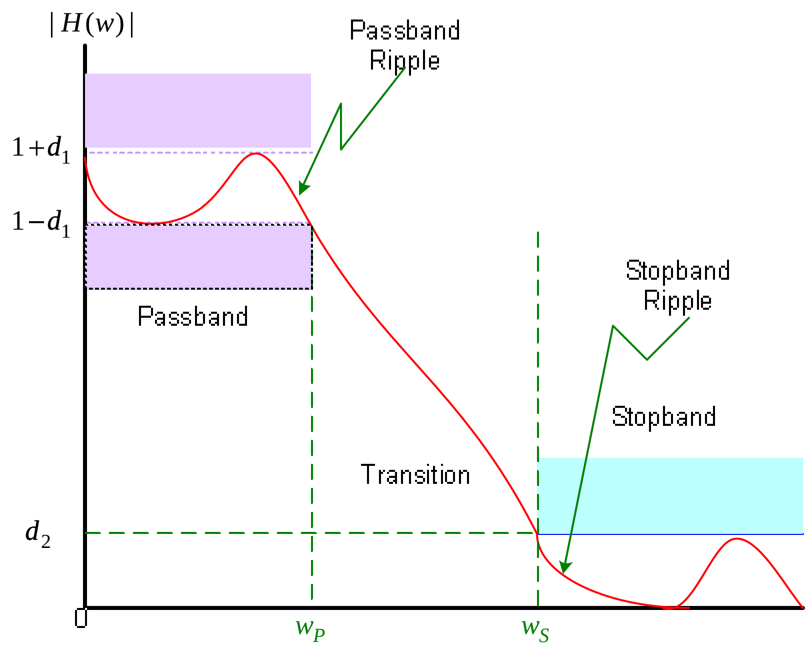
Chapter 7: Filter Design

7.1 Practical Filter Terminology

Analog and digital filters and their designs constitute one of the major emphasis areas in signal processing and communication systems. This is due to the fact that most of the LTI system blocks have a transfer function of a LP, HP, BP, BS or all pass nature. Traditionally, filters and their implementations have been in the analog realm but more and more digital versions become the staple of the engineering design tools. We have presented the spectral descriptions of ideal filters.

Terminology of a practical analog LP filter:



Observations:

- In addition to passband extending to w_P and stopband starting from w_S, there is a transition region $w_S - w_P$ rad/s or Hz, depending on the units of the horizontal axis.
- The ripple of $\pm d_1$ around the nominal value of 1.0 in the passband is called the passband or in-band ripple.

- Similarly, d_2 at the highest level that the spectrum can reach in the stopband is called stopband ripple.
- Both of these ripple parameters are expressed either in db or in percentage.
- An analog LP filter is expressed by:

$$\begin{aligned} 1-d_1 \le |H(w)| \le 1+d_1 \quad & if\ |w| \le w_P \\ |H(w)| \le d_2 \quad & if\ |w| \ge |w_S| \end{aligned} \tag{7.1}$$

- A digital LP filter is expressed by:

$$\begin{aligned} ||H(\Omega)|-1| \le d_1 \quad & if\ |\Omega| \le \Omega_P \\ |H(\Omega)| \le d_2 \quad & if\ \Omega_S \le |\Omega| \le p \end{aligned} \tag{7.2}$$

- Smaller the transition better the performance or higher the roll-off of a filter.
- Filters are specified in terms of these characteristics and there are both FIR and IIR structures for realization.
- Some filter classes have no ripple in the passband and some others exhibit equi-ripple characteristics.
- Almost all the time LP filters or LPF equivalent of other filters are designed and special transformations are used for obtaining actual systems.

7.2 Filter Parameter Transformations

There are standard transformations for converting a frequency-selective filter from one type to another one both for the continuous systems and the digital ones. Given a LP transfer function $H(s)$ with a normalized cut-off frequency unity (1.0), and convert it to a LP filter with a cutoff of w_C via:

$$s^{\#} = s.w_C \tag{7.3}$$

where $s^{\#}$ represents the transformed frequency variable. Since $w^{\#} = w.w_C$, then frequency range $0 \le |w| \le 1$ is mapped to $0 \le |w^{\#}| \le w_C$. Therefore, the transformation

$$s^{\#} = s.\frac{w_C^{\#}}{w_C} \qquad (7.4)$$

transforms a LP filter to a new LP filter with a cutoff frequency $w_C^{\#}$. There are similar transformations for other filter classes as shown in the table below.

Filter Type	**Transformation**
Low Pass	$\frac{s^{\#}}{w_C}$
High Pass	$\frac{w_C}{s^{\#}}$
Band Pass	$\frac{w_0}{BW}(\frac{s^{\#}}{w_0} + \frac{w_0}{s^{\#}}), \quad w_0 = \sqrt{w_{C_1}.w_{C_2}}$
Band Stop	$\frac{BW}{w_0.(\frac{s^{\#}}{w_0} + \frac{w_0}{s^{\#}})}, \quad BW = w_{C_2} - w_{C_1}$

Similarly, the discrete-time case is handled in terms of z-transforms. Given a LP filter with a cutoff frequency Ω_C, we want to obtain another one with a cutoff frequency $\Omega_C^{\#}$.

$$z^{\#} = \frac{z-a}{1-az} \quad \text{or equivalently} \quad (z^{\#})^{-1} = \frac{z^{-1}-a}{1-az^{-1}} \qquad (7.5)$$

By using the form: $z = e^{j\Omega}$ we have:

$$z^{\#} = \exp[j.\tan^{-1}\frac{(1-a^2)Sin\Omega}{2a + (1+a^2).Cos\Omega}\} \qquad (7.6)$$

the transformation maps the unit circle in the z-plane into a unit circle in the $z^{\#}$ –plane. The required a is given by:

$$a = \frac{Sin[(\Omega_C - \Omega_C^{\#})/2]}{Sin[(\Omega_C + \Omega_C^{\#})/2]} \quad (7.7)$$

Transformations are tabulated below:

Assume that $\Omega_C^{\#}$ is the desired cutoff frequency for LP/HP filters and $\Omega_{C_1}^{\#}, \Omega_{C_2}^{\#}$ are the desired lower and upper frequencies for the BP/BS filters.

