Set Theory and Logic

Supplementary Materials

Math 103: Contemporary Mathematics with Applications

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Chapter 1

Set Theory

1.1 Basic definitions and notation

A **set** is a collection of objects. For example, a deck of cards, every student enrolled in Math 103, the collection of all even integers, these are all examples of **sets** of things. Each object in a set is an **element** of that set. The two of diamonds is an element of the set consisting of a deck of cards, one particular student is an element of the set of all students enrolled in Math 103, the number 4 is an element of the set of even integers.

We often use capital letters such as A to denote sets, and lower case letters such as a to denote the elements.

Definition 1. Given a set A, if u is an element of A we write

$$u \in A.$$

If the element u is not in the set A we write

$$u \notin A.$$

Some sets that you may have encountered in mathematics courses before are:

- The integers $\mathbb{Z}$
- The even integers $2\mathbb{Z}$
- The set of rational numbers $\mathbb{Q}$
- The set of real numbers $\mathbb{R}$.

We can now practice using our element notation:

Example 1.1.1. We have $4 \in 2\mathbb{Z}$.

Example 1.1.2. $16 \in \mathbb{Z}$,

Example 1.1.3. $3 \notin 2\mathbb{Z}$.

Example 1.1.4. $\sqrt{3} \notin \mathbb{Q}$

So far, we have been defining sets by describing them in words. We can also specify some sets by listing their elements. For example, define the set T by writing

$$T = \{a, b, c, d, e\}.$$

When defining a set by listing, always use the brackets $\{,\}$. Another set that we can define by listing is the set of **natural numbers**

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\},$$

where we have indicated a general pattern (hopefully easily regognized!) by writing $\cdots$. Many sets cannot be listed so easily (or at all for that matter), and in many of these cases it is convenient to use a **rule** to specify a set. For example, suppose we want to define a set S that consists of all real numbers between -1 and 1, inclusive. We use the notation

$$S = \{x | x \in \mathbb{R} \text{ and} -1 \leq x \leq 1\}.$$

We read the above as "S equals the set of all x such that x is a real number and x is greater than or equal to -1, and less than or equal to 1." What happens if someone specifies a set by a rule like "x is a negative integer greater than 1000"? What should we do? There are no numbers that are negative and greater than 1000. We allow examples of rules of this kind, and make the following definition:

Definition 2. The **empty set** is the set with no elements, and is denoted by the symbol ϕ, or by $\{\ \}$.

Thus, the above set $\{x | x \in \mathbb{Z}, x < 0 \text{ and } x > 1000\} = \{\ \} = \phi$.

Definition 3. Two sets are **equal** if they have exactly the same elements, denoted

$$A = B.$$

If A and B are not equal, we write $A \neq B$.

Example 1.1.5. Let $T = \{a, b, c, d, e\}$ and let $R = \{e, d, a, c, b\}$. We can check that T and R have exactly the same elements, so $T = R$.

Example 1.1.6. Let $S = \{x | x \in \mathbb{Z} \text{ and } x \leq 0\}$, and let $A = \{3n | n \in \mathbb{Z}\}$. We can see that $S \neq A$ because A consists of all integer multiples of 3, hence $3 \in A$ but $3 \notin S$. This shows $S \neq A$.

As we have seen from our examples, sets may contain a finite number of elements, or an infinite number of elements. Examples of finite sets include T from Example 1.1.5, and also the set of students enrolled in Math 103. Examples of infinite sets are $\mathbb{Z}$ and $\mathbb{R}$.

Definition 4. If a set S is finite, we let $n(S)$ denote the **number of elements** in S.

Example 1.1.7. Let T be as in Example 1.1.5, then $n(T) = 5$.

1.2 Subsets

One important relation between sets is the idea of a subset. Given sets A and B, we say B is a **subset** of A if every element of B is also an element of A. We denote this as

$$B \subseteq A.$$

Example 1.2.1. $\{2, 4, 6\} \subseteq 2\mathbb{Z}$.

Example 1.2.2. Let $A = \{a, b, c, d, e\}$, and $B = \{a, e\}$ then $B \subseteq A$.

Example 1.2.3. Let's list all subsets of A from Example 1.2.2 that have four elements:

$$\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}.$$

For any set A, since every element of A is in A we have $A \subseteq A$. This says that a set is always a subset of itself. We also consider the empty set to be a subset of any set A, $\phi \subseteq A$.

Let $S = \{a, b, c, d\}$, let's list **all** subsets of the set $S = \{a, b, c, d\}$. To organize our work, we will list them by size.

Table 1.1: Subsets of S

number of elements	subsets
0	ϕ
1	$\{a\}, \{b\}, \{c\}, \{d\}$
2	$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\{c, d\}$
3	$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
4	$\{a, b, c, d\}$

We have listed all of the subets of S. Notice that there are 16 of them. In fact, one can prove the following theorem by using methods of counting covered later in this course.

Theorem 1.2.4. *Let S be a set having N elements. Then there are 2^N subsets of S.*

1.3 Union, Intersection, and Complement

Let U be a set. Given two subsets A and B of U we define the **union** of A and B to be the subset of U that contains all elements that are in A, or in B, or possibly in both. The union of A and B is denoted

$$A \cup B.$$

In our "rule" notation $A \cup B = \{x \in U | x \in A \text{ or } x \in B, \text{ or both}\}$.

Example 1.3.1. Let $U = \{1, 2, 3, \cdots 10\}$. Let $S = \{2, 4, 6, 8, 10\}$, $T = \{5, 6, 7, 8\}$. Then

$$S \cup T = \{2, 4, 5, 6, 7, 8, 10\}.$$

We often use what is known as a **Venn diagram** to illustrate sets. In a Venn diagram circles are used to represent subsets of a set U (denoted by a large rectangle). Here is a Venn diagram illustrating $A \cup B$.

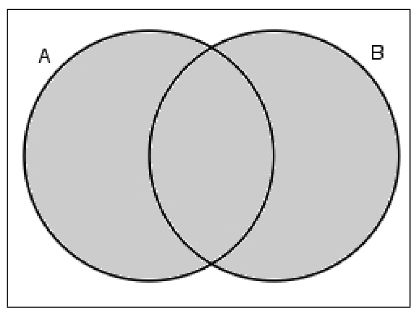


Figure 1.1: $A \cup B$

We have the following facts about the union:

1. $A \cup \phi = A$
2. $A \cup A = A$
3. $A \cup B = B \cup A$
4. $(A \cup B) \cup C = A \cup (B \cup C)$

We define the **intersection** of subsets A and B of U to be the subset of U that contains all of the elements that are in both A and B. The intersection of A and B is denoted

$$A \cap B.$$

We have $A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$.

Here is a Venn diagram illustrating the intersection:

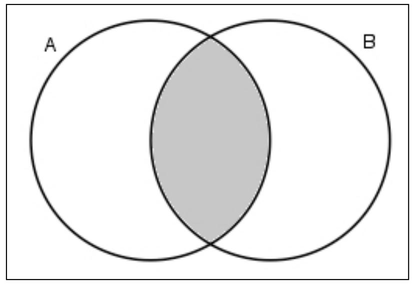


Figure 1.2: $A \cap B$

Example 1.3.2. If U, S and T are given as in Example 1.3.1 above, then $S \cap T = \{6, 8\}$.

We have the following facts about the intersection:

1. $A \cap \phi = \phi$
2. $A \cap A = A$
3. $A \cap B = B \cap A$
4. $(A \cap B) \cap C = A \cap (B \cap C)$

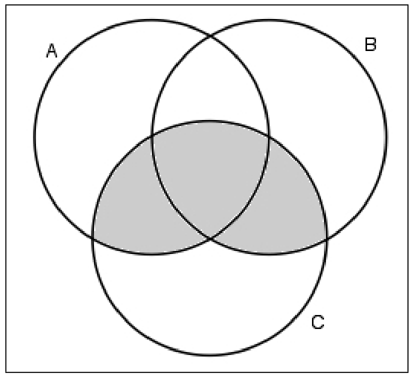
Given the two operations $\cup, \cap$ we can apply them in combination, as long as we remember to use parenthesis to indicate in what order the operations should be performed.

Example 1.3.3. Let $U = \{a, b, c, d, e, f, g\}$, let $S = \{a, e\}, H = \{a, b, c, d\}, K = \{a, c, e, f\}$. Then

$$(S \cap H) \cup K = \{a\} \cup K = \{a, c, e, f\},$$
$$S \cap (H \cup K) = S \cap \{a, b, c, d, e, f\} = \{a, e\},$$
$$(S \cap H) \cap K = \{a\}.$$

Notice that $(S \cap H) \cup K \neq S \cap (H \cup K)$. It is important to always use parenthesis in the appropriate place when working with three or more sets, statements like "$A \cap B \cup C \cap D$" do not have one interpretation so do not actually specify a set. The exception is when all operations are the same, as in properties (4) of intersection and union. For example $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$.

Example 1.3.4. Venn diagrams illustrating the sets $(A \cup B) \cap C$, and $A \cup (B \cap C)$.



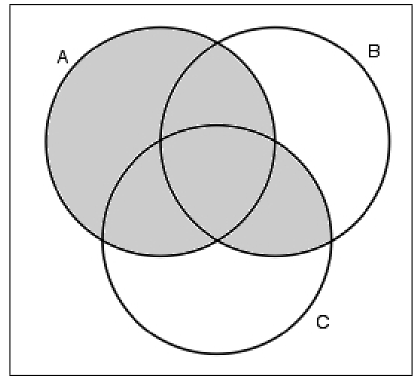


Figure 1.3: $(A \cup B) \cap C$ $\qquad A \cup (B \cap C)$

Given A a subset of U, the **complement** of A is the subset of U consisting of all elements not in A. The complement of A is denoted A'.

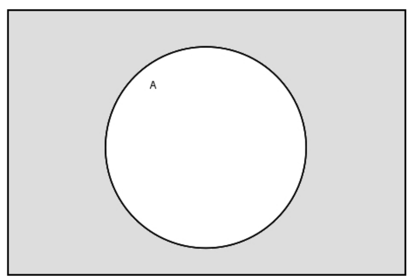


Figure 1.4: A'

Example: Let $U = \{\cdots -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$, let $\mathbb{N} = \{0, 1, 2, 3 \cdots\}$. Then

$$N' = \{-1, -2, -3, -4, \cdots\}.$$

The complement satisfies the following rules:

1. $(A')' = A$
2. $U' = \phi$ and $\phi' = U$
3. $A \cup A' = U$
4. $A \cap A' = \phi$

Example 1.3.5. Let D be the set of a standard deck of cards. Let R be the subset of red cards, let F be the subset of face cards. (The face cards include all suits of K, Q, J.) Find the following sets: $(R \cup F)'$, $R' \cap F'$, $R' \cup F'$.

