Sine and Cosine Series - (12.3)

1. **Sine and Cosine Series Expansions:**
Let $f(x)$ be **an even function** on $(-p,\ p)$. $f(x)$ can be expanded to **an even periodic function** with period $2p$:

$$f_1(x) = f(x) \text{ for } x \text{ in } (-p,\ p) \text{ with period } T = 2p.$$

Then the Fourier series of $f_1(x)$

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) \text{ where}$$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx, \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx$$

is called **the cosine series expansion** of $f(x)$ or $f(x)$ is said to be expanded in **a cosine series**. Similarly, let $f(x)$ be **an odd function** on $(-p,\ p)$. $f(x)$ can be expanded to **an odd periodic function** with period $2p$:

$$f_2(x) = f(x) \text{ for } x \text{ in } (-p,\ p) \text{ with period } T = 2p.$$

Then the Fourier series of $f_2(x)$

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) \quad \text{where} \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx$$

is called **the sine series expansion** of $f(x)$ or $f(x)$ is said to be expanded in **a sine series**.

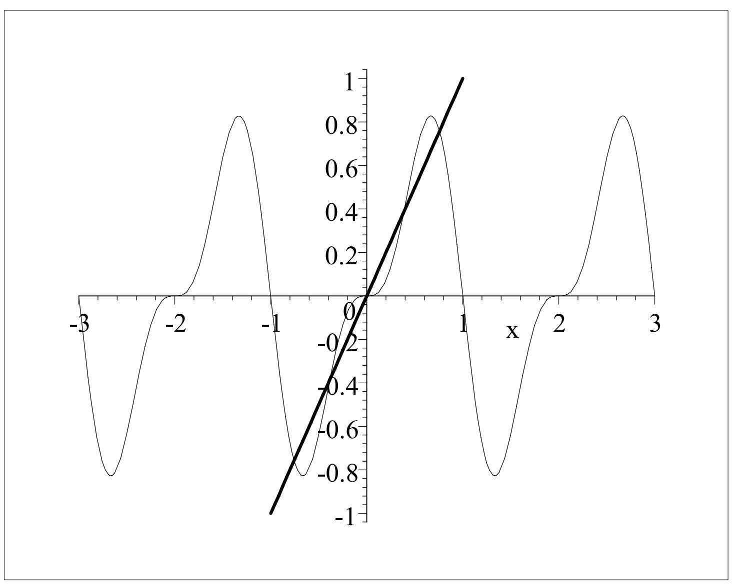
Example *Let $f(x) = x,\ -1 < x < 1$. Find the cosine or sine series expansion of $f(x)$.*

Since $f(x)$ is an odd function, it has a sine series expansion.

$$b_n = 2\int_0^1 x\sin(n\pi x)dx = -2\frac{n\pi\cos n\pi}{n^2\pi^2} = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

$$f_{\exp}(x) = \sum_{n=1}^{\infty} b_n\sin(n\pi x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)$$

$$= \frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \frac{1}{4}\sin(4\pi x) + \ldots\right)$$



$f(x) = x,\ -1 \le x \le 1$

Example *Let* $f(x) = cos(x), \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. *Find the cosine or sine series expansion of* $f(x)$.

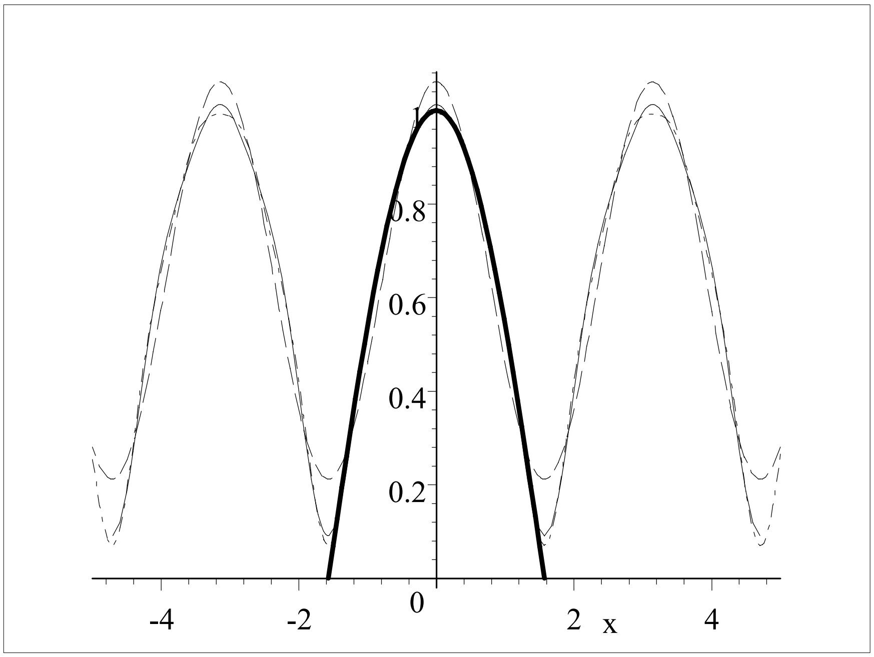
Since $\cos x$ is even on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, it has a cosine series expansion.