Filter Type	**Transformation**	**Associated Formulas**
Low Pass	$(z^{\#})^{-1} = \frac{z^{-1} - a}{1 - az^{-1}}$	$a = \frac{Sin[(\Omega_C - \Omega_C^{\#})/2]}{Sin[(\Omega_C + \Omega_C^{\#})/2]}$
High Pass	$-\frac{z^{-1} + a}{1 + az^{-1}}$	$a = -\frac{Sin[(\Omega_C - \Omega_C^{\#})/2]}{Sin[(\Omega_C + \Omega_C^{\#})/2]}$
Band Pass	$\frac{z^{-2} - \frac{2a.k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2a.k}{k+1} z^{-1} + 1}$	$a = \frac{Cos[(\Omega_{C_2}^{\#} + \Omega_{C_1}^{\#})/2]}{Cos[(\Omega_{C_2}^{\#} - \Omega_{C_1}^{\#})/2]}$ $k = Cot[(\Omega_{C_2}^{\#} - \Omega_{C_1}^{\#})/2].Tan\frac{\Omega_C}{2}$
Band Stop	$\frac{z^{-2} - \frac{2a.}{k+1} z^{-1} - \frac{k-1}{k+1}}{-\frac{k-1}{k+1} z^{-2} - \frac{2a.}{k+1} z^{-1} + 1}$	$a = \frac{Cos[(\Omega_{C_2}^{\#} + \Omega_{C_1}^{\#})/2]}{Cos[(\Omega_{C_2}^{\#} - \Omega_{C_1}^{\#})/2]}$ $k = Tan[(\Omega_{C_2}^{\#} - \Omega_{C_1}^{\#})/2].Tan\frac{\Omega_C}{2}$

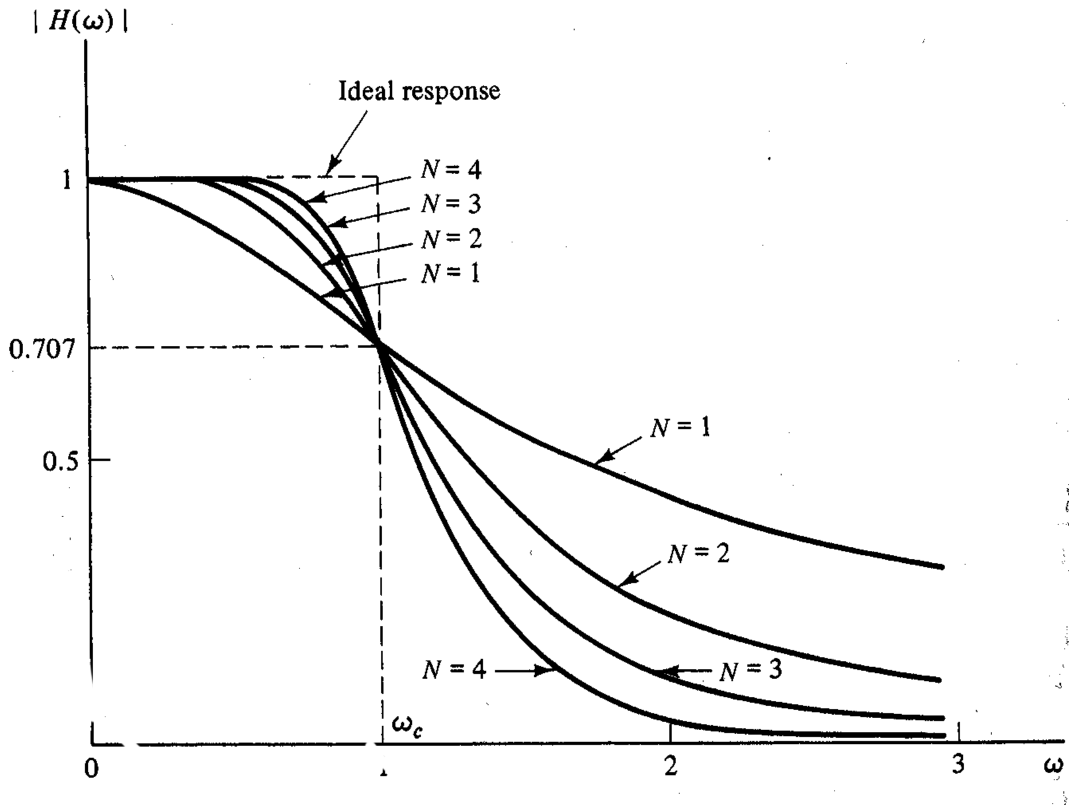
7.3 Butterworth Filters

The Butterworth filter class is an infinite impulse response filter. Let us recall the governing difference equation of IIR systems:

$$y[n] = \sum_{l=1}^{N} a_l . y[n-l] + \sum_{k=0}^{M} b_k . x[n-k] \qquad \text{and} \qquad H(z) \equiv \frac{Y(z)}{X(z)} = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k . z^{-k}}{\sum_{l=1}^{N} a_l . z^{-l}} \tag{7.8}$$