The set

$$R \cup F = \{A\heartsuit, A\diamondsuit, 2\heartsuit, 2\diamondsuit, \cdots K\heartsuit, K\diamondsuit, K\spadesuit, K\clubsuit, Q\spadesuit, Q\clubsuit, J\spadesuit, J\clubsuit\}$$

i.e. consists of all cards that are either red, or black face cards. The complement of $R \cup F$ consists of the cards **not** listed above and is

$$(R \cup F)' = \{A\clubsuit, A\spadesuit, 2\clubsuit, 2\spadesuit, \cdots, 10\clubsuit, 10\spadesuit\}.$$

The set R' is the set of black cards, the set F' is the set of non-face cards (of any suit), so the intersection is the set of black non-face cards:

$$R' \cap F' = \{A\spadesuit, A\clubsuit, 2\spadesuit, 2\clubsuit, \cdots, 10\spadesuit, 10\clubsuit\}.$$

This is the same set as $(R \cup F)'$. Now let's find $R' \cup F'$ the union of the black cards and the non-face cards.

$$R' \cup F' = \{A\heartsuit, A\diamondsuit, 2\heartsuit, 2\diamondsuit, \cdots 10\heartsuit, 10\diamondsuit, A\clubsuit, A\spadesuit, 2\clubsuit, 2\spadesuit, \cdots J\clubsuit, J\spadesuit, Q\clubsuit, Q\spadesuit, K\clubsuit, K\spadesuit\}.$$

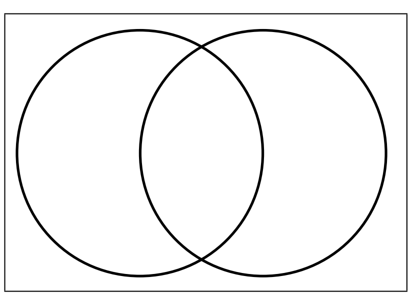
We see that $R' \cup F'$ is **not** equal to $(R \cup F)'$.

Theorem 1.3.6. *De Morgan's Laws: Given two sets $A, B \subset U$,*

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Example 1.3.7. Fill in the Venn diagrams for $(A \cup B)'$, and for $A' \cap B'$.



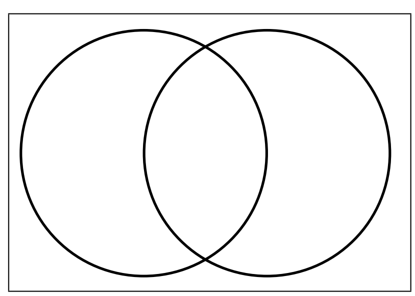


Figure 1.5: $(A \cup B)'$ $\qquad A' \cap B'$

If we are interested in elements of a set A that are not contained in a set B, we can write this set as $A \cap B'$. This concept comes up so often we define the **difference** of two sets A and B:

$$A - B = A \cap B',$$

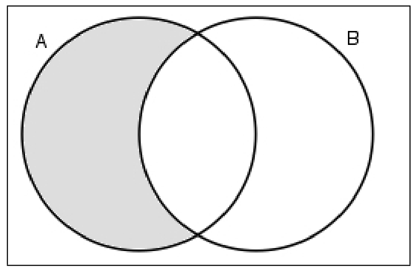


Figure 1.6: $A - B$

For example, if S is the set of all juices in the supermarket, and T is the set of all foodstuffs in the supermarket with added sugar, then $S - T$ is the set of all juices in the market without added sugar.

1.4 Cardinality and Survey Problems

If a set S is finite, recall that $n(S)$ denotes the number of elements in S.

Example 1.4.1. Let D denote a standard deck of cards. $n(D) = 52$.

Theorem 1.4.2. *If A and B are both finite sets, then*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

To see how this theorem works, lets consider our set D. Let the set of all red cards be denoted R, and let the set of face cards be denoted F. How many elements are in $R \cup F$? We can count them as listed in Example 1.3.5, or we can use the formula. The intersection consists of the six red face cards: $\{K\heartsuit, K\diamondsuit, Q\heartsuit, Q\diamondsuit, J\heartsuit, J\diamondsuit\}$. Using the formula gives

$$n(R \cup F) = n(R) + n(F) - n(R \cap F) = 26 + 12 - 6 = 32.$$

What we should **not** do is simply add the number of red cards to the number of face cards, if we do that we have counted the red face cards twice.

We can use our formula for the number of elements to analyze surveys.

Example 1.4.3. Suppose Walter's online music store conducts a customer survey to determine the preferences of its customers. Customers are asked what type of music they like. They may choose from the following categories: Pop (P), Jazz (J), Classical (C), and none of the above (N). Of 100 customers some of the results are as follows:

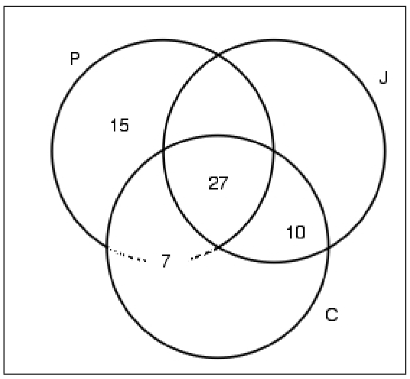
44 like Classical

27 like all three

15 like only Pop

10 like Jazz and Classical, but not Pop

How many like Classical but not Jazz? We can fill in the Venn diagram below to keep track of the numbers. There are $n(C) = 44$ total that like Classical, and $n(C \cap J) = 27 + 10 = 37$ that like both Jazz and Classical, so $44 - 37 = 7$ like Classical but not Jazz.



Example 1.4.4. Let's look at some more survey results from Example 1.4.3:

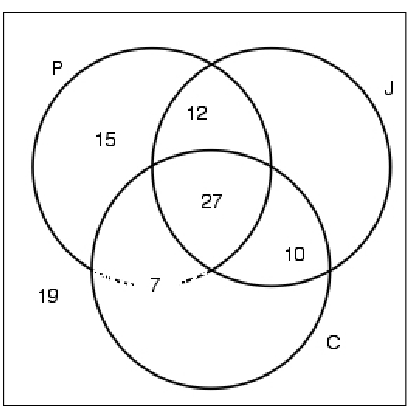
78 customers like Jazz or Pop (or possibly both).

19 customers marked "None of the above" when asked what they like.

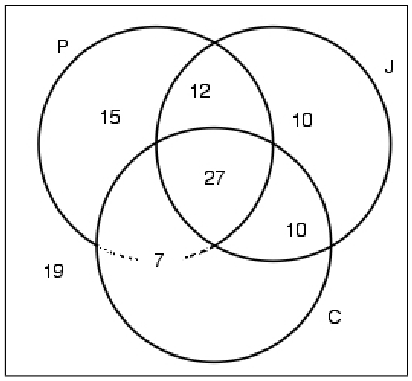
12 like Jazz and Pop, but not classsical.

How many like only Jazz?

To answer this, let's fill in more of the diagram:



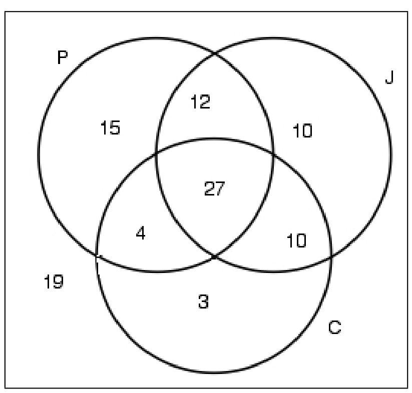
We have $n(C) = 44$, $n((P \cup J \cup C)') = 19$. If we let j be the number of surveyed customers who like only Jazz, then because there are 100 surveyed customers, we see $19 + 44 + 15 + 12 + j = 100$. Solving for j gives $j = 10$.



How many like Pop and Classical, but not Jazz?

We know that $n(P \cup J) = 78$. Using the diagram, the number who like Pop and Classical, but not Jazz is

$$78 - 10 - 27 - 12 - 15 - 10 = 4.$$



1.5 Cartesian Products

You may recall the Cartesian plane $\mathbb{R}^2$ which is the set of all points in the plane. This set consists of **ordered** pairs of numbers (x, y) where x and y are real numbers. The point $(1, 2)$ is not the same as $(2, 1)$. We use round brackets $(,)$ to denote ordered pairs, reserving the brackets $\{,\}$ for sets.

We can make a more general definition involving ordered pairs: Given two sets A, B we define the *Cartesian product* to be

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Example 1.5.1.

$$\{2, 3, 4\} \times \{7, 9, 10\} = \{(2, 7), (2, 9), (2, 10), (3, 7), (3, 9), (3, 10), (4, 7), (4, 9), (4, 10)\}$$

Theorem 1.5.2. *If A and B are two finite sets, then the number of elements in the Cartesian product $A \times B$ is given by*

$$n(A \times B) = n(A) \times n(B).$$

Example 1.5.3. If we roll two dice, and create a set of all possible results. How many elements are there?

We can think of the possible results of rolling dice as a set of ordered pairs. Let $D1$ denote the set of possible results of rolling the first die $D1 = \{1, 2, \cdots 6\}$, and let $D2$ denote the set of possible results of rolling the second die, $D2 = \{1, 2, \cdots 6\}$. There are $6 \times 6 = 36$ possible results from rolling the pair:

$$\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Example 1.5.4. Write out the subset of $D1 \times D2$ that represents cases where the sum of the numbers showing is either 7 or 11. How many elements are in this set?

The subset we are looking for is

$$\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \cup \{(5, 6), (6, 5)\}$$

$$= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (5, 6), (6, 5)\}.$$

and the number of elements is $6 + 2 = 8$.

1.6 Excercises

1. Let $U = \{1, 2, 3, 4, 5, \cdots, 10\}$ $A = \{2, 4, 6, 8, 10\}$ $B = \{3, 6, 9\}$ $C = \{1, 2, 3, 8, 9, 10\}$ perform the indicated operations

   (a) $A \cap B$

   (b) $A \cup B$

   (c) $A' \cap C$

   (d) $(A \cap C)'$

   (e) $(A \cup B) \cap C$

   (f) $(A \cup B) \cap A$

2. Determine if the following statements are true or false. Here A represents any set.

   (a) $\phi \subseteq A$

   (b) $A' \subseteq A$

   (c) $(A')' = A$

3. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 4, 5, 9\}$.

   (a) Find $A \cup B$

   (b) Find $A \cap B$

   (c) Use a Venn diagram to represent these sets.

4. Let U be the set of integers. Let $A = \{3, 6\}, B = \{3, 8, 10, 12\}$ and $C = \{6, 8, 10\}$. Perform the indicated operations.

