Math Logic

1 Introduction

Math is like a foreign language. It is used to describe the world around us. In order to understand math, like any other language you must learn the vocabulary and how to express ideas with that vocabulary. Math logic is the structure that allows us to describe concepts in terms of math. We will start with very basic ideas and build on them. Starting in this way, we will be able to build an understanding of math in manageable steps. With this foundation even more complicated problems won't be so difficult.

2 Propositions

A proposition is a declaration that can be either true or false, but not both. For example, "Today is Friday" is a proposition. This statement can be true or false, but not both. It is common to define a shorthand notation for propositions: Let P be the proposition "Today is Friday." If the statement is true, then P has truth value true. If it is false, then P has truth value false. It is also common notation to use a "place filler" in a proposition. For example let $P(x)$ be "x is a an odd number." Then $P(x)$ is a proposition depending on x.

Sample problems: Determine which of the following are propositions.

1. $x = 5$.
2. The grey dog.
3. The dog is brown.
4. The real numbers.

Of course the propositions that we have been using here aren't sufficiently complicated to develop meaningful mathematical ideas. So we must describe interactions with propositions. Naturally, we define an operation called negation. This idea is exactly what you would expect it to be. For example, let P denote "Today is Friday". Then the negation of P, written $-P$, is "Today is not Friday." Notice that P and $-P$ cannot both have the same truth value. This is called the law of non contradiction.

Sample problems: Find the negation of each of the previous sample problems determined to be propositions.

Next we define two operations on propositions. Again these operations are intuitive based on every day vernacular. The operations are "and", denoted "$\wedge$", and "or", denoted "$\vee$." In order for the proposition $P \wedge Q$ to be true, both P and Q must be true. In order for $P \vee Q$ to be true either P must be true or Q must be true, not necessarily both. For example, let P be "$x < 4$" and Q be "$x > 2$." Then $P \wedge Q$ is the proposition "$x < 4$ and $x > 2$." And $P \vee Q$ is the proposition "$x < 4$ or $x > 2$."

Sample problems: Write the following in words based on the propositions: Let P be "I have a sister," Q be "I have a brother," and N be "I have a cousin."

1. $P \wedge Q \wedge N$
2. $P \wedge -Q$
3. $P \vee Q$
4. $(P \vee Q) \wedge N$

3 Implication

Implication or logical implication is another relationship between two propostions. Implication is basically the idea of an if, then statemnt. For example, let P be "Today is Saturday" and Q be "It is the weekend." Then the implication, "P implies Q", written $P \Rightarrow Q$, means "If it is Saturday, then it is the weekend." Then we can determine that $P \Rightarrow Q$ is true. Note that $Q \Rightarrow P$, called the converse, is false.

Sample problems: Write the following in words based on the propositions: Let P be "$x = 1$" and Q be "$x^2 = 1$." Then determine which are true.

1. $P \Rightarrow Q$
2. $Q \Rightarrow P$

It is also possible for $P \Rightarrow Q$ and $Q \Rightarrow P$ to both be true. So when $P \Rightarrow Q$ and $Q \Rightarrow P$, we write $P \Leftrightarrow Q$ and say "P if and only if Q." When this condition is true, we also say that P and Q are equivalent.

Implication can be written in terms of the previously defined operations. The symbolic definition of implication is: $P \Rightarrow Q = -P \vee Q$. Then it can be seen that $P \Rightarrow Q = -P \vee Q = Q \vee -P = -Q \Rightarrow -P$. So $P \Rightarrow Q$ and $-Q \Rightarrow -P$ are equivalent or $(P \Rightarrow Q) \Leftrightarrow (-Q \Rightarrow -P)$. This equivalent statement is called the contrapositive.

Sample Problems: Write the contrapositive in words of the implications in the previous sample problems.

4 Quantifiers

The previous sections have made it possible to write a lot of different propositions, but we are still not able to write all of the statements that we will want. Quantifiers allow us to specify the scope of the propositions that we can write. There are two quantifiers that we will be concerned with. The first is called the universal quantifier, written $\forall$. This quantifiers is used to define the scope of a proposition. For example, the proposition "All people are good" can be written symbolically using the universal quantifier. Let $P(x)$ be the statement "x is good." Then the previous proposition can be written as "$\forall$ x is a person, $P(x)$."
The other quantifier that we introduce is the existential quantifier, written $\exists$. This quantifier is also used to define the scope of a proposition. For example the proposition "There exist good people" can be written symbolically "$\exists$ a person x such that $P(x)$."
It is often useful to use these two together to write propositions. For example the proposition "For any real number x, there is a real number that is greater than x" can be written symbolically as: "$\forall$ real number x, $\exists$ a real number y such that $y > x$."

Sample problems: Define propositions and use quantifiers to write the following symbolically.

1. The square of every real number is non-negative.
2. The square root of any positive number is a real number.
3. Every real number has an additive inverse.
4. Every non-constant linear function has a zero.