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} \cos(x)dx = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \cos(x)\cos(2nx)dx = -\frac{4}{\pi}\frac{\cos n\pi}{-1+4n^2} = \frac{4}{\pi}(-1)^{n+1}\frac{1}{4n^2-1}$$

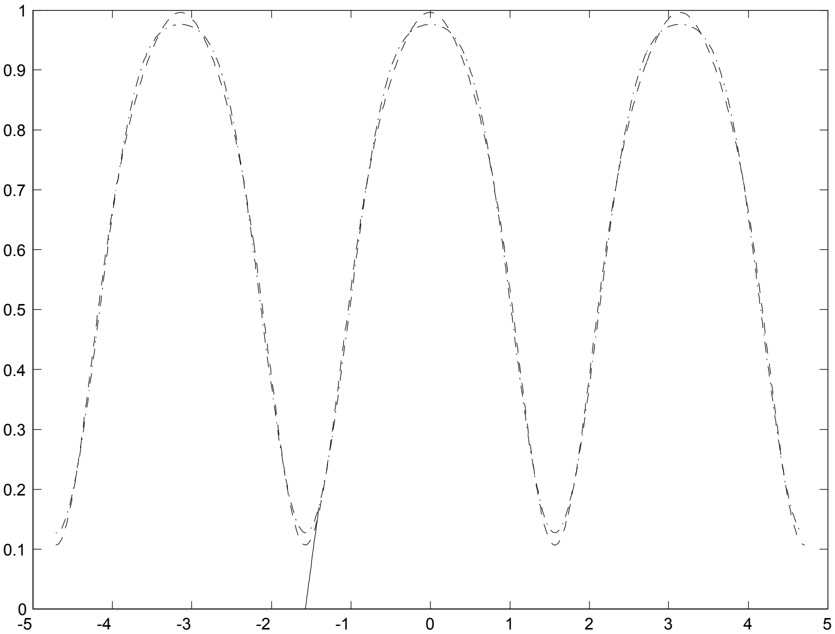
$$f_{\exp} = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{4}{\pi}(-1)^{n+1}\frac{1}{4n^2-1}\cos(2nx)$$

$$= \frac{4}{\pi}\left(\frac{1}{2} + \frac{1}{3}\cos(2x) - \frac{1}{15}\cos(4x) + \frac{1}{35}\cos(6x) - \frac{1}{63}\cos(8x) + \ldots\right)$$



In MatLab:

```
>> x1=-pi/2:.01:pi/2;
>> y1=cos(x1);
>> x2=-3*pi/2:.01:3*pi/2;
>> y2=4/pi*(1/2+cos(2*x2)/3-cos(4*x2)/15);
>> y3=y2+4/pi*(cos(6*x2)/63);
>> clf
>> plot(x1,y1)
>> hold
>> plot(x2,y2,'-.',x2,y3,'–')
>> hold off
```



2. **Sine and Cosine for Half-range Expansions:**
   Let $g(x)$ be defined for $0 < x < p$, where $p > 0$. Define

$$f_1(x) = \begin{cases} g(x) & \text{if } 0 < x < p \\ g(-x) & \text{if } -p < x < 0 \end{cases}, \text{ with period } 2p$$

$$f_2(x) = \begin{cases} g(x) & \text{if } 0 < x < p \\ -g(-x) & \text{if } -p < x < 0 \end{cases}, \text{ with period } 2p.$$

Note that:

a. $f_1(x)$ is a**n even periodic function** and $f_2(x)$ is **an odd periodic function**.
b. $f_1(x)$ has **an cosine series expansion** and $f_2(x)$ has **an sine series expansion**.

