The Work of Maryam Mirzakhani

Maryam Mirzakhani has made striking and highly original contributions to geometry and dynamical systems. Her work on Riemann surfaces and their moduli spaces bridges several mathematical disciplines—hyperbolic geometry, complex analysis, topology, and dynamics—and influences them all in return. She gained widespread recognition for her early results in hyperbolic geometry, and her most recent work constitutes a major advance in dynamical systems.

Riemann surfaces are named after the 19th century mathematician Bernhard Riemann, who was the first to understand the importance of abstract surfaces, as opposed to surfaces arising concretely in some ambient space. Mathematicians building on Riemann's insights understood more than 100 years ago that such surfaces can be classified topologically, i.e. up to deformation, by a single number, namely, the number of handles. This number is called the *genus* of the surface. The sphere has genus zero, the surface of a coffee cup has genus one, and the surface of a proper pretzel has genus three. Provided that one disregards the precise geometric shape, there is exactly one surface of genus g for every positive integer g.

A surface becomes a Riemann surface when it is endowed with an additional geometric structure. One can think of this geometric structure as a so-called complex structure, which allows one to do complex analysis on the abstract surface. Since the complex numbers involve two real parameters, a surface, which is two-dimensional over the real numbers, has only one complex dimension and is sometimes called a complex curve. The following fact links the theory of Riemann surfaces to algebraic geometry: Every complex curve is an algebraic curve, meaning that the complex curve, although defined abstractly, can be realized as a curve in a standard ambient space, in which it is the zero set of suitably chosen polynomials. Thus, although a Riemann surface is a priori an analytic object defined in terms of complex analysis on abstract surfaces, it turns out to have an algebraic description in terms of polynomial equations.

An alternative but equivalent way of defining a Riemann surface is through the introduction of a geometry that allows one to measure angles, lengths, and areas. The most important such geometry is *hyperbolic geometry*, the original example of a non-Euclidean geometry discovered by Bolyai, Gauss, and Lobatchevski. The equivalence between complex algebraic and hyperbolic structures on surfaces is at the root of the rich theory of Riemann surfaces.

Mirzakhani's early work concerns closed geodesics on a hyperbolic surface. These are closed curves whose length cannot be shortened by deforming them. A now-classic theorem proved more than 50 years ago gives a precise way of estimating the number of closed geodesics whose length is less than some bound L. The number of closed geodesics grows exponentially with L; specifically, it is asymptotic to e^L/L for large L. This theorem is called the "prime number theorem for geodesics", because it is exactly analogous to the usual "prime number theorem" for whole numbers, which estimates the number of primes less than a given size. (In that case the number of primes less than e^L is asymptotic to e^L/L for large L.)

Mirzakhani looked at what happens to the "prime number theorem for geodesics" when one considers only the closed geodesics that are *simple*, meaning that they do not intersect themselves. The behavior is very different in this case: the growth of the number of geodesics of length at most L is no longer exponential in L but is of the order of L^{6g-6}, where g is the genus. Mirzakhani showed that in fact the number is asymptotic to $c \cdot L^{6g-6}$ for large L (going to infinity), where the constant c depends on the hyperbolic structure.

While this is a statement about a single, though arbitrary, hyperbolic structure on a surface, Mirzakhani proved it by considering all such structures simultaneously. The complex structures on a surface of genus g form a continuous, or non-discrete, space, since they have continuous deformations. While the underlying topological surface remains the same, its geometric shape changes during a deformation. Riemann knew that these deformations depend on $6g-6$ parameters or "moduli", meaning that the "moduli space" of Riemann surfaces of genus g has dimension $6g-6$. However, this says nothing about the global structure of moduli space, which is extremely complicated and still very mysterious. Moduli space has a very intricate geometry of its own, and different ways of looking at Riemann surfaces lead to different insights into its geometry and structure. For example, thinking of Riemann surfaces as algebraic curves leads to the conclusion that moduli space itself is an algebraic object called an algebraic variety.