They have no passband ripple and the magnitude square for order N is described by:

$$|H(w)|^2 = 1/1 + (\frac{w}{w_C})^{2N} \tag{7.9}$$



- Magnitude is a monotonically decreasing function of w. It has a -3.0 dB at the normalized frequency of 1.0, which is also known as the 3 dB bandwidth.
- Butterworth is called a maximally flat approximation to ideal low-pass filters.
- The filter transfer function is obtained from:

$$H(s).H(-s)\,|_{s=jw} = |H(w)|^2 = \frac{1}{1+[\frac{(jw)^2}{j^2}]^N} = \frac{1}{1+[\frac{s}{j}]^{2N}} \qquad (7.10)$$

It is easy to see that the poles of $H(s)$ are given by the roots of:

$$[\frac{s}{j}]^{2N} = -1 = e^{j(2k-1)p} \quad \text{for } k = 0,1,2,\cdots,2N-1 \qquad (7.11)$$

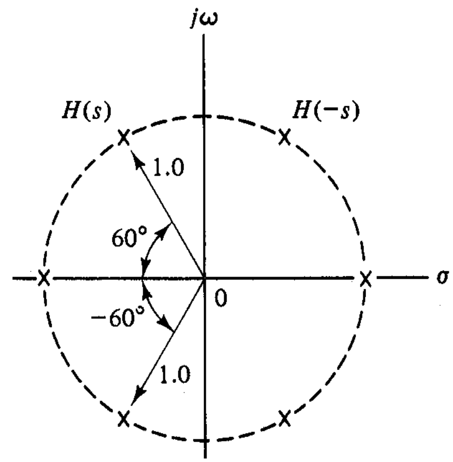
$$s_k = e^{j(2k+N-1)p\,/\,2N} = s_k s_k + jw_k \quad \text{for } k = 0,1,2,\cdots,2N-1 \qquad (7.12)$$

$$s_k = Cos(\frac{2k+N-1}{2N}p) = Sin(\frac{2k-1}{2N}.\frac{p}{2}) \quad \text{and} \quad w_k = Sin(\frac{2k+N-1}{2N}p) = Cos(\frac{2k+-1}{2N}\frac{p}{2}) \qquad (7.13)$$

For instance, Butterworth filter for N=3 has poles on the unit circle:

$s_0 = e^{jp/3}$; $\qquad s_1 = e^{j2p/3}$; $\qquad s_2 = e^{j3p/3} = e^{jp}$

$s_4 = e^{j4p/3}$; $\qquad s_5 = e^{j5p/3}$; $\qquad s_6 = e^{j6p/3} = 1$



To get a stable transfer function we MUST chose the poles of $H(s)$ in the LH plane, which results in:

$$H(s) = \frac{1}{(s - e^{j2p/3})(s - e^{jp})(s - e^{j4p/3})} = \frac{1}{(s^2 + s + 1)(s + 1)}$$

The coefficients of this third order Butterworth filter is given by a table:

Butterworth Filter Coefficients for orders 1 through 8								
Order	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1							
2	$\sqrt{2}$	1						
3	2	2	1					
4	2.613	3.41	2.613	1				
5	3.236	5.236	5.236	3.236	1			
6	3.864	7.464	9.141	7.464	3.864	1		
7	4.494	10.103	14.606	14.606	10.103	4.494	1	
8	5.126	13.128	21.828	25.691	21.828	13.128	5.126	1

To obtain a particular filter with a 3dB cutoff frequency at w_C, we replace s in $H(s)$ with a s / w_C .The corresponding magnitude characteristic is then expressed by:

$$H(s) = \frac{1}{1 + (w / w_C)^{2N}}$$

with the two constraints: $|H(w_P)| = 1 - d_1$ and $|H(w_S)| = d_2$

We must satisfy: $\left(\frac{w_P}{w_C}\right)^{2N} = \left(\frac{1}{1 - d_1}\right)^2 - 1$ and $\left(\frac{w_S}{w_C}\right)^{2N} = \left(\frac{1}{d_2}\right)^2 - 1$

To results in the order of the filter: $$N = Int\left[\frac{1}{2}\left[\left\{\frac{Log\,\dfrac{d_1(2 - d_1)d_2^2}{(1 - d_1)^2(1 - d_2)^2}}{Log(w_P / w_S)}\right\}\right]\right] \qquad (7.14)$$

Design Steps:

1. Determine equation N from (7.14), which must be the nearest integer.
2. Determine w_C from the expressions of constraints.
3. For the computed N determine the denominator polynomial of normalized $H(s)$ from the table.
4. Find the unnormalized transfer function by replacing s in $H(s)$ with s/w_C, which will be a unity gain filter.
5. Adjust the gain of the filter by the desired amplification factor, if needed.

Example 7.1: Design a low-pass Butterworth filter to have an attenuation no more than 1.0 dB for $|w| \leq 2000\ rad/s$ and at least 15 dB for $|w| \geq 5000\ rad/s$.

