Lecture 38:
Examples of Laurent Series

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38.1 Examples of Laurent series

Example 38.1. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

for all $z \in \mathbb{C}$, we have

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \cdots$$

for all z with $|z| > 0$. We shall see later that Laurent series expansions are unique, and so this must be the Laurent series representation for $e^{\frac{1}{z}}$. In particular, we know that if C is a simple closed contour about the origin, with positive orientation, then the coefficient of $\frac{1}{z}$ is

$$b_1 = \frac{1}{2\pi i} \int_C e^{\frac{1}{z}} dz.$$

Since $b_1 = 1$, we have

$$\int_C e^{\frac{1}{z}} dz = 2\pi i.$$

Example 38.2. Let

$$f(z) = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)}.$$

From the theory of partial fractions, we know there exist constants A and B such that

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}.$$

Letting $z = 1$, we see that $1 = -A$, and letting $z = 2$, we see that $2 = B$. Hence $A = -1$ and $B = 2$, so

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}.$$

Let $D_1 = \{z \in \mathbb{C} : |z| < 1\}$, $D_2 = \{z \in \mathbb{C} : 1 < |z| < 2\}$, and $D_3 = \{z \in \mathbb{C} : |z| > 2\}$.

For $z \in D_1$, we find a Maclaurin series for $f(z)$:

$$\begin{aligned} f(z) &= \frac{2}{z-2} - \frac{1}{z-1} \\ &= \frac{1}{1-z} - \frac{1}{1-\frac{z}{2}} \\ &= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^n} \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) z^n. \end{aligned}$$

Note that these expansions are valid since, for $z \in D_1$, $|z| < 1$ and $\left|\frac{z}{2}\right| < 1$.

For $z \in D_2$, we find a Laurent series for $f(z)$:

$$\begin{aligned} f(z) &= -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} - \frac{1}{1-\frac{z}{2}} \\ &= -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^n} \\ &= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{z^n}{2^n} \end{aligned}$$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

Note that these expansions are valid since, for $z \in D_2$, $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

For $z \in D_3$, we have

$$\begin{aligned}
f(z) &= -\frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} + \frac{2}{z} \cdot \frac{1}{1 - \frac{2}{z}} \\
&= -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} + \frac{2}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} \\
&= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{2^{n+1} - 1}{z^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{2^n - 1}{z^n}.
\end{aligned}$$

Note that these expansions are valid since, for $z \in D_3$, $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$