   (a) $B \cap C'$

   (b) $(A' \cup C')'$

   (c) $(A \cup C) \cap B$

   (d) $B - C$

   (e) $C - B$

   (f) $B \cup C'$

   (g) $A \times C$

5. Use Venn diagrams to verify DeMorgan's laws.

6. Represent the following sets with a Venn diagram

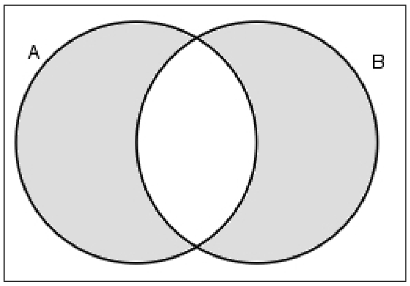
   (a) $(B \cup C) \cap A'$

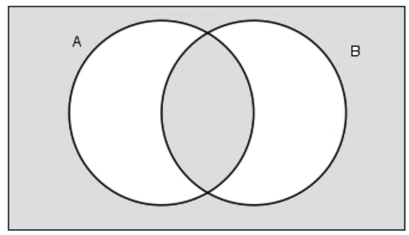
(b) $(A \cap B) \cup C$

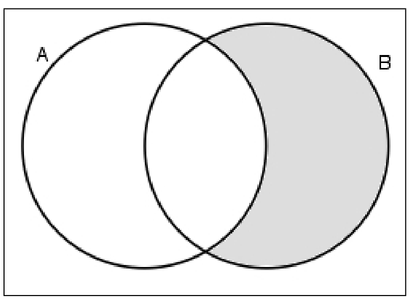
7. Denote the set $A = \{x | x \in \mathbb{Z} \text{ and } x < 3\}$ by the listing method.

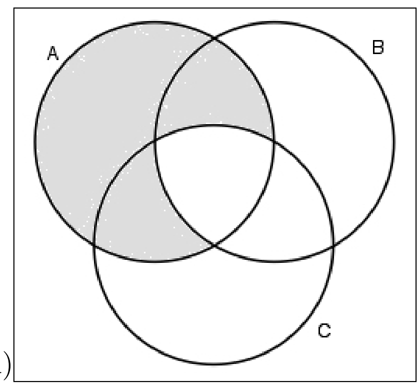
8. A **proper** subset of a set A is one that is not equal to the set A itself. If a set has 6 elements, how many proper subsets does it have?

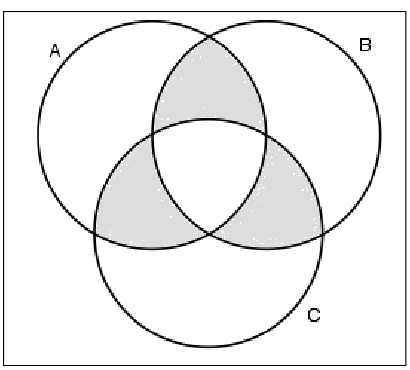
9. Describe the shaded region using $\cap, \cup, ', -$:

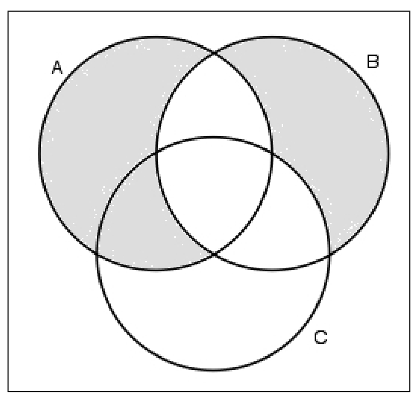
a) 

b) 

c) 

d) 

e) 

f) 

10. One hundred students were surveyed and asked if they are currently taking math (M), English (E) and/or History (H) The survey findings are summarized here:

Table 1.2: Survey Results

$n(M) = 45$	$n(M \cap E) = 15$
$n(E) = 41$	$n(M \cap H) = 18$
$n(H) = 40$	$n(M \cap E \cap H) = 7$
$n[(M \cap E) \cup (M \cap H) \cup (E \cap H)] = 36$	

    (a) Use a Venn diagram to represent this data.

    (b) How many students are only taking math?

11. Ninety people at a Superbowl party were surveyed to see what they ate while watching the game. The following data was collected:

    48 had nachos.

    39 had wings.

    35 had a potato skins.

    20 had both wings and potato skins.

    19 had both potato skins and nachos.

    22 had both wings and nachos.

    10 had nachos, wings and potato skins.

    (a) Use a Venn diagram to represent this data.

    (b) How many had nothing?

12. In Example 1.5.3 how many pairs sum to an even number, or one greater than 9?

13. Let D be a standard deck of cards, let $S\heartsuit = \{A\heartsuit, K\heartsuit, Q\heartsuit, J\heartsuit\}$. a) List all the subsets of $S\heartsuit$ that contain both the $A\heartsuit$ and $K\heartsuit$. How many subsets of this type are there? Discuss why this is the same number as **all** the subsets of $\{Q\heartsuit, J\heartsuit\}$.

    b) How many subsets of D contain both the $A\heartsuit$ and $K\heartsuit$?

14. Answer the following True or False.

    (a) $\{1, 2, 3\}$ is a subset of $\{3, 2, 1, 4\}$.

    (b) $\{3, 2, 1, 4\}$ is a subset of $\{1, 2, 3\}$.

(c) The empty set is a subset of every set.

(d) 1 is an element of $\{3, 2, 1, 4\}$.

(e) $\{1\}$ is an element of $\{3, 2, 1, 4\}$.

(f) $\{1\}$ is a proper subset of $\{3, 2, 1, 4\}$

(g) $\{3, 2, 1, 4\} = \{1, 2, 3, 4\}$

(h) $(0, 1/2)$ is an element of $\mathbb{Q} \times \mathbb{Z}$

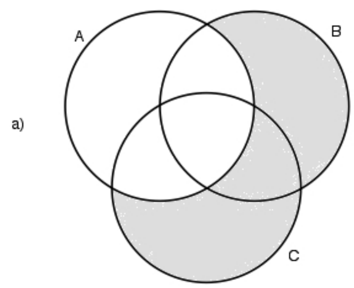
(i) $(0, 1/2)$ is an element of $\mathbb{Z} \times \mathbb{Q}$

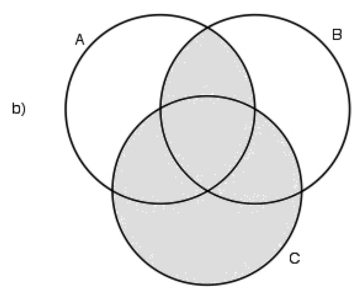
(j) $(-7/8, 0)$ is an element of $\mathbb{Q} \times \mathbb{Q}$

(k) $(-7/8, 0)$ is an element of $\mathbb{Z} \times \mathbb{Z}$

1.7 Solutions to exercises

1. a) $\{6\}$, b)$\{2,3,4,6,8,9,10\}$, c) $\{1,3,9\}$, d)$\{1,3,4,5,6,7,9\}$, e) $\{2,3,8,9,10\}$, f) A.
2. a) T b) F c) T.
3. a) $\{1,3,4,5,7,9\}$, b) $\{1,5,9\}$
4. a) $\{3,12\}$, b) $\{6\}$, c) $\{3,8,10\}$, d) $\{3,12\}$, e) $\{6\}$, f) all integers except 6,
   g) $\{(3,6),(3,8),(3,10),(6,6),(6,8),(6,10)\}$
6.





7. $\{\cdots,-4,-3,-2,-1,0,1,2\}$
8. 63
9. a) $(A-B)\cup(B-A)$ b) $(A\cup B)'\cup(A\cap B)$ c) $B-A$ d) $A-(A\cap B\cap C)$
   e) $[(A\cap B)\cup(A\cap C)\cup(B\cap A)]-[A\cap B\cap C]$ f) $[(A\cup B)-(A\cap B)]-(B\cap C)$
10. b) 19
11. b) 19
12. $18+6-4=20$
13. a) $\{A\heartsuit,K\heartsuit\}$, $\{A\heartsuit,K\heartsuit,Q\heartsuit\}$, $\{A\heartsuit,K\heartsuit,J\heartsuit\}$, $\{A\heartsuit,K\heartsuit,Q\heartsuit,J\heartsuit\}$. b) 2^{50}.
14. a) True, b) False, c) True, d) True, e) False, f) True, g) True, h) False, i) True, j) True, k) False

Chapter 2

Logic

2.1 Statements and Connectives

Symbolic logic studies some parts and relationships of the natural language by representing them with symbols. The main ingredients of symbolic logic are *statements* and *connectives.*

A *statement* is an assertion that can be either true or false.

Examples. The following sentences:

It is sunny today;
Ms. W. will have a broader audience next month;
I did not join the club;

are statements, while questions (e.g. *How's the weather*?), interjections (e.g. *Cool!*) and incomplete sentences (e.g. *If I could ...*) are not considered to be statements unless rephrased appropriately.

Simple statements do not contain other statements as their parts. (All of the examples above are simple statements.) We typically represent simple statements using lower-case letters p, q, r, ...; for example

s= *Your bicycle is slick*;
c= *I like its color.*

Connectives join simple statements into more complex statements, called *compound statements.* The most common connectives and their symbols are:

$$and/but = \wedge; \qquad or = \vee; \qquad if \ldots, then = \rightarrow .$$

Example. $Your\ bicycle\ is\ slick\ and\ I\ like\ its\ color = s \wedge c.$

The "operation"

$$not = \neg$$

turns a single statement into its negation and it is not a connective.

The symbols representing statements, connectives, and the negation operation form our dictionary. Parentheses are used for punctuation.

Simple statements.

p	*p is true*	(Assertion)
$\neg p$	*p is false*	(Negation)

Connectives and compound statements.

$p \wedge q$	*p and q*	(Conjunction)
$p \vee q$	*either p or q, or both*	(Disjunction)
$p \to q$	*if p then q*	(Conditional)

Notes.

1. The connective *or*, in logic, has an inclusive meaning. For example, *Bob will play tennis or go to the movies* is interpreted as follows: *Bob will either play tennis, or go to the movies, or do both.*

2. The connective *but* has an identical role as the connective *and*, thus the same symbol $\wedge$ is used for both. For example, *Your bicycle is slick, but I don't like its color* is written symbolically as $b \wedge \neg c$.

Parentheses.

The use of parentheses is important and needs particular attention. Suppose we want to convert the following compound statement into symbolic form:

If I do a web search for pages containing the terms "termites" or "cattle", then I will search for pages containing "global warming".

First we identify the simple statements present in these expression and assign letters to each of them (we can rephrase them slightly without modifying their meaning):

t=I search for pages containing "termites;"
c=I search for pages containing "cattle;"
g=I search for pages containing "global warming."

Connectives and punctuation help with splitting compound statements into simple ones. For example, the particle *then* splits the compound statement into two parts: the first part is the disjunction $t \vee c$, and the second is the simple statement g. The symbolic form of the compound statement is then

$$(t \vee c) \rightarrow g.$$

Note. If we had accidentally skipped the parenthesis, we would have created the compound statement $t \vee c \rightarrow g$, which could be read as: *I search for pages containing "termites" or if I search for pages containing "cattle", then I search for those containing "global warming".* This has a rather different meaning from the original statement!

