find by looking at the functions in the complex plane $\mathbb{C}$ instead of in $\mathbb{R}$. That was the case with $f(x) = 1/(x^2+1)$, because we can see $f(z) = 1/(z^2+1)$ has clear problems at $z = \pm i$, but when we look instead at functions such as logarithms, the definitions of which are somewhat complicated in $\mathbb{C}$, it is perhaps better to use real-number methods (such as the Ratio Test), though it should be noted that $f(x) = \ln x$ has a discontinuity at $x = 0$, so we expect the same of $f(z) = \ln z$ (whatever that means), and so the disc in $\mathbb{C}$ in which a series centered at $z = 1$ cannot extend more than a distance of 1 in any direction, so clearly neither can the interval of convergence in $\mathbb{R}$.

Piecewise-defined functions have the other difficulty discussed in this Section 11.6, in that a Taylor Series that works very well for the formula for one piece is unlikely to extend to the other pieces, which we expect to have different formulas for their definitions there.

With these two ideas in mind (being wary of piece-wise defined functions, and the possibility of looking into $\mathbb{C}$ to find where a real Taylor Series converges), one can avoid some common mistakes of scientific researchers who assume a series expansion of a function in order to fit data to polynomials. That assumption is often correct, but not always, and it is important to be able to detect when function input values lie outside the interval where a Taylor Series is valid.

Exercises

1. Show by direct computation that if $z = (1+i)/\sqrt{2}$, then $z^2 = i$.

2. Find the four fourth roots of -16, using the technique in Example 11.6.4, page 11.6.4. Graph all the roots together.

3. Where will the Maclaurin Series for $f(x) = 1/(x^4+16)$ be valid? Use two different methods for solving this:
   (a) using geometric series arguments, and
   (b) using the previous problem and a complex plane argument.

4. Consider the *complex conjugate* of a complex number $z \in \mathbb{C}$ defined by $\overline{z}$ as below:
$$\begin{aligned} & z = x + iy \\ \iff & \overline{z} = x - iy. \end{aligned} \tag{11.57}$$
   This is also written $\overline{x+iy} = x - iy$.
   (a) Show that $z\overline{z} = |z|^2$.
   (b) Show how to use this with division, where
$$\frac{a+bi}{c+di} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

5. Show that when we divide z_1 by z_2, the quotient z_1/z_2 has angle $\theta = \theta_1 - \theta_2$, there z_1, z_2 have angles θ_1, θ_2, respectively, in the sense of Figure 11.5, page 805.

6. Show that z and $\frac{1}{z}$ will have angles whose terminal rays point in opposite directions, assuming $z \neq 0$.