In Mirzakhani's proof of her counting result for simple closed geodesics, another structure on moduli space enters, a so-called symplectic structure, which, in particular, allows one to measure volumes (though not lengths). Generalizing earlier work of G. McShane, Mirzakhani establishes a link between the volume calculations on moduli space and the counting problem for simple closed geodesics on a single surface. She calculates certain volumes in moduli space and then deduces the counting result for simple closed geodesics from this calculation.

This point of view led Mirzakhani to new insights into other questions about moduli space. One consequence was a new and unexpected proof of a conjecture of Edward Witten (a 1990 Fields Medalist), one of the leading figures in string theory. Moduli space has many special loci inside it that correspond to Riemann surfaces with particular properties, and these loci can intersect. For suitably chosen loci, these intersections have physical interpretations. Based on physical intuition and calculations that were not entirely rigorous, Witten made a conjecture about these intersections that grabbed the attention of mathematicians. Maxim Kontsevich (a 1998 Fields Medalist) proved Witten's conjecture through a direct verification in 1992. Fifteen years later, Mirzakhani's work linked Witten's deep conjecture about moduli space to elementary counting problems of geodesics on individual surfaces.

In recent years, Mirzakhani has explored other aspects of the geometry of moduli space. As mentioned before, the moduli space of Riemann surfaces of genus g is itself a geometric object of $6g - 6$ dimensions that has a complex, and, in fact, algebraic structure. In addition, moduli space has a metric whose geodesics are natural to study. Mirzakhani and her co-workers have proved yet another analogue of the "prime number theorem for closed geodesics", in which they count closed geodesics in moduli space, rather than on a single surface. She has also studied certain dynamical systems (meaning systems that evolve with time) on moduli space, proving in particular that the system known as the "earthquake flow", which was introduced by William Thurston (a 1982 Fields Medalist), is chaotic.

Most recently, Mirzakhani, together with Alex Eskin and, in part, Amir Mohammadi, made a major breakthrough in understanding another dynamical system on moduli space that is related to the behavior of geodesics in moduli space. Non-closed geodesics in moduli space are very erratic and even pathological, and it is hard to obtain any understanding of their structure and how they change when perturbed slightly. However, Mirzakhani et al have proved that *complex* geodesics and their closures in moduli space are in fact surprisingly regular, rather than irregular or fractal. It turns out that, while complex geodesics are transcendental objects defined in terms of analysis and differential geometry, their closures are algebraic objects defined in terms of polynomials and therefore have certain rigidity properties.

This work has garnered accolades among researchers in the area, who are working to extend and build on the new result. One reason the work sparked so much excitement is that the theorem Mirzakhani and Eskin proved is analogous to a celebrated result of Marina Ratner from the 1990s. Ratner established rigidity for dynamical systems on homogeneous spaces—these are spaces in which the neighborhood of any point looks just the same as that of any other point. By contrast, moduli space is totally inhomogeneous: Every part of it looks totally different from every other part. It is astounding to find that the rigidity in homogeneous spaces has an echo in the inhomogeneous world of moduli space.

Because of its complexities and inhomogeneity, moduli space has often seemed impossible to work on directly. But not to Mirzakhani. She has a strong geometric intuition that allows her to grapple directly with the geometry of moduli space. Fluent in a remarkably diverse range of mathematical techniques and disparate mathematical cultures, she embodies a rare combination of superb technical ability, bold ambition, far-reaching vision, and deep curiosity. Moduli space is a world in which many new territories await discovery. Mirzakhani is sure to remain a leader as the explorations continue.

References

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Biography

Born in 1977 in Tehran, Iran, Maryam Mirzakhani received her Ph.D. in 2004 from Harvard University, where her advisor was Curtis McMullen. From 2004 to 2008 she was a Clay Mathematics Institute Research Fellow and an assistant professor at Princeton University. She is currently a professor at Stanford University. Her honors include the 2009 Blumenthal Award for the Advancement of Research in Pure Mathematics and the 2013 Satter Prize of the American Mathematical Society.