2.1.1 Exercises

1. Convert the following compound statements into symbolic statements, by assigning symbols to each simple statements (for example, f="*my favorite dish has lots of anchovies*") and using the appropriate connectives:

   (a) *Jim is a lawyer, yet he is not a crook.*

   (b) *Although our professor is young, he is knowledgeable.*

   (c) *My favorite dish has lots of anchovies or is not spicy and also it comes in a large portion.*

   (d) *My favorite dish has lots of anchovies and it comes in a large portion or it is spicy and it also comes in a large portion.*

   (e) *If you do not attend class, then either you read a book or you will not pass the exam.*

   (f) *I am doing a web search for pages containing the terms "global warming", but not for pages containing both "termites" and "cattle".*

   (g) *I am doing a web search for pages containing the terms "global warming", but not for pages containing the word "termites" and not for pages containing the word "cattle".*

2. Convert the following symbolic statement into words if
   s=*The sunroof is extra*,
   r=*The radial tires are included*,
   w=*Power windows are optional.*

   (a) $(s \wedge w) \rightarrow r$.

   (b) $r \wedge [s \vee (\neg w)]$.

   (c) $(\neg r) \rightarrow [(\neg s) \vee (\neg w)]$.

   (d) $\neg(w \wedge s)$.

2.2 Truth Values and Truth Tables

Every logical statement, simple or compound, is either true or false. We say that the *truth value* of a statement is true (represented by the letter T) when the statement is true, and false (represented by the letter F) when the statement is false. The truth value of a compound statement can always be deduced from the truth values of the simple statements that compose it.

Example. If p=*I play the piano* is false, and q=*I study logic* is true, then the conjunction $p \wedge q$ =*I play the piano and study logic* is a false statement.

A *truth table* summarizes all possible truth values of a statement. For example, p can only either be true (T) or false (F), so its truth table (the simplest of all) is:

p
T
F

The next simplest truth table is the truth table for the negation, whose truth values are always the opposite as those of the original statement:

p	$\neg p$
T	F
F	T

The truth tables for the conjunction and the disjunction are shown next. An easy way to remember them is to note that the statement $p \wedge q$ (conjunction) is true only when p and q are both true; while the statement $p \vee q$ (disjunction) is false only when p and q are both false.

(Conjunction)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(Disjunction)

Example. Construct the truth table for the compound statement $\neg(p \vee q) \wedge p$.

We will first break down this statement in components of increasing complexity: the simple statements p and q, the disjunction $p \vee q$, its negation $\neg(p \vee q)$, and finally the statement $\neg(p \vee q) \wedge p$. We will create one column for each of these components:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg(p \vee q) \wedge p$
T	T			
T	F			
F	T			
F	F			

and fill them out according to the

2.2.1 Basic Rules

1. *The negation $\neg$ reverses truth values.*
   (So the values in the fourth column are the opposite as the values in the third column.)

2. *The only case in which a conjunction $\wedge$ of two statements is true is when both statements are true.*

3. *The only case in which a disjunction $\vee$ of two statements is false is when both statements are false.*

At the end, we obtain the completed truth table:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg(p \vee q) \wedge p$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

2.2.2 Exercises

1. Fill in the missing values in the following truth table:

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$(\neg p) \vee q$	$[p \wedge \neg q] \vee [(\neg p) \vee q]$
T	T	F	F	F	T	T
T	F	F		T		T
F	T	T	F	F	T	T
F	F	T	T		T	

2. Construct truth tables for the following compound statements

   (a) $p \wedge \neg q$.
   (b) $\neg(\neg p) \wedge p$.
   (c) $\neg(p \vee q)$.
   (d) $(\neg p) \wedge \neg q$.
   (e) $\neg(p \vee \neg q) \vee p$.

3. If p is a true statement and q is a false statement, what are the truth values of the following statements? (For such problems, you need only construct one row of the truth table.)

   (a) $\neg(p \wedge q)$.
   (b) $\neg(p \wedge \neg q)$.

2.2.3 Big Truth Tables

Compound statements may contain several simple statements; in order to figure out how many columns you need in the truth table:

Break the compound statement into "building blocks" of increasing complexity, starting with a column for each letter, and ending with the compound statement itself.

We illustrate the procedure for the following compound statement containing three simple statements (the most difficult case we will encounter):

$$(p \wedge \neg r) \vee (q \vee r).$$

If we break the statement into increasingly complex pieces, we will need one column for each of the following (in order):

- p, q, r (the simplest building blocks)
- $\neg r$
- $p \wedge \neg r$ and $q \vee r$
- $(p \wedge \neg r) \vee (q \vee r)$ (the most complex block)

So that the truth table will contain the **seven** columns:

p	q	r	$\neg r$	$p \wedge \neg r$	$q \vee r$	$(p \wedge \neg r) \vee (q \vee r)$

Next, we fill in the first three columns. Since each of the 3 statements p, q, r is either true (T) or false (F), there are $2^3 = 8$ possibilities. (In fact, r can be true or false for each of the 4 possible pairs of truth values associated with p and q.) Here is the truth table with the first three columns filled in

p	q	r	$\neg r$	$p \wedge \neg r$	$q \vee r$	$(p \wedge \neg r) \vee (q \vee r)$
T	T	T				
T	F	T				
F	T	T				
F	F	T				
T	T	F				
T	F	F				
F	T	F				
F	F	F				

In order to fill in the remaining columns, we follow the Basic Rules 2.2.1. Here is a partially completed truth table, complete the rest on your own.

p	q	r	$\neg r$	$p \wedge (\neg r)$	$q \vee r$	$(p \wedge \neg r) \vee (q \vee r)$
T	T	T	F	F	T	T
T	F	T		F		T
F	T	T	F	F	T	
F	F	T			T	T
T	T		T	T		T
T	F	F	T	T	F	
F	T	F	T	F	T	T
F	F	F	T		F	F

2.2.4 Exercises

1. Construct the truth tables for the following compound statements:

   (a) $p \wedge (q \vee r)$.

   (b) $\neg(p \vee (\neg q)] \wedge r$.

   (c) $(r \wedge p) \vee \neg q$.

   (d) $(p \wedge q) \vee (p \wedge r)$.

2.3 Conditional statements and their truth tables

A compound statement of the form "*If p then q*", written symbolically as

$$p \rightarrow q,$$

is called a *conditional statement*; p is called the *antecedent*, and q is called the *consequent* of the conditional statement.

Example. Consider the conditional statement: *If M is a human being, then M is mortal.* In this statement, p=*"M is a human being"* is the antecedent, and q=*"M is mortal"* is the consequent. The truth table for a conditional statement is shown below:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

and it is best justified by looking at the special case in which the conditional statement is in the form of a promise, such as

If you deliver this pizza by 7PM, then I will give you a \$5 bonus.

For this conditional statement, the antecendent is p=*you deliver this pizza by 7PM* and the consequent is q=*I will give you a \$5 bonus.* Clearly, if you deliver the pizza by the stated time ($p = T$) and I give you \$5 ($q = T$), then the conditional is a true promise (i.e. $p \to q$ is true). It is also clear that, if you deliver the pizza by 7PM ($p = T$), but I do not give you \$5 ($q = F$), then the promise was broken (i.e. $p \to q$ is false). Now, suppose you do not deliver the pizza by 7PM ($p = F$), then, whatever my decision is (to give you or not give you the \$5), then my original promise is still standing, (i.e. $p \to q$ is true).

2.3.1 Exercises

1. Given the conditional statement *If your course average is better than* 98% *then you will earn an A in the class*,
   (a) Identify the antecedent and the consequent, and write the statement in symbolic form.
   (b) Examine each of the four possible scenarios described in the truth table (p and q both true, p=T and q=F, etc.) and explain why the corresponding truth value for this conditional statement makes sense.
2. Construct a truth table for each of the following compound statements.
   (a) $(p \wedge q) \to q$.
   (b) $(\neg p) \to (p \to q)$.
   (c) $[p \wedge (\neg q \vee r)] \to (q \vee \neg p)$.

2.4 Tautologies and Contradictions

Anything that happens, happens. Anything that in happening causes something else to happen, causes something else to happen. Anything that in happening happens again, happens again. Though not necessarily in that order.
(From Mostly Harmless, by Douglas Adams.)

A *tautology* is a statement that is always true. The expression "*A is A*" (often attributed to Aristotle), is one of the most common tautologies. Consider these examples:

- A quote from a student: *The main idea behind data compression is to compress data.*
- A quote from George W. Bush concerning Native American tribes: *Tribal sovereignty means that, it's sovereign.*

Both statements are certainly true, but ... rather pointless.

The way we check whether a statement is a tautology is by using truth tables. Let us look at the following paragraph taken from *Through the Looking Glass* by Lewis Carrol:

*"You are sad," the Knight said in an anxious tone: "let me sing you a song to comfort you. . . Everybody that hears me sing it –**either it brings the tears into their eyes, or else** –"*

"Or else what?" said Alice, for the Knight had made a sudden pause.

*"**Or else it doesn't**, you know."*

The statement in bold face can be written in symbolic form as $p \vee \neg p$, where $p =$*It brings tears into their eyes.* The truth table for this statement is

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

which says that any statement of the form $p \vee \neg p$ is a tautology. Now fill out the following truth table for the statement $p \vee \neg(p \wedge q)$ and show that this statement is a tautology:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T			
T	F			
F	T			
F	F			

As opposed to tautologies, *contradictions* are statements that are always false. The expression "A *and* $\neg A$" is a contradiction.

I don't believe in reincarnation, but I did in my past life. (Anonymous)

Another example of a contradiction is $\neg[p \vee (\neg p)]$ (show why). For more practice, complete the following truth table for the statement $p \wedge \neg(p \vee q)$ and show that this statement is a contradiction.

p	q	$p \vee q$	$\neg(p \vee q)$	$p \wedge \neg(p \vee q)$
T	T			
T	F			
F	T	T	F	F
F	F			

2.4.1 Exercises

1. Show that the statement $(p \vee q) \vee [(\neg p) \wedge \neg q]$ is a tautology by constructing its truth table.

2. Use a truth table to check whether the following are tautologies, contradictions, or neither.

(a) $p \wedge (\neg p)$.

(b) $p \wedge p$.

(c) $p \vee \neg(p \vee q)$.

(d) $q \vee \neg[p \wedge \neg p]$.

(e) $q \wedge \neg(p \vee p)$.

(f) $\neg[(p \wedge q) \to q]$.

3. (a) If a proposition is neither a tautology nor a contradiction, what can be said about its truth table?

   (b) If A and B are two (possibly compound) statements such that $A \vee B$ is a contradiction, what can you say about A and B?

   (c) If A and B are two (possibly compound) statements such that $A \wedge B$ is a tautology, what can you say about A and B?

2.5 Logical Equivalence

Two statements are said to be *logically equivalent* when they have the same logical content. The simplest example of two logically equivalent statements is that of any statement p and its *double negation* $\neg(\neg p)$. In fact, if p is true (or false) then so is its double negation and viceversa. It is easy to check whether two statements are logically equivalent by using truth tables

Two statements are logically equivalent when their truth tables are identical.