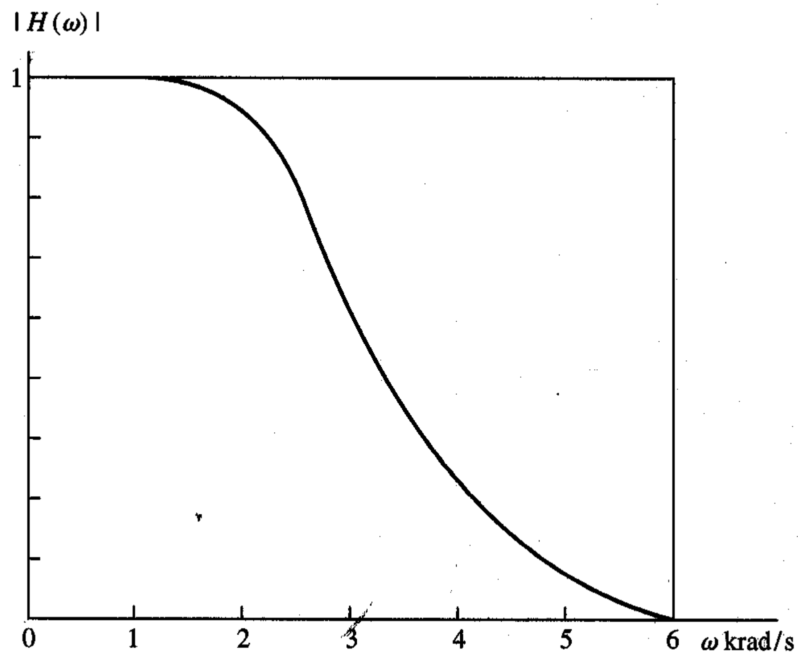
$20\ Log(1-d_1) = -1 \quad \Rightarrow d_1 = 0.1087$ and $20\ Log(d_2) = -15 \quad \Rightarrow d_2 = 0.1778$

These result at $N \geq 2.6045 \Rightarrow \quad N = 3$. From the table we have:

$H(s) = 1/(s^3 + 2s^2 + 2s + 1)$ and the second constraint yields $w_C = 2826.8\ rad/s$.

$$H(s) = (2826.8)^3 / [s^3 + 2*(2826.8).s^2 + 2*(2826.8)^2.s + (2826.8)^3]$$

When implemented using matlab we have:



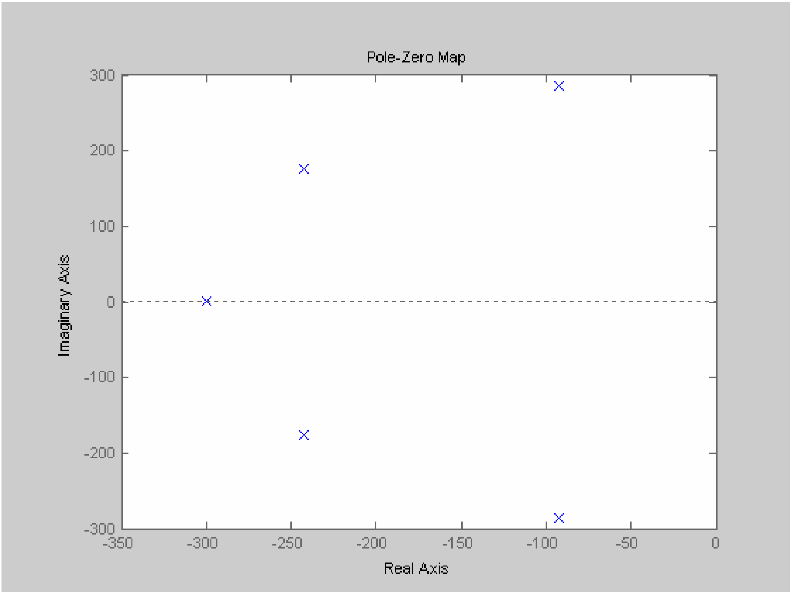
Example 7.2: An electro-cardiogram produces a voltage proportional to the heartbeat potentials. The doctor uses the plots after the raw data is filtered by an experimentally determined filter with the following impulse response:

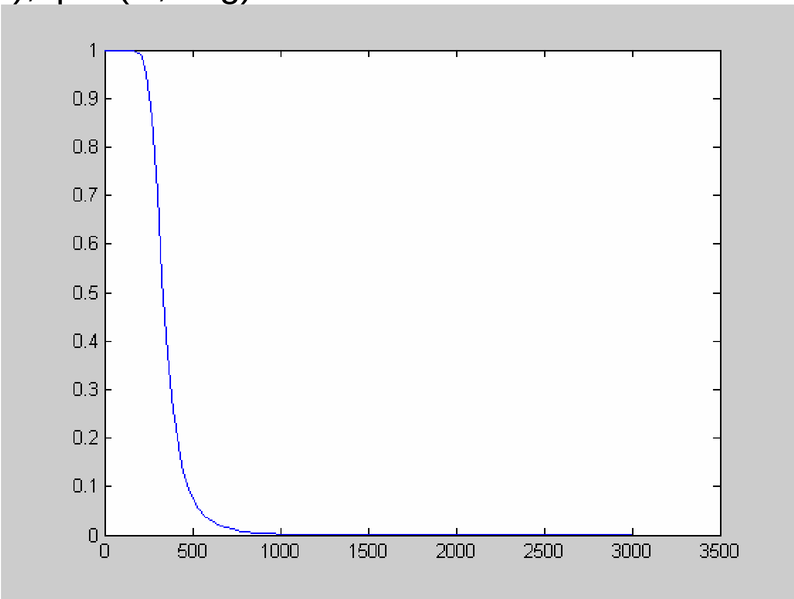
$$h(t) = 568.e^{-300\,t} - .e^{-243t}[485Cos176t - 668Sin176t] - e^{-93t}[83Cos285t - 255Sin285t]$$