Fourier series of $f_1(x)$ and $f_2(x)$ are of the forms:

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right), \text{ where } a_0 = \frac{2}{p}\int_0^p f(x)dx, \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx;$$

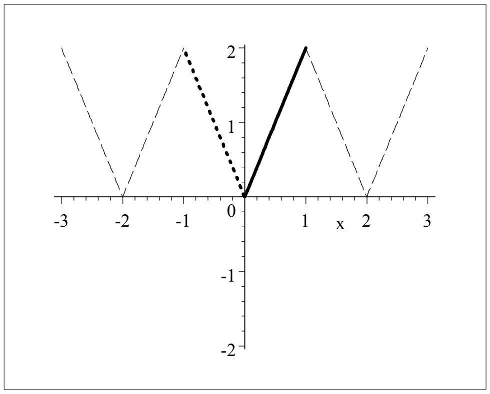
$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right), \quad \text{where } b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)dx.$$

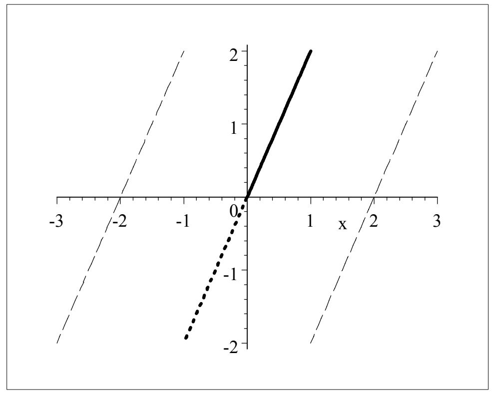
$f_1(x)$ is represented by **a cosine series** and $f_2(x)$ is represented by **a sine series**. Both $f_1(x)$ and $f_2(x)$ are called **half-range expansions** of $g(x)$.

Example *For each of the following function, Find its even and odd expansions and sketch the graph of each expansion in three periods.*

(1) $g(x) = 2x, \quad 0 < x < 1.$ $f_1(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ -2x & \text{if } -1 < x < 0 \end{cases}$ and $f_2(x) = 2x,$ for $-1 < x < 1.$

The period: 2 $(p = 1)$



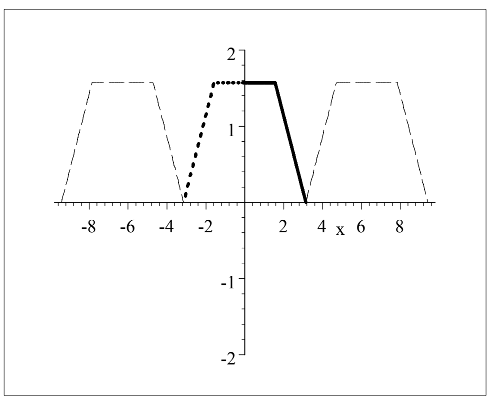


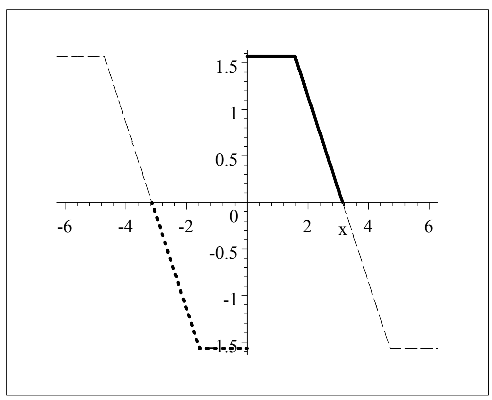
– $y = g(x)$, – – $y = f_1(x)$ – $y = g(x)$, – – $y = f_2(x)$

(2) $g(x) = \begin{cases} \frac{\pi}{2} & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$.

$$f_1(x) = \begin{cases} x + \pi & \text{if } -\pi < x < -\frac{\pi}{2} \\ \frac{\pi}{2} & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \text{ and } f_2(x) = \begin{cases} -x - \pi & \text{if } -\pi < x < -\frac{\pi}{2} \\ -\frac{\pi}{2} & \text{if } -\frac{\pi}{2} < x < 0 \\ \frac{\pi}{2} & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

Period $T = 2\pi$ and $p = \pi$.