Example. Any two statements of the form $p \to q$ and $(\neg p) \vee q$ are logically equivalent. To show this we construct the following truth table (note that we duplicated a column for q since it makes it easier to fill out the last column):

p	q	$p \to q$	$\neg p$	q	$(\neg p) \vee q$
T	T	**T**	F	T	**T**
T	F	**F**	F	F	**F**
F	T	**T**	T	T	**T**
F	F	**T**	T	F	**T**

The third and last columns (set in boldface) are identical, showing that the two statements have identical truth values regardless of their contents, and thus are logically equivalent. Note that this implies that, for example, the statements *"If the price is right, I will buy this"* and *"The price is not right or I will buy this"* have the same logical content.

2.5.1 Exercises

1. Determine whether the following pairs of statements are logically equivalent:

   (a) $p \vee (p \wedge q)$ and p.

   (b) $\neg(p \vee q)$ and $(\neg p) \vee \neg q$.

   (c) $\neg(p \to q)$ and $(\neg p) \to \neg q$.

2. Are the statements *" If I am not in Charleston, then I am in Italy"*, and *"I am in Charleston or I am not in Italy"* logically equivalent? (To answer this question, first convert them into symbolic statements.)

3. Show that $p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$.
   (Equivalent queries: if you search for web pages containing the term *logic* and the terms *reason or politics*, you will get the same results by searching for web pages containing both terms *logic and reason* or both terms *logic and politics*.

4. Use a truth table to verify the following logical equivalences:

   (a) $p \vee q = q \vee p$. (*Commutative Law for disjunction.*)

   (b) $p \wedge q = q \wedge p$. (*Commutative Law for conjunction.*)

   (c) $(p \wedge q) \wedge r = p \wedge (q \wedge r)$. (*Associative Law for conjunction.*)

   (d) $(p \vee q) \vee r = p \vee (q \vee r)$. (*Associative Law for disjunction.*)

   (e) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ (*Distributive Laws.*)

5. Use logical equivalence to rewrite each of the following sentences. If possible, rewrite them more simply.

   (a) Either she is late and has forgot to call, or she is late and has had an accident.

   (b) You must take either Math or Logic, and you must take either Math or French.

6. If two propositions are logically equivalent, what can be said about their truth tables?

2.5.2 De Morgan's Laws

The following long truth table:

p	q	$p \wedge q$	$p \vee q$	$\neg(p \wedge q)$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \vee \neg q$	$(\neg p) \wedge \neg q$
T	T	T	T	**F**	***F***	F	F	**F**	***F***
T	F	F	T	**T**	***F***	F	T	**T**	***F***
F	T	F	T	**T**	***F***	T	F	**T**	***F***
F	F	F	F	**T**	***T***	T	T	**T**	***T***

shows an important pair of logical equivalences known as *De Morgan's Laws:*

(A) $\neg(p \wedge q)$ *is logically equivalent to* $(\neg p) \vee \neg q$.

(B) $\neg(p \vee q)$ *is logically equivalent to* $(\neg p) \wedge \neg q$.

Example. *It is not true that: today is Wednesday and it is raining* is equivalent to asserting that *Today is not Wednesday or it is not raining.*

2.5.3 Exercises

1. Joe tells you that *he is an actor and went to India last summer.* You know that he is lying to you, what compound statement about Joe is true?

2. Simplify the expression $\neg[(\neg p) \wedge q]$ using De Morgan's Laws and the double negation.

3. Use logical equivalence to rewrite the statement: *It is not true that both you are a billionaire and I am crazy."*

4. Use logical equivalence to rewrite each of the following sentences. If possible, rewrite more simply.

   (a) *It is not true that both I am wise and you are a fool.*

   (b) *It is not true that either I am wise or you are a fool.*

2.6 Conditionals in the English Language

Michael Jordan eats Wheaties. (From a 1989 series of commercials.)

Conditional statements appearing throughout the English language do not often come in the form

If p then q.

Here you will find a list of examples of commonly used forms of conditional statements.

(A) *If you build it, he will come.* (The Voice in the movie *Field of Dreams.*)

Here *then* is simply missing, but this statement is clearly recognizable as a conditional.

(B) *When you are distracted, it is difficult to study.*

This statement can be rewritten as: *If you are distracted, then it is difficult to study.*

(C) *It must be alive if it is breathing.*

This statement can be rewritten as: *If it is breathing, then it must be alive.*

(D) *Michael Jordan eats Wheaties.*

This is a more subtle case. Surely the goal of this commercial is not just to inform you that: *If you are Michael Jordan, then you eat Wheaties.* (This is the literal meaning of the statement.) What they would like you to (mis-)understand is: *If you eat Wheaties, then you will become like Michael Jordan.* We will discuss this common fallacy later on.

(E) *No Koalas live in Texas*

This can be rewritten as: *If it is a Koala, then it does not live in Texas.*

(F) *To earn this scholarship, it is necessary to have a 3.0 GPA.*

This is a widely used alternative form of conditional. It often occurs in legal documents and written rules. We can translate it as: *If you earned this scholarship, then you had a 3.0 GPA.* A word of caution: this does not mean that a 3.0 GPA is the only requirement for earning the scholarship, in other words, it may not be sufficient. Check out the following examples.

(G1) *To earn this scholarship, it is sufficient to have a 3.0 GPA and a gross yearly income of below $15,000.* Or,

(G2) *You have a 3.0 GPA and a gross yearly income of below $15,000 only if you earn this scholarship.*

Both these statements mean: *If you have a 3.0 GPA and a gross yearly income of below $15,000, then you will earn this scholarship.*

Summarizing the last two examples, the conditional

$$p \to q$$

can be equivalently translated as

(F) q is necessary for p or (G) p is sufficient for q (or p only if q) .

2.6.1 Exercises

Rewrite the following statements using the form *if … then …*.

1. Every picture tells a story.
2. No gunea pigs are scholars.
3. You can believe it if you see it on the internet.
4. It is necessary to be 18 in order to be able to vote.
5. Doing crossword puzzles is sufficient for driving me crazy.

6. I can come with you only if I can find some time.

7. Jesse will be a liberal when pigs fly.

2.6.2 How to negate a conditional statement

My Lord, I reject your proposition that "If we lose the war, then our heads will fall"!

To find out how we can rephrase a statement like the one above, i.e. how to a negate conditional statement, we first complete the following truth table:

p	q	$\neg q$	$p \wedge (\neg q)$	$p \rightarrow q$	$\neg(p \rightarrow q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

The completed truth table shows that:

The negation $\neg(p \rightarrow q)$ is logically equivalent to the conjunction $p \wedge \neg q$.

We can now rephrase the negation of the statement *"If we lose the war, then our heads will fall"*, as *"We lost the war, but our heads did not fall."*

2.6.3 Exercises

Write the negation of the following statements:

1. If you decide to go to the party, then I will go with you.

2. If you say "I do", then you will be happy for the rest of your live.

3. I'll be surprised, if that is an authentic Persian rug.

4. If wanting peace is wrong, I do not want to be right.

2.7 Related Conditionals

In this section we introduce three important forms of conditional statements related to $p \to q$.

2.7.1 Converse, Inverse, Contrapositive

We begin with the statement

If you stay, then I leave.

The antecedent of this statement is p=*you stay*, and the consequent is q= *I leave*. The symbolic form of the conditional statement is then

$$p \to q.$$

The Converse. The converse statement is obtained by interchanging antecedent and consequent. In symbols, the converse is simply

$$q \to p.$$

Its English form is

If I leave, then you stay.

The Inverse. The inverse statement is obtained by negating the antecedent and the consequent. In symbols, the inverse is

$$(\neg p) \to \neg q.$$

Its English form is

If you do not stay, then I do not leave.

The Contrapositive. The contrapositive statement is one of the most important related conditionals (it will return in the study of validity of logical arguments). It is obtained by both negating the antecedent and the consequent *and* by interchanging them. In symbols, the contrapositive is

$$(\neg q) \to \neg p.$$

Its English form is

If I do not leave, then you do not stay.

2.7.2 Exercises

1. For each of the statements written below:

   –If I live in Miami, then I live in Florida;

   –If you do not agree, then the deal will fall apart;

   –It will bloom, if we water it;

   (a) Write it in symbolic form, identifying the antecendent p and the consequent q.

   (b) Write its converse, first in symbols, then in English.

   (c) Write its inverse, first in symbols, then in English.

   (d) Write its contrapositive, first in symbols, then in English.

2. Given the conditional statement $(\neg p) \to q$, write the following related conditionals in symbolic form (simplifying when possible):

   (a) The converse.

   (b) The contrapositive.

   (c) The inverse.

3. For each of these conditional statements, write the converse, inverse, and contrapositive in the *"if . . . then"* form. In some cases it may be useful to restate the original conditional in the *"if . . . then"* form first.

   (a) If you lead, then I will follow.

   (b) If it ain't broken, don't fix it.

   (c) I will go to the party, if I finish studying.

   (d) Walking in front of a moving car is dangerous.

   (e) Milk contains calcium.

   (f) If you built it, he will come.

   (g) $p \to \neg q$.

   (h) $\neg q \to \neg p$.

   (i) $p \to (q \vee r)$. (Hint: use one of De Morgan's Laws.)

   (j) Doing crossword puzzles is sufficient for driving me crazy.

2.7.3 Logical equivalence of related conditionals

From the truth table

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$(\neg p) \to \neg q$	$(\neg q) \to \neg p$
T	T	**T**	T	F	F	T	**T**
T	F	**F**	T	F	T	T	**F**
F	T	**T**	F	T	F	F	**T**
F	F	**T**	T	T	T	T	**T**

we deduce two important observations:

The conditional $p \to q$ and its converse $q \to p$ are not logically equivalent.

Example. The statements *If you are Michael Jordans, then you eat Wheaties* and *If you eat Wheaties, then you are (like) Michael Jordan* do not have the same meaning!

The conditional $p \to q$ and its contrapositive $(\neg q) \to \neg p$ are logically equivalent.

Example. *If Mr. X is the murderer, then his right index finger is missing* and *If his right index finger is not missing, then Mr. X is not the murderer* have the same logical content.

Note. We also observe that the converse $q \to p$ and the inverse $(\neg p) \to \neg q$ are logically equivalent statements. This should be no surprise since they are the contrapositives of each other. (See exercise 1.)

2.7.4 Exercises

1. Show that the contrapositive of the inverse of $p \to q$ is the converse of $p \to q$.

2. Which of the following statements are logically equivalent to *The effects of global warming will be mitigated if governments act quickly*? (Select one or more and justify your answer using truth tables.)

   (a) *If the effects of global warming are mitigated, then governments have acted quickly.*

   (b) *Governments will not have acted quickly if the effects of global warming will not be mitigated.*

   (c) *Governments will not have acted quickly or the effect of global warming will be mitigated.*

2.8 Negating Statements with Quantifiers

Sentences containing **universal quantifiers** such as *all, each, every, no(one)*, or **existential quantifiers** such as *some, there exists, at least one*, require some thinking when we write their negations.

Example. The statement

All men in this group are named Bob,

can be applied to various situations, and can be true in some and false in others. Take a look at the following groups of men illustrating all possible scenarios (followed by their descriptions in parentheses):

A: Bob Hope, Bob Cape, Bob Sullivan. (All men are named Bob.)