An analysis yields a 5th Order Butterworth filter with a cutoff frequency of 47.75 Hz and unity gain.

```
% Example 7.2 5-th Butterworth filter for electro-cardiogram plots.
    [n,d]=butter(5,2*pi*47.75,'s');
    for i=1:size(n,2)-1; n(i)=0; end;
    pzmap(n,d);
    [r,p,k]=residue(n,d)
    delta=0.1*2*pi*47.75

    w=0:delta:100*delta;  h=freqs(n,d,w);  mag=abs(h);  plot(w,mag)
```






```
r = [1.0e+002 *;  -0.4146 + 1.2761i;  -0.4146 - 1.2761i;  -2.4272 - 3.3408i;  -2.4272 + 3.3408i;   5.6837]
p= [1.0e+002 *;  -0.9271 + 2.8534i;  -0.9271 - 2.8534i;  -2.4272 + 1.7635i;  -2.4272 - 1.7635i;  -3.0002]
k =  [ ]
```

7.4 Chebychev Filters

The Chebychev filter class si based on Chebychev Cosine polynomials:

$$C_N(w) = \begin{cases} Cos(N.Cos^{-1}w) & |w| \le 1 \\ Cosh(N.Cosh^{-1}w) & |w| > 1 \end{cases} \qquad (7.15)$$

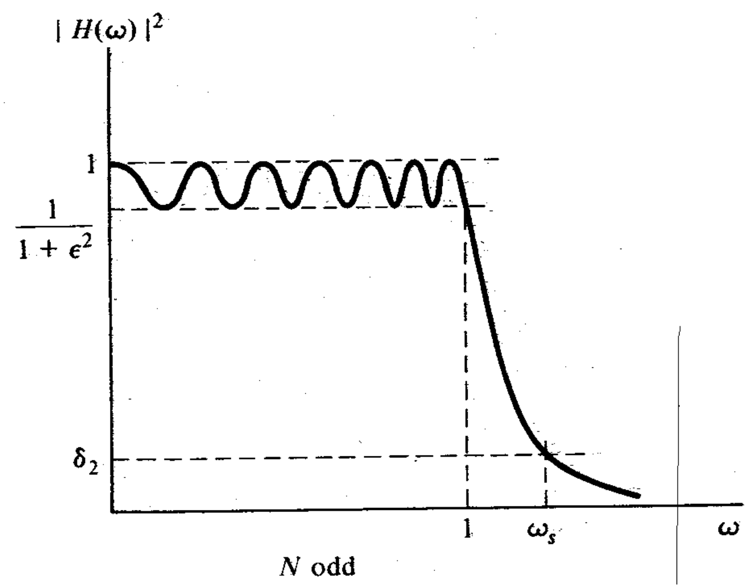
and we can use the following recursion to generate higher-order terms:

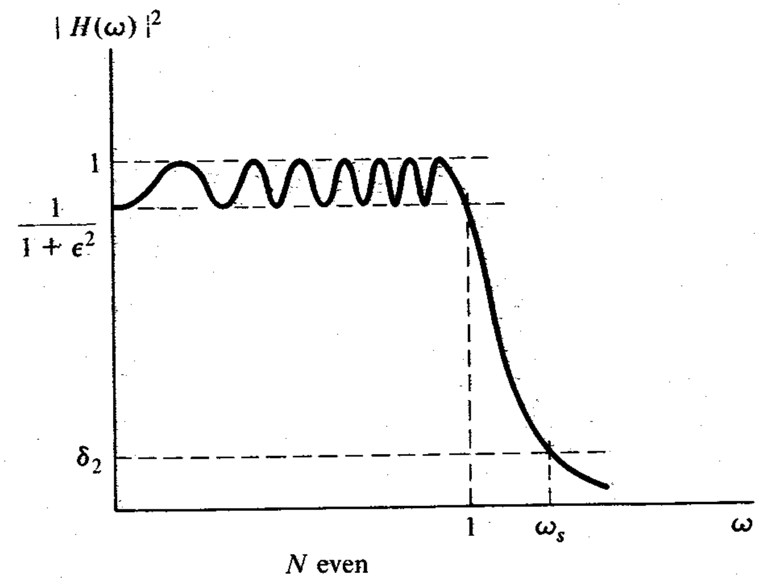
$$C_N(w) = 2.w.C_{N-1}(w) - C_{N-2}(w); \quad C_0(w) = 1; \quad C_1(w) = w \qquad (7.16)$$

The resulting filter has a low-pass characteric or order N:

$$|H(s)|^2 = \frac{1}{1 + e^2 C_N^2(w)} \qquad (7.17)$$

The magnitude characteristics:





$|H(s)|$ has ripples between 1 and $1/\sqrt{1+e^2}$ and for large values of w, stop-band region:

$$|H(w)| \cong 1/e.C_N(w); \qquad (7.18)$$

Attenuation (loss) $\quad loss = 20.Log_{10}e + 20.Log_{10}C_N(w)$ (7.19)

for large w: $\quad loss = 20.Log_{10}e + 6(N-1) + 20.N.Log_{10}w$ (7.20)

Design Steps:

1. From the passband specification determine equation e from (7.18), which must be the nearest integer.
2. Using the stop-band specification and (7.20) determine N.
3. Let us introduce a new parameter:

$$b = \frac{1}{N}.Sinh^{-1}(\frac{1}{e}) \qquad (7.21)$$

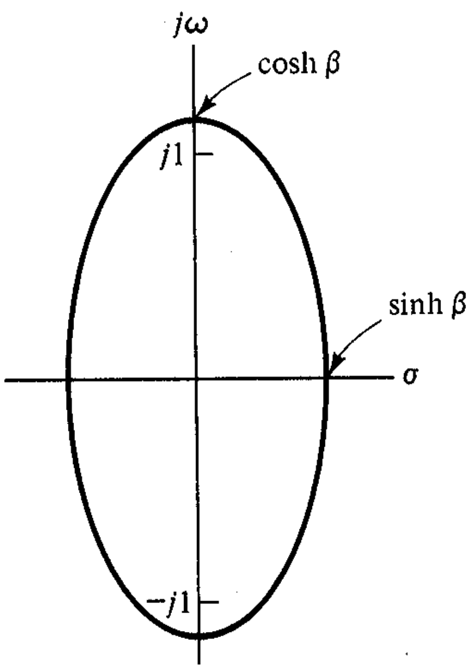
4. The poles of $H(s)$ are:

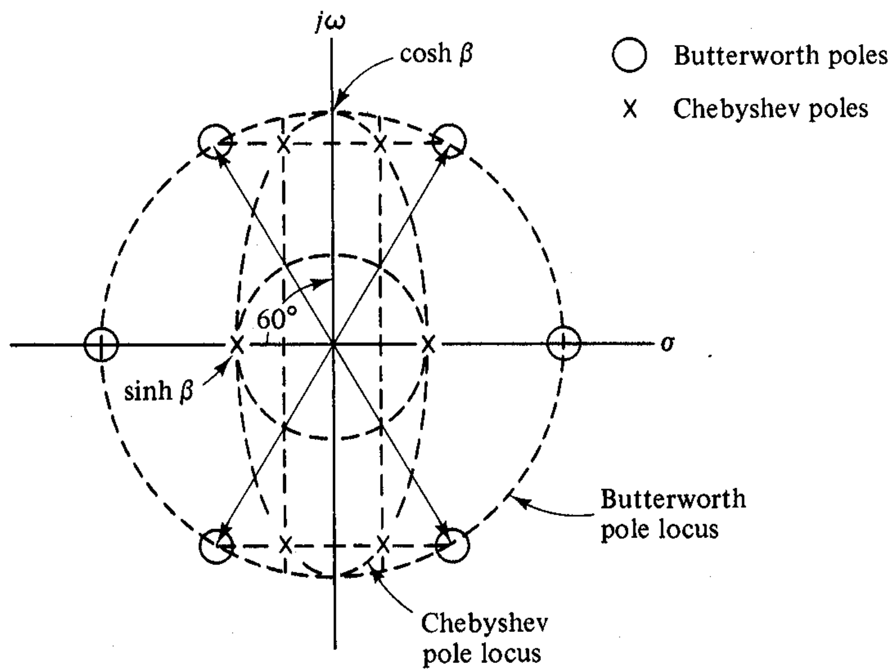
$$s_k = s_k + jw_k \quad for \quad k = 0,1,2,\cdots,N-1 \qquad (7.22)$$

$$s_k = Sin(\frac{2k-1}{N}).\frac{p}{2}.Sinh(b) \quad and \quad w_k = Cos(\frac{2k-1}{N}).\frac{p}{2}.Cosh(b) \qquad (7.23)$$

which are located on an ellipse in the s-plane with an equation:

$$\frac{s_k^2}{Sinh^2(b)} + \frac{w_k^2}{Cosh^2(b)} = 1 \qquad (7.24)$$





Example 7.3: Consider a Chebychev filter with attenuation not more than 1.0 dB for $|w| \le 1000\ rads/s$ and at least 10 dB for $|w| \ge 5000\ rads/s$.

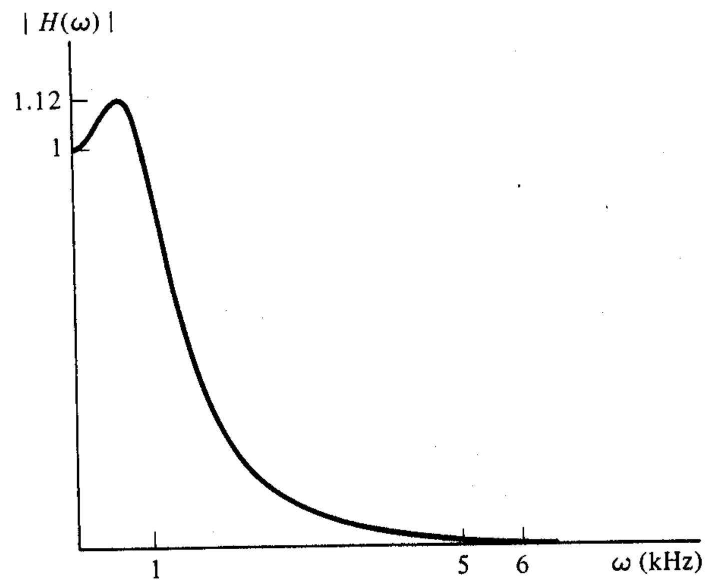
Let us normalize the parameters: $w_P = 1.0$ and $w_S = 5.0$.