– $y = g(x)$, – – $y = f_1(x)$ – $y = g(x)$, – – $y = f_2(x)$

Example *Let* $g(x) = 2x, 0 < x < 1$. *Find t**he even half-range expansion** of* $g(x)$.

According to Example (1) above, $f_1(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ -2x & \text{if } -1 < x < 0 \end{cases}$, with period 2 $(p = 1)$.

$$a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 2xdx = 2$$

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_0^1 2x\cos(n\pi x)dx = \frac{4}{\pi^2 n^2}(\cos \pi n - 1)$$

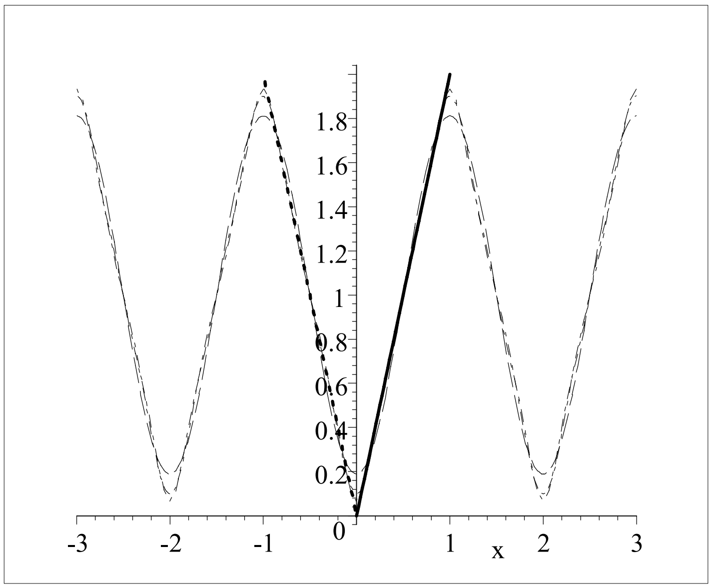
$$= \begin{cases} 0 & \text{if } n = 2m \\ \dfrac{-8}{\pi^2 n^2} & \text{if } n = 2m+1 \end{cases}$$

$$f_1(x) = 1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos((2m+1)\pi x)$$

$$F_1(x) = 1 - \frac{8}{\pi^2}\cos(\pi x)$$

$$F_2(x) = 1 - \frac{8}{\pi^2}[\cos(\pi x) + \frac{1}{9}\cos(3\pi x)]$$

$$F_3(x) = 1 - \frac{8}{\pi^2}[\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x)]$$



y=$f_1(x), F_1(x), F_2(x), F_3(x)$

Example *Let* $g(x) = \begin{cases} \dfrac{\pi}{2} & \text{if } 0 < x < \dfrac{\pi}{2} \\ \pi - x & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$. *Find* ***the odd half-range expansion*** *of* $g(x)$.

According to Example (2) above, $f_2(x) = \begin{cases} -x-\pi & \text{if } -\pi < x < -\dfrac{\pi}{2} \\ -\dfrac{\pi}{2} & \text{if } -\dfrac{\pi}{2} < x < 0 \\ \dfrac{\pi}{2} & \text{if } 0 < x < \dfrac{\pi}{2} \\ \pi - x & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$ with period 2π $(p = \pi)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{\pi}{2}\sin(nx)dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\frac{\pi}{n} + \frac{\sin\frac{1}{2}\pi n}{n^2}\right]$$

$$= \frac{1}{n} + \frac{2\sin\frac{1}{2}\pi n}{\pi n^2}, \qquad \text{recall} \begin{cases} \sin\frac{1}{2}\pi n = 0 \text{ if } n = 2m \\ \sin\frac{1}{2}\pi n = (-1)^{m-1} \text{ if } n = 2m-1 \end{cases}$$