B: Mark Joos, Bob Hope, Richard Smith. (Some men are named Bob and some are not.)

C: Mark Joos, Rich Smith, Sam Tines. (No men are named Bob.)

The statement *All men in this group are named Bob* is clearly true for group A, and it is false for both group B and group C.

When negating a statement with a quantifier, the statement and its negation must apply to all scenarios and have opposite truth values in each scenario. Therefore, correct negations of the statement

p=*All men in this group are named Bob*

are the statements

$\neg p$=*Some men in this group are not named Bob,*
$\neg p$ =*Not all men in this group are named Bob.*

Caution: The statement *All men in this group are not named Bob* is **not** the negation of the original statement, because there is one scenario (B) in which they are both false. So they cannot be the negation of each other.

Similarly, correct negations of of the statement

$p =$ *Some of us have headaches*

are the statements

$\neg p$ =*None of us has headaches,*
$\neg p$ =*All of us do not have headaches.*

The Venn diagrams shown below in Figures 2.1 and 2.2 are helpful for remembering these rules of negation.

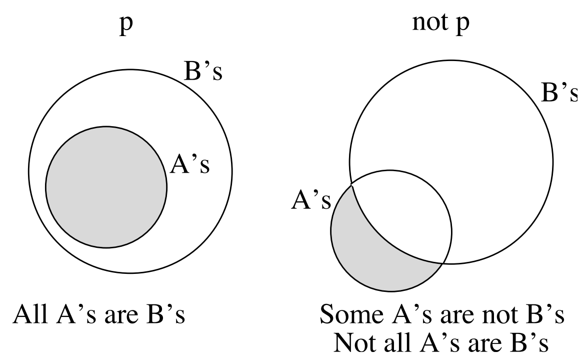


Figure 2.1: Negating statements with universal quantifiers.

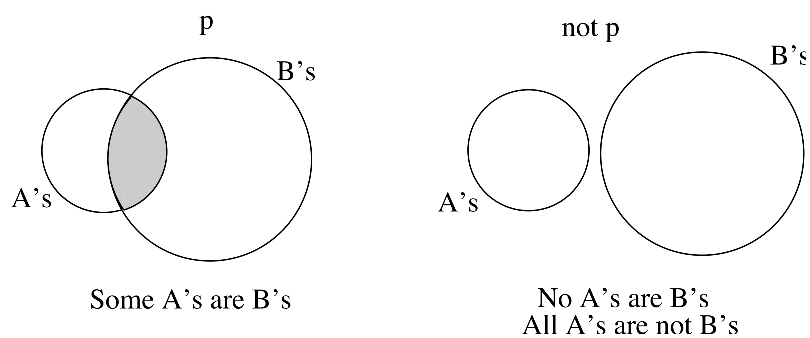


Figure 2.2: Negating statements with existential quantifiers.

Notes:

1. The negation of a statement containing a universal quantifier (e.g. p =*All animals can swim*, q =*No one can hear*) is a statement containing an existential quantifier ($\neg p$ =*Some animals can not swim*, $\neg q$=*Someone can hear.*) This reflects the fact that the negation of a universal truth requires at least one exception.

2. Similarly, the negation of a statement containing an existential quantifier (e.g. p=*There is a man who is three-meter tall*) is a statement containing a universal quantifier ($\neg p$ =*No man is three-meter tall*).

3. Recalling the fact that the double negation of a statement is the statement itself, we can now easily write the negation of statements such as p =*No one can sing* and q =*Some of us will not join* by simply looking at the Venn diagrams in Fig. 2.1 and 2.2. The correct statements are $\neg p$ =*Some person can sing* and $\neg q$ =*All of us will join.*

2.8.1 Exercises

1. Write the negations of the following statements:
   (a) Some books are more interesting than this book.
   (b) No gunea pigs are scholars.
   (c) Every dog has its day.
   (d) There exists a two-meter long dog.
   (e) All losers will get another chance.
   (f) A few of us are tall.
   (g) Not everybody is a born singer.
   (h) Some of these flowers are not yellow.
   (i) Roses are red and violets are blue. (Review DeMorgan's Laws before attempting this.)

2. Write the negation of the following statement (Review DeMorgan's Laws before attempting this.)

No man is an island, and every man is a piece of the continent.
(From *Devotions* by John Donne.)

2.9 Logical Arguments and Venn Diagrams

A logical arguments is a list of **premises** (such as assumptions, rules, facts and observations), followed by a single statement called the **conclusion**. Here are two examples of logical arguments:

All giraffes have long necks
I have a pet giraffe
Therefore, one of my pets has a long neck

Most kids love chocolate
I am no longer a kid
Therefore, I hate chocolate

Notice that each of these two arguments has two premises (the first two statements) and one conclusion, preceeded by the word *therefore*.

2.9.1 Validity of a logical argument

A logical argument is valid if, whenever all premises are true, then the conclusion is also true.

In other words, validity of an argument means that true premises guarantee a true conclusion.

A logical argument containing quantifiers is called a *categorical syllogism*. Recall that quantifiers are terms such as *all, every, each, none* (the *universal quantifiers*), and *some, most, at least one, there is* (the *existential quantifiers*). The validity of a categorical syllogism can be tested in a particularly simple way using *Venn diagrams*.

Let us examine the first of the two arguments given above. This argument has two premises:

P1: *All giraffes have long necks.*

P2: *I have a pet giraffe.*

We first "draw" the first premise using two Venn diagrams: the first circle represents the group of all giraffes, and the second circle represents the group of animals with long necks. Premise 1 tells us that the set of all giraffes is contained within the set of animals with long necks, as illustrated on the left of Figure 2.3.

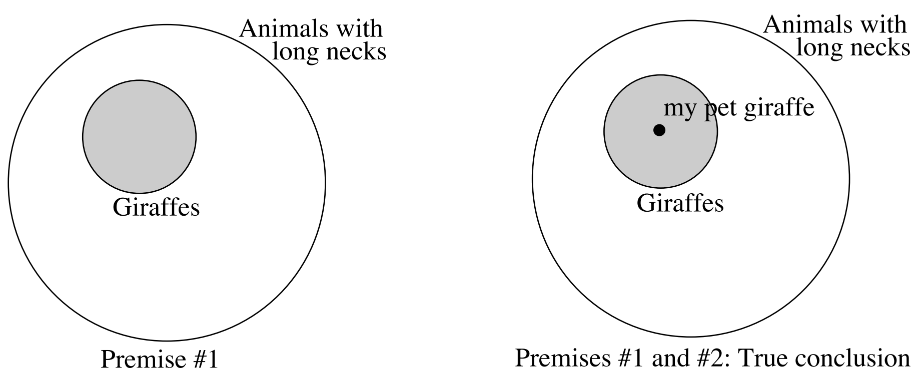


Figure 2.3: An example of valid argument

Next, we complete the drawing by "adding" the second premise. This can be done simply by placing a dot (representing *my pet giraffe*) inside the Venn diagram describing the group of all giraffes (inner circle), as shown on the right of Figure 2.3.

Finally, we check whether the drawing forces the conclusion to be true: obviously, since your pet giraffe is inside the set of all giraffes, which is inside the set of animals with long necks, your pet giraffe has to have a long neck too! So the conclusion (**C**: *one of my pets has a long neck*) follows from the premises. We conclude that the first argument is *valid.*

We now move on to examine the validity of the second argument. This argument has two premises:

P1: *Most kids love chocolate.*

P2: *I am no longer a kid.*

First note that sometimes premises and conclusion need to be rephrased to better understand how to proceed. For example, Premise 1 *Most kids love chocolate* can be rephrased as *Some kids love chocolate* without substantially changing its meaning, Premise 2 can also be rephrased as *I am not a kid.* If we represent Premise 1 using Venn diagrams, we obtain the two circles at the top of Figure 2.4.

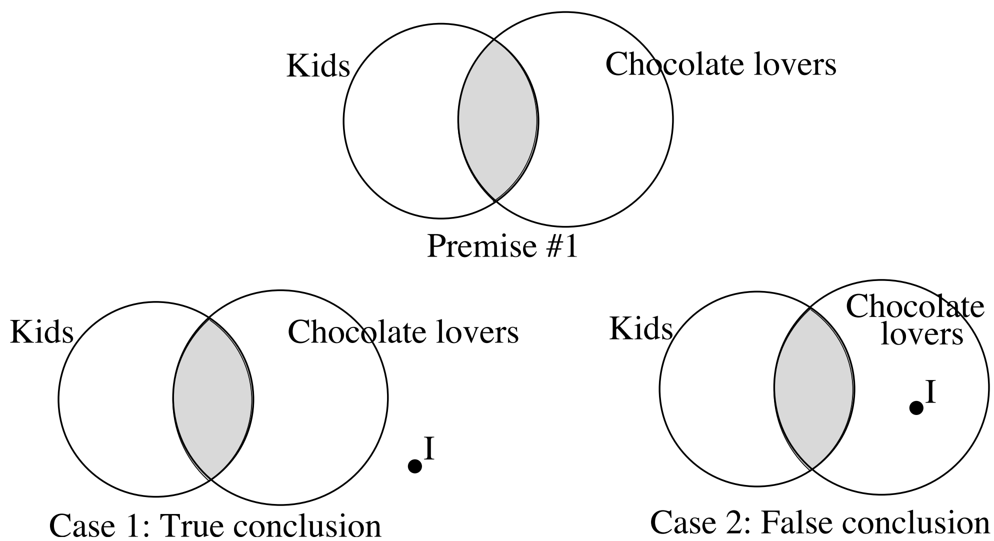


Figure 2.4: Negating statements with existential quantifiers.

When drawing the second premise, we notice that there are two possible options. The dot representing "I" must be placed outside of the circle representing *Kids*, however it may be outside of (left bottom diagram) or inside (right bottom diagram) the circle representing *Chocolate lovers*. In the first case (outside), the conclusion is true, while for the second scenario the conclusion is clearly false. Since we found a case in which *true premises do not force a true conclusion*, we conclude that this argument is *invalid.*

Notes:

1. To show that an argument is invalid it is sufficient to draw one scenario representing each of the premises, in which the conclusion is false. You will need to build up your intuition by looking at several examples. A good one is:

   Some marigolds are yellow.
   All lemons are yellow.
   Therefore, some lemons are marigolds.

   One way of representing the premises which shows that, *for the selected scenario*, the conclusion is false is illustrated in Figure 2.5.

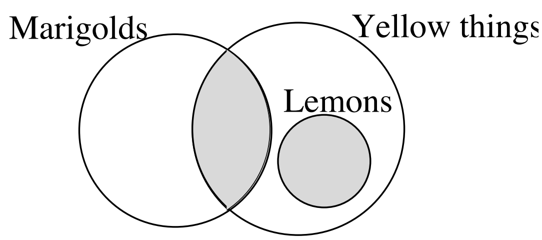
   

   Figure 2.5: Another example of invalid argument.