$$20.Log_{10}\frac{1}{\sqrt{1+e^2}} = -1 \quad \Rightarrow \quad e = 0.509$$

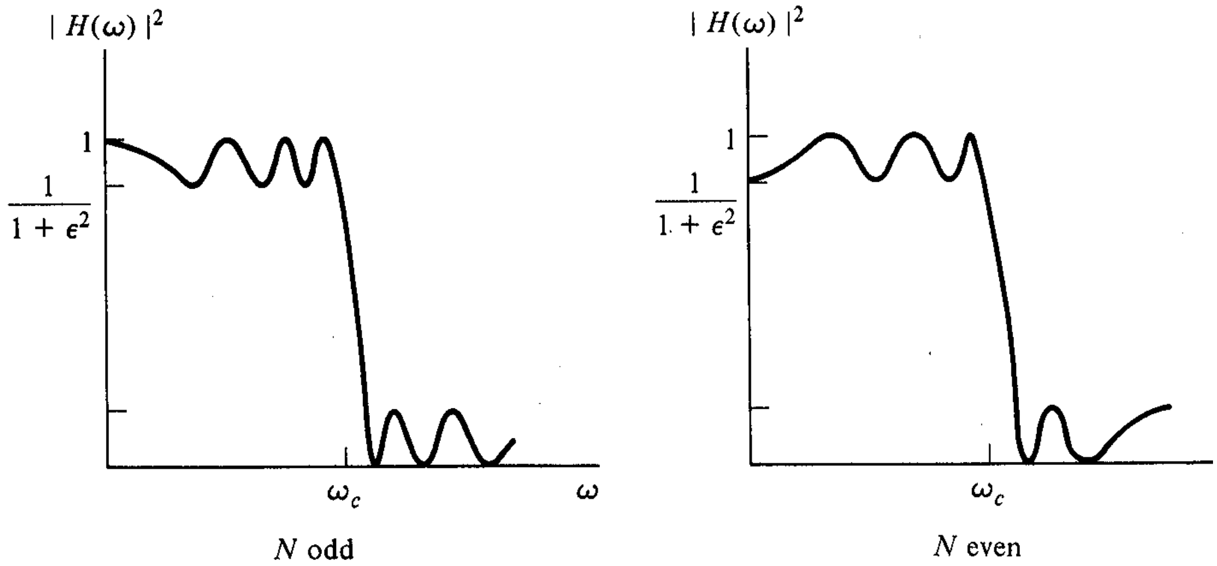
$$10 = 20.Log_{10}0.509 + 6(N-1) + 20.Log_{10}5 \quad \Rightarrow \quad N = [1.092] = 2$$

$$\frac{s}{w_P} = -1/\sqrt{2} \pm j1/\sqrt{2} \quad w_P = 1000 \quad \Rightarrow s_{1,2} = -545.31 \pm j892.92$$

$$H(s) = \frac{1}{(s+545.31)^2 + (892.92)^2}$$



Finally, an approximation to the ideal low-pass filter can be made in terms of Jacobi elliptic sine functions, which results in a smaller order or a sharper roll-off for a given order. The cost of this improvement is the presence of ripples in both the passband and stopband and stability issues.



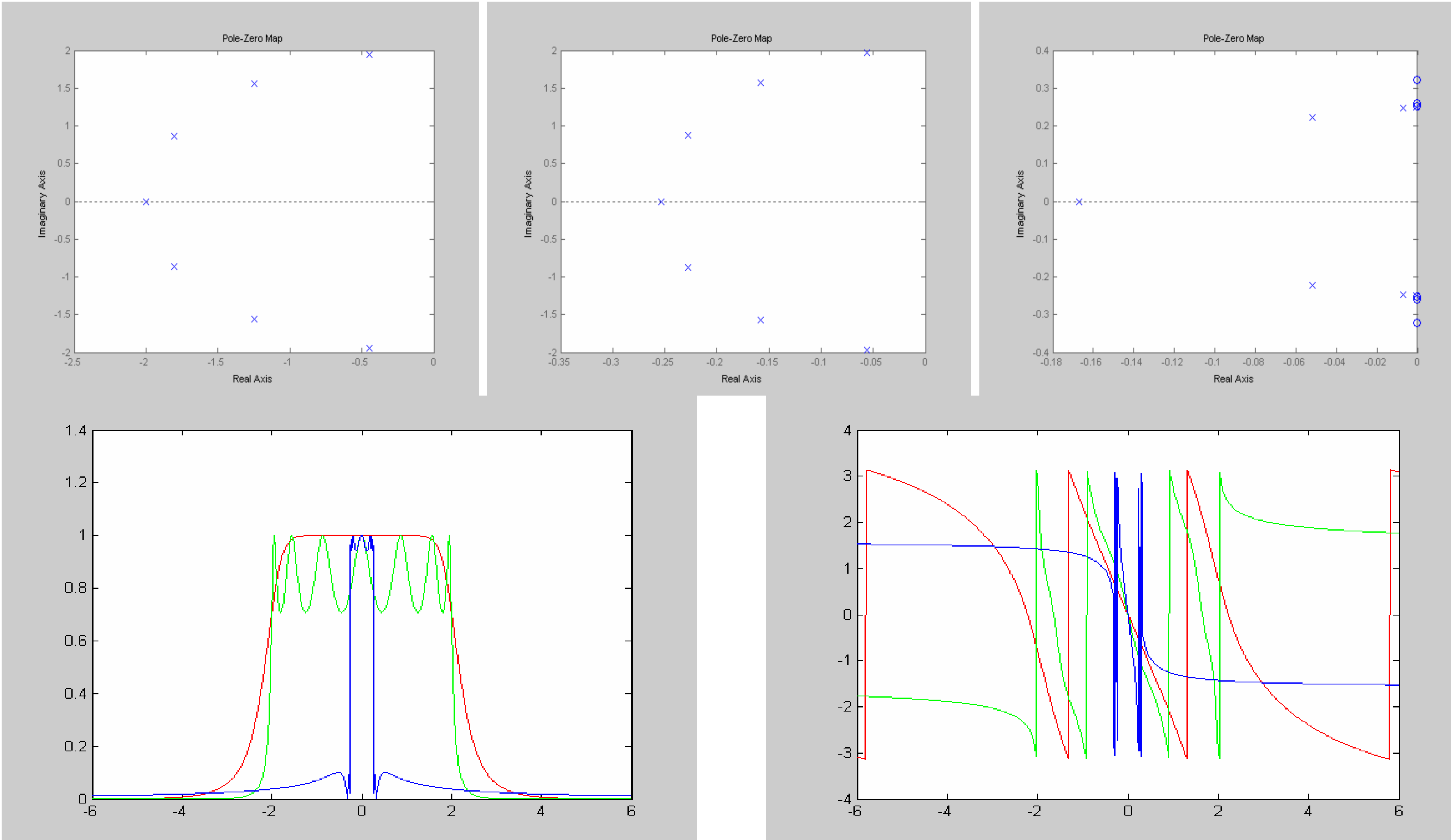
Example 7.4: 7^{th} Order Butterworth, Chebychev I and Elliptic Filters

```
% Example 7.4 Seventh Order IIR filters
% Filter design functions
        [nb,db]=butter(7,2,'s');  [nc,dc]=cheby1(7,3,2,'s');  [ne,de]=ellip(7,0.5,20,0.25,'s');

%Pole-zero maps
        pzmap(nb,db); figure; pzmap(nc,dc); figure; pzmap(ne,de); figure;

% Magnitude and phase computations
        w=-6:0.01:6
        hb=freqs(nb,db,w); hc=freqs(nc,dc,w); he=freqs(ne,de,w);
        magb=abs(hb); magc=abs(hc); mage=abs(he);
        phaseb=angle(hb); phasec=angle(hc); phasee=angle(he);

% Plotting functions
        plot(w,magb,'r',w,magc,'g',w,mage,'b'); figure; plot(w,phaseb,'r',w,phasec,'g',w,phasee,'b');
end;
```