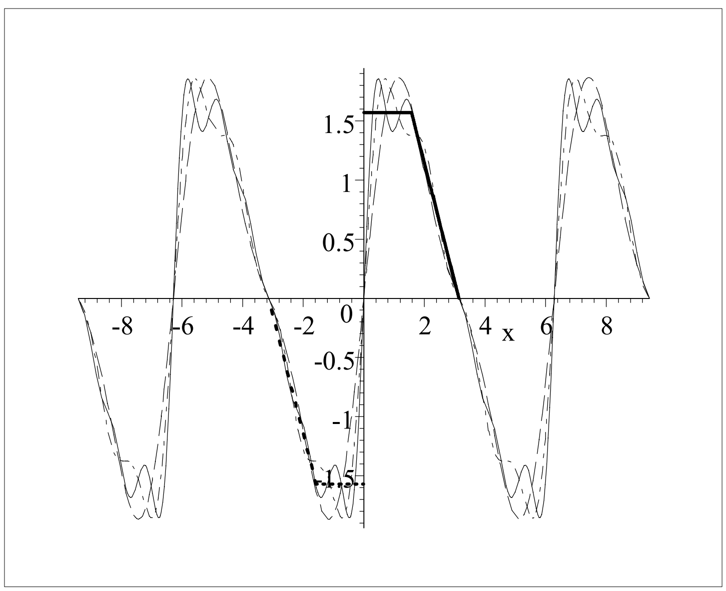
$$= \begin{cases} \frac{1}{2m} & \text{when } n = 2m \\ \frac{1}{2m-1}[1 + \frac{2(-1)^{m-1}}{\pi(2m-1)}] & \text{when } n = 2m-1 \end{cases}$$

$$f_2(x) = \sum_{m=1}^{\infty}\left[\frac{1}{2m-1}\left(1 + \frac{2(-1)^{m-1}}{\pi(2m-1)}\right)\sin((2m-1)x) + \frac{1}{2m}\sin(2mx)\right]$$

$$F_1(x) = (1 + \frac{2}{\pi})\sin(x) + \frac{1}{2}\sin(2x)$$

$$F_2(x) = F_1(x) + \frac{1}{3}(1 - \frac{2}{3\pi})\sin(3x) + \frac{1}{4}\sin(4x)$$

$$F_3(x) = F_2(x) + \frac{1}{5}(1 + \frac{2}{5\pi})\sin(5x) + \frac{1}{6}\sin(6x)$$



$y = f_2(x), F_1(x), F_2(x), F_3(x)$

Example *Find a particular solution to the differential equation:*

$$m\frac{d^2y}{dt^2} + ky = f(t)$$

where $m = \frac{1}{16}$, $k = 4$, $f(t) = \pi t$, *for* $0 < t < 1$ *with* $T = 2$.

$f_{\exp}(t) = \pi t, \quad -1 < t < 1$ with period $T = 2$. Find the Fourier series for $f_{\exp}$. Since $f_{\exp}$ is an odd function, its Fourier series is a sine series.

$$b_n = \frac{1}{1}\int_{-1}^{1} \pi t \sin(n\pi t)\ dt = -\frac{2n\pi\cos n\pi}{\pi n^2} = \frac{2}{n}(-1)^{n+1}$$

$$f_{\exp}(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(n\pi t)$$

The general solution for

$$\frac{1}{16}\frac{d^2y}{dt^2} + 4y = 0 \Leftrightarrow \frac{d^2y}{dt^2} + 64y = 0, \quad \lambda^2 + 64 = 0, \quad \lambda = \pm i8$$

$$y = C_1\cos(8t) + C_2\sin(8t)$$

A particular solution for the nonhomogeneous differential equation:

$$\frac{1}{16}\frac{d^2y}{dt^2} + 4y = B_n\sin(n\pi t)$$

is

$$y_n = A_n\sin(n\pi t).$$

Solve A_n by the method of undetermined coefficients:

$$y_n' = n\pi A_n\cos(n\pi t), \quad y_n'' = -(n\pi)^2 A_n\sin(n\pi t)$$

$$-\frac{1}{16}(n\pi)^2 A_n\sin(n\pi t) + 4A_n\sin(n\pi t) = B_n\sin(n\pi t)$$

$$A_n = \frac{16B_n}{64-(n\pi)^2}, \qquad y_n = \frac{16B_n}{64-(n\pi)^2}\sin(n\pi t) = \frac{32(-1)^{n+1}}{n\left(64-(n\pi)^2\right)}\sin(n\pi t)$$

A particular solution for the differential equation:

$$\frac{1}{16}\frac{d^2y}{dt^2} + 4y = f_{\exp}(t)$$

is

$$y_{\text{par}} = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n\left(64-(n\pi)^2\right)}\sin(n\pi t)$$

The general solution of the differential equation:

$$y = C_1\cos(8t) + C_2\sin(8t) + \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n\left(64-(n\pi)^2\right)}\sin(n\pi t)$$