2. The fact that an argument has a true conclusion does not guarantee its validity. For example,

   Some mammals have horns.
   Cows have horns.
   Therefore, cows are mammals.

   is an invalid argument even if its conclusion is clearly true. In fact, this argument has the same form as the one we discussed in Note 1. Show its invalidity using a Venn diagram representation similar to the one in Figure 2.5.

2.9.2 Exercises

1. Use Venn diagrams to determine whether the following logical arguments containing quantifiers are valid or not valid. Make sure to first identify the premises and the conclusion.

   (a) Every men is mortal.
   Socrates is a man.
   Therefore, Socrates is mortal.

(b) Some philosophers are absent-minded.
Amanda is absent-minded.
Therefore, Amanda is a philosopher.

(c) All tigers are meat eaters.
Simba is a meat eater.
Therefore, Simba is a tiger.

(d) All vitamins are healthy.
Caffeine is a vitamin.
Therefore, caffeine is healthy.

(e) All A's are B's.
Some B's are C's.
Therefore, some A's are C's.

(f) No iMacs have floppy drives.
My computer has no floppy drive.
Therefore, my computer is an Imacs.

(g) Some investors are wealthy.
All wealthy people are happy.
Therefore, some investors are happy.

(h) No fish is a mammal.
Cows are mammals.
Therefore, cows are not fish.

(i) All people who drive contribute to air pollution.
All people who contribute to air pollution make life a little worse.
Some people who live in a suburb make life a little worse.
Therefore, some people who live in a suburb drive.

2. Complete the argument by adding a conclusion that makes the argument valid. (There may be several correct answers.)

(a) Some rules are unfair.
All unfair rules should be eliminated.
Therefore, . . .

(b) Some mathematicians are fine musicians.
All fine musicians are intelligent.
Therefore, . . .

(c) No team which plays in a domed stadium has ever won the Super Bowl.
Some teams that wear red uniforms have won the Super Bowl.
Therefore, . . .

2.10 Analyzing Logical Arguments with Truth Tables.

Recall that a *logical argument* consists of a list of *premises* (there may be two, three, or more premises in an argument), followed by a single *conclusion*, and that *validity* means that whenever all the premises are true, then the conclusion is also true. An argument is, on the other hand, *invalid* when all true premises do not force a true conclusion.

In this section, we will learn how to use truth tables to determine the validity of some arguments.

Example 1.

If you win the game, then I will celebrate.
You won the game.
Therefore, I am celebrating.

We rewrite the two premises and the conclusion in symbolic form, introducing the symbols:

p=*You win the game,* and q=*I celebrate.*

Then, the symbolic form of this argument is:

$$\begin{array}{c} p \to q \\ p \\ \therefore\ q \end{array}$$

Note: The symbol $\therefore$ represents the word *therefore*, which introduces the conclusion.

We then construct a truth table whose *last three columns* contain Premise 1, Premise 2, and the Conclusion *in this exact order*.

p	q	$p \to q$	p	q
T	T	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Note: We duplicated the column for statement q as the last column. It is important that the last columns is always the one representing the conclusion.

Looking at the truth table above, we observe that the first row (shown in boldface) is the only row in which *both premises are true*. We check the truth value of the conclusion *in this row* and find that it is true as well. We conclude that, since true premises guarantee a true conclusion, this argument is *valid*.

Summary 1. Any argument of the form

$$\begin{gathered} p \to q \\ p \\ \therefore\ q \end{gathered}$$

is a *valid argument.* It is known as the *Law of Detachment* or *Modus Ponens* (its Latin name).

Example 2.

If you are a hero, then you wear a Rolex.
You wear a Rolex.
Therefore, you are a hero.

We introduce the symbols:

p=*You are a hero,* and q=*You wear a Rolex.*

Then, the symbolic form of this argument is:

$$\begin{gathered} p \to q \\ q \\ \therefore\ p \end{gathered}$$

The last three columns of the following truth table represent Premise 1, Premise 2, and the Conclusion in this order:

p	q	$p \to q$	q	p
T	T	**T**	**T**	**T**
T	F	F	F	T
F	T	**T**	**T**	**F**
F	F	T	F	F

There are now *two* rows in which *both premises are true*: the first and third rows. We check the truth value of the conclusion *in each of these rows* and find that the conclusion is true in the first row, *but* the conclusion is *false in the third row.* Since true premises do *not* guarantee a true conclusion, this argument is *invalid.*

Summary 2. Any argument of the form

$$\begin{gathered} p \to q \\ q \\ \therefore\ p \end{gathered}$$

is an *invalid argument.* It is known as *The Fallacy of the Converse* (because it incorrectly suggests that a conditional statement and its converse are equivalent statements).

Example 3.

If the test is positive, then you will require treatment.
You did not require treatment.
Therefore, the test was negative.

We introduce the symbols:

p=*The test is positive,* and q=*You will require treatment.*

Then, the symbolic form of this argument is:

$$\begin{array}{c} p \to q \\ \neg q \\ \therefore \neg p \end{array}$$

The last three columns of the following truth table represent Premise 1, Premise 2, and the Conclusion in this order:

p	q	$p \to q$	$\neg q$	$\neg p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	**T**	**T**	**T**

The last row is the only row in which both premises are true. In this row the conclusion is also true, showing that this is a *valid* argument.

Summary 3. Any argument of the form

$$\begin{array}{c} p \to q \\ \neg q \\ \therefore \neg p \end{array}$$

is a *valid argument* and is known as the *Law of Contraposition* or *Modus Tollens.*

Example 4.

If you pay your taxes late, then you will pay a late penalty.
You do not pay your taxes late.
Therefore, you will not pay a late penalty.

We introduce the symbols:

p=*You pay your taxes late,* and q=*You will pay a late penalty.*

Then, the symbolic form of this argument is:

$$\begin{array}{c} p \to q \\ \neg p \\ \therefore \neg q \end{array}$$

The last three columns of the following truth table represent Premise 1, Premise 2, and the Conclusion in this order:

p	q	$p \to q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	**T**	**T**	**F**
F	F	**T**	**T**	**T**

Both premises are true in the third and last row. In the last row the conclusion is also true. However, the conclusion is false in the third row. Since true premises do not guarantee a true conclusion, this argument is *invalid.*

Summary 4. Any argument of the form

$$\begin{array}{c} p \to q \\ \neg p \\ \therefore \neg q \end{array}$$

is an *invalid argument*, and is called the *Fallacy of the Inverse* (because it incorrectly suggests that a conditional statement and its inverse are equivalent statements).

Example 5.

I will learn logic or I will eat my hat.
I will not eat my hat.
Therefore, I will learn logic.

We introduce the symbols:

p=*I will learn logic,* and q=*I will eat my hat.*

Then, the symbolic form of this argument is:

$$\begin{array}{c} p \vee q \\ \neg q \\ \therefore\ p \end{array}$$

The last three columns of the following truth table represent Premise 1, Premise 2, and the Conclusion in this order:

p	q	$p \vee q$	$\neg q$	p
T	T	T	F	T
T	F	**T**	**T**	**T**
F	T	T	F	F
F	F	F	T	F

The second row is the only one in which both premises are true. Checking the conclusion in this row, we find that it is also true. Just like in Example 1, we conclude that this argument is valid.

Summary 5. Any argument of the form

$$\begin{aligned}&p \vee q\\&\neg q\\&\therefore\ p\end{aligned}$$

is a *valid argument* and is known as *Disjunctive Syllogism.*

2.10.1 Exercises

1. Use a truth table to determine whether the following arguments are valid or not valid.

   (a) *If the test is negative, then you will not require treatment.*
   The test was positive.
   Therefore, you will need to be treated.

   (b) *If you love me, then you will do everything I ask.*
   You do not do everything I ask.
   Therefore, you do not love me.

   (c) *If June 1 is Monday, then June 2 is Friday.*
   If June 2 is Friday, then June 5 is Wednesday.
   Therefore, if June 1 is Monday, then June 5 is Wednesday.

   Note: There are three distinct statements in this argument. How many rows will your truth table have? This form of logical argument is sometimes known as the *Chain Rule.*

   (d) *If a car has airbags, then it is safe.*
   This car has airbags.
   Therefore, it is safe.

   (e) *Jamie is fluent in Spanish.*
   If Jamie is fluent in Spanish, then she will work in Madrid.
   She will not visit Mexico or she will not work in Madrid.
   Therefore, she will visit Mexico.

(f)

$$\begin{array}{c} p \\ q \to \neg p \\ \therefore \neg q \end{array}$$

2. Determine whether each of the following arguments is valid or invalid.

   (a) *If I were a chicken, then I would lay eggs. However, I am not a chicken, so I don't lay eggs.*

   (b) If interest rates go down, then I will buy a house. If I buy a house, I will need a loan. Therefore, I will not need a loan if I do not buy a house.

   (c) If I am honest, you are lying; either you are lying or you have your facts wrong. You cannot have done your research, if you have your facts wrong. You did your research. Therefore, I am honest.

   (d) If you like apples, you will like this pie. If you like this pie, then you will like the bakery. Therefore, if you do not like the bakery, you do not like apples.

3. Given the premises:

   P1: *If today is saturday, then we do not have class today.*
   P2: *If we do not have class today, then I will go shopping or I will go to the picnic.*

   (a) Write a valid conclusion (in English) for the argument.

   (b) Write the entire argument in symbolic form.

4. Decide whether each of the following is a valid argument. If it is valid, give a proof. If it is invalid, give a counterexample. In any case, supply verbal statements that make all the premises true; if the argument is invalid make sure that they also make the conclusion false.

   (a)

$$\begin{array}{c} p \to r \\ (\neg q) \to \neg r \\ \therefore\ p \to q \end{array}$$

   (b)

$$\begin{array}{l} p \to (q \vee r) \\ \neg q \\ \therefore \neg p \end{array}$$

5. Is the following logical argument in symbolic form valid? Explain your reasoning or use a truth table to analyze it.

$$\begin{gathered} r \\ r \to q \\ (\neg p) \vee \neg q \\ \therefore \neg p \end{gathered}$$

6. Is the following argument valid or invalid?

   My stereo system is faulty: there is no sound coming out of the left speaker. Switching the speaker leads will not bring sound to the left speaker if and only if the left speaker is faulty. If switching the speaker leads causes the right speaker to fail, then there is a fault with either the amplifier or the CD player. Switching the leads from the CD player has no effect if and only if there is no problem with the CD player. I discovered the following: switching the leads to the speakers resulted in both channels failing, and switching the leads from the CD player reversed the problem from the left to the right speaker. Therefore replacing the CD player and the left speaker will solve the problem.

2.11 Solutions to Selected Exercises

Exercises 2.1.1

1. (a) j=*Jim is a lawyer*, c=*Jim is a crook.*

$$j \wedge \neg c.$$

   (c) a=*my favorite dish has lots of anchovies*,
   s=*my favorite dish is spicy*,
   l=*my favorite dish comes in a large portion.*

$$(a \vee \neg s) \wedge l.$$

   (e) a=*you attend class*, r=*you read the book*, p=*you pass the exam.*

$$(\neg a) \to (r \vee \neg p).$$

   (g) g=*my websearch is for pages containing "global warming"*,
   t=*my websearch is for pages containing "termites"*,
   c=*my websearch is for pages containing "cattle".*

$$g \wedge (\neg t) \wedge (\neg c).$$

(Note: for each problem there are several correct answers, for example, in part (a), one could also let c=*Jim is a not crook*. In this case the symbolic representation is $j \wedge c$.)

2. (a) If the sunroof is extra and the power windows are optional, then the radial tires are included.

   (c) If the radial tires are not included, then either the sunroof is not extra or the power windows are not optional.

Exercises 2.2.2

1.

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$(\neg p) \vee q$	$[p \wedge \neg q] \vee [(\neg p) \vee q]$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

2. (a)

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

   (c)

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

   (e)

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg(p \vee \neg q) \vee p$
T	T	F	T	F	T
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	T	F	F

3. (a) T.

Exercises 2.2.4

1. (a)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(c)

p	q	r	$r \wedge p$	$\neg q$	$(r \wedge p) \vee \neg q$
T	T	T	T	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	T	T

Exercises 2.3.1

1. (a) antecedent: p=*Your course average is better than 98%*, consequent q=*You will earn an A in the class.*

2. (a)

p	q	$p \wedge q$	$(p \wedge q) \to q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

(c)

p	q	r	$\neg q$	$(\neg q) \vee r$	$p \wedge (\neg q \vee r)$	$\neg p$	$q \vee \neg p$	$[p \wedge (\neg q \vee r)] \rightarrow (q \vee \neg p)$
T	T	T	F	T	T	F	T	T
T	T	F	F	F	F	F	T	T
T	F	T	T	T	T	F	F	F
T	F	F	T	T	T	F	F	F
F	T	T	F	T	F	T	T	T
F	T	F	F	F	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

Exercises 2.4.1

1.

p	q	$p \vee q$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$	$(p \vee q) \wedge [(\neg p) \vee \neg q]$
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	T

2. (a)

p	$\neg p$	$p \wedge \neg p$
T	F	F

Contradiction

(c)

p	q	$\neg p$	$p \vee \neg p$	$\neg(p \wedge \neg p)$	$q \vee \neg(p \wedge \neg p)$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	F	T	T

Tautology

Exercises 2.5.1

1. (a)

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	**T**
T	F	F	**T**
F	T	F	**F**
F	F	F	**F**

Logically equivalent

(c)

p	q	$p \to q$	$\neg(p \to q)$	$\neg p$	$\neg q$	$(\neg p) \to \neg q$
T	T	T	**F**	F	F	**T**
T	F	F	**T**	F	T	**T**
F	T	T	**F**	T	F	**F**
F	F	T	**F**	T	T	**T**

Not logically equivalent

3.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**T**	T	F	**T**
T	F	T	T	**T**	F	T	**T**
T	F	F	F	**F**	F	F	**F**
F	T	T	T	**F**	F	F	**F**
F	T	F	T	**F**	F	F	**F**
F	F	T	T	**F**	F	F	**F**
F	F	F	F	**F**	F	F	**F**

Logically equivalent

4. (b)

p	q	$p \wedge q$	$q \wedge p$
T	T	**T**	**T**
T	F	**F**	**F**
F	T	**F**	**F**
F	F	**F**	**F**

(d)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**T**	T	**T**
T	F	T	T	**T**	T	**T**
T	F	F	T	**T**	F	**T**
F	T	T	T	**T**	T	**T**
F	T	F	T	**T**	T	**T**
F	F	T	F	**T**	T	**T**
F	F	F	F	**F**	F	**F**

5. (a) She is late and has either forgot to call or has had an accident.

6. Their last columns are identical.

Exercises 2.5.3

1. Either Joe is not an actor or he did not go to India last summer.

3. Either you are not a billionaire or I am not crazy.

4. (b) I am not wise and you are not a fool.

Exercises 2.6.1

1. If something is a picture, then it tells a story.

3. If you see it on the internet, then you can believe it.

5. If I do crossword puzzles, then I will go crazy.

7. If pigs fly, then Jesse will be a liberal.

Exercises 2.6.3

1. You decide to go to the party and I do not go with you.

2. That is an authentic Persian rug and I am not surprised.

Exercises 2.7.2

1. First statement:

   (a) antecedent p=*I live in Miami*, consequent q=*I live in Florida*.

   $$p \rightarrow q.$$

   (b) $q \rightarrow p$
   If I live in Florida, then I live in Miami.

   (c) $(\neg p) \rightarrow \neg q$
   If I do not live in Miami, then I do not live in Florida.

   (d) $(\neg q) \rightarrow \neg p$
   If I do not live in Florida, then I do not live in Miami.

2. (a) $q \rightarrow \neg p$.

   (c) $p \rightarrow \neg q$.

3. (a) Converse: *If I follow, then you lead.*
   Contrapositive: *If I do not follow, then you do not lead.*
   Inverse: *If you do not lead, then I will not follow.*

   (c) Original: *If I finish studying, then I will go to the party.*
   Converse: *If I go to the party, then I finished studying.*
   Contrapositive: *If I do not go to the party, then I did not finish studying.*
   Inverse: *If I do not finish studying, then I will not go to the party.*

(e) Original: *If it is milk, then it contains calcium.*
Converse: *If it contains calcium, then it is milk.*
Contrapositive: *If it does not contain calcium, then it is not milk.*
Inverse: *If it is not milk, then it does not contain calcium.*

(g) Converse: $(\neg q) \to p$.
Contrapositive: $q \to \neg p$.
Inverse: $(\neg p) \to q$.

(i) Converse: $(q \vee r) \to p$.
Contrapositive: $[\neg(q \vee r)] \to \neg p$ or equivalently, $(\neg q \wedge \neg r) \to \neg p$.
Inverse: $(\neg p) \to \neg(q \vee r)$ or equivalently, $\neg p \to (\neg q \wedge \neg r)$.

Exercises 2.7.4

1. By definition, the inverse of is $p \to q$ is $(\neg p) \to \neg q$. The contrapositive of $(\neg p) \to \neg q$ is $(\neg\neg q) \to \neg\neg p$, which is the same as $q \to p$, the converse of $p \to q$.

2. (b) This is the contrapositive of the original statement, so it is logically equivalent.

Exercises 2.8.1

1. (a) No book is more interesting than this book.
   (c) Some dogs do not have their day.
   (e) Some losers wil not get another chance.
   (g) Everybody is a born singer.
   (h) Either some roses are not red or some violets are not blue.

Exercises 2.9.2

1. (a) Valid: see Figure 2.6, left.

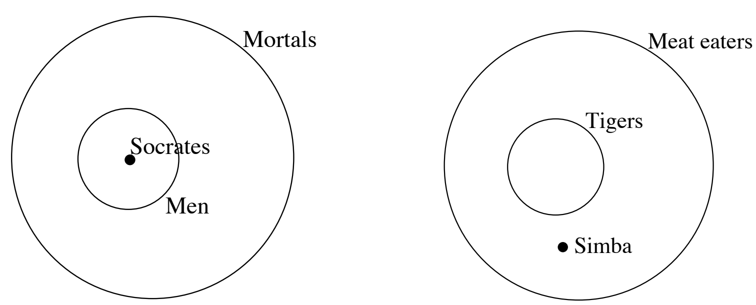
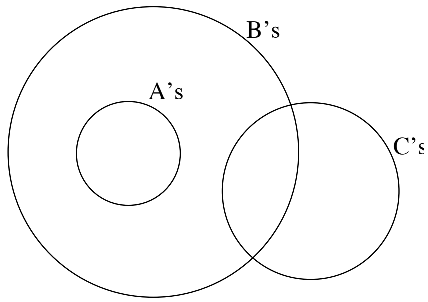


Figure 2.6: Left: valid argument. Right: invalid argument.



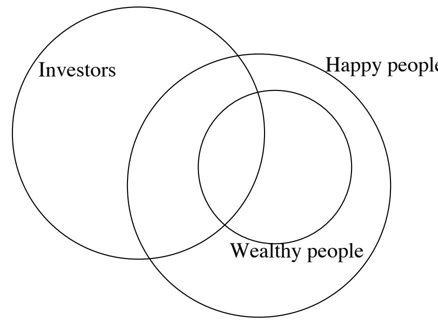


Figure 2.7: Left: invalid argument. Right: valid argument.

(c) Invalid: see Figure 2.6, right.

(e) Invalid: see Figure 2.7, left.

(g) Valid: see Figure 2.7, right.

(h) Invalid: see Figure 2.8.

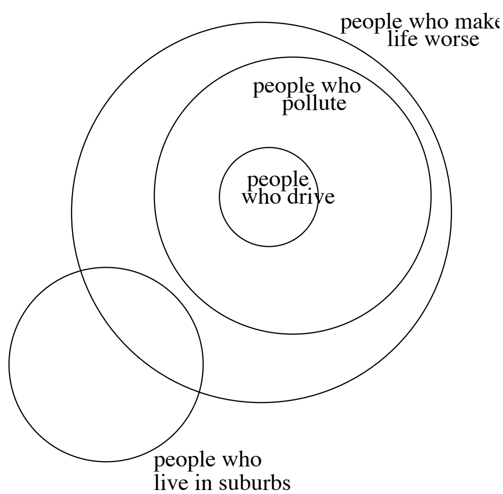


Figure 2.8: Invalid argument.

2. (b) Therefore, some mathematicians are intelligent.

Exercises 2.10.1

1. (a) We can represent the argument symbolically as:

$$\begin{aligned} &p \to q \\ &\neg p \\ \therefore\ &\neg q \end{aligned}$$

p	q	$p \to q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Invalid

(c) We can represent the argument symbolically as:

$$\begin{aligned} &p \to q \\ &q \to r \\ \therefore\ &p \to r \end{aligned}$$

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Valid

Note: This valid argument is known as the *chain rule* or *law of transitivity.*

(e) We can represent the argument symbolically as:

$$\begin{aligned} &p \\ &p \to q \\ &(\neg r) \vee (\neg q) \\ \therefore\ &r \end{aligned}$$

p	q	r	$\neg r$	$\neg q$	p	$p \to q$	$(\neg r) \vee \neg q$	r	
T	T	T	F	F	T	T	F	T	
T	T	F	T	F	T	T	T	F	
T	F	T	F	T	T	F	T	T	
T	F	F	T	T	T	F	T	F	*Invalid*
F	T	T	F	F	F	T	F	T	
F	T	F	T	F	F	T	T	F	
F	F	T	F	T	F	T	T	T	
F	F	F	T	T	F	T	T	F	

2. (a) Invalid (Fallacy of the Inverse).

   (c) Invalid.

3. (a) Therefore, if today is Saturday, then I will go shopping or I will go to the picnic.

4. (a) Valid: $(\neg q) \to \neg r$ is the contrapositive of $r \to q$, hence is equivalent. So we can rewrite the argument as:

$$\begin{aligned} & p \to r \\ & r \to q \\ \therefore\ & p \to q \end{aligned}$$

   This is valid by the law of transitivity. (See problem 1c above.)