7.5 Impulse Invariant Design of Digital IIR Filters

The digital versions of the previous set of IIR filters are obtained by equating the impulse responses at the sampling instances:

$$h[n] = h_a(nT) \qquad (7.25)$$

and the digital frequency variable is related to its analog counterpart via:

$$\Omega = w.T \tag{7.26}$$

Design Steps:

1. From the specified pass-band and stop-band cutoff frequencies: Ω_P, Ω_S and 7.26 find their analog counterparts.
2. Determine the analog transfer function $H_a(s)$ as it was done in Sections 7.3 or 7.4.
3. Expand $H_a(s)$ in partial fractions and obtain z-transform of each term from a s-transform to z-transform conversion tables[1].
4. Combine the terms to obtain $H(z)$.

Example 7.5: Let us re-visit the filtering problem of Example 7.1. Using $w_C = 2836.8$ and $H(s)$ from that exercise and partial fraction expansion:

$$H(s) = \frac{(2826.8)^3}{[s^3 + 2*(2826.8).s^2 + 2*(2826.8)^2.s + (2826.8)^3}$$

$$= \frac{2826.8}{s+2826.8} - \frac{2826.8*(s+1413.4)}{(s+1413.4)^2+(2448.1)^2} + \frac{0.5*(2826.8)^2}{(s+1413.4)^2+(2448.1)^2}$$

From the conversion table, we have get the transfer function in z-domain and for $T = 1.0\,ms$

$$H(z) = 2826.8*[\frac{z}{z-e^{-1413.4T}} - \frac{z^2 - z.e^{-1413.4T}\{Cos(2448.1T)+ZSin(2448.1T)\}}{z^2 - 2z.e^{-1413.4T}.Cos(2448.1T)+e^{-2826.8TT}}$$

$$= 2826.8*[\frac{z}{z-0.2433} - \frac{z^2 - 0.0973.z}{z^2 - 0.3742.z + 0.0592}]$$

which can be amplitude normalized and implemented in terms of canonical building blocks.

[1] These tables can be found in CRC and other mathematical tables or in many texts including Soliman & Srinath, Continuous and Discrete Signals and Systems, 2nd Edition, Prentice-Hall, 1998 The table has been included as a courtesy of the publishers.

Laplace transforms and their *Z*-transform equivalents

Laplace Transform, $H(s)$	*Z*-Transform, $H(z)$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{2}{s^3}$	$\frac{T^2 z(z+1)}{(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-\exp[-aT]}$
$\frac{1}{(s+a)^2}$	$\frac{Tz\exp[-aT]}{(z-\exp[-aT])^2}$
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}\left(\frac{z}{z-\exp[-aT]}-\frac{z}{z-\exp[-bT]}\right)$
$\frac{a}{s^2(s+a)}$	$\frac{Tz}{(z-1)^2}-\frac{(1-\exp[-aT])z}{a(z-1)(z-\exp[-aT])}$
$\frac{1}{(s+a)^2}$	$\frac{Tz\exp[-aT]}{(z-\exp[-aT])^2}$
$\frac{a^2}{s(s+a)^2}$	$\frac{z}{z-1}-\frac{z}{z-\exp[-aT]}-\frac{aT\exp[-aT]z}{(z-\exp[-aT])^2}$
$\frac{\omega_0}{s^2+\omega_0^2}$	$\frac{z\sin\omega_0 T}{z^2-2z\cos\omega_0 T+1}$
$\frac{s}{s^2+\omega_0^2}$	$\frac{z(z-\cos\omega_0 T)}{z^2-2z\cos\omega_0 T+1}$
$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\frac{z\exp[-aT]\sin\omega_0 T}{z^2-2z\exp[-aT]\cos\omega_0 T+\exp[-2aT]}$
$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\frac{z^2-z\exp[-aT]\cos\omega_0 T}{z^2-2z\exp[-aT]\cos\omega_0 T+\exp[-2aT]}$

Example 7.6: Let us design a Butterworth LP filter with specifications: Passband amplitude to be within 2.0 dB for frequencies below $0.2p\ rads/s$ and the stopband magnitude to be less than -10 dB for $0.4p\ rads/s$ or above. Assume unity gain at D.C.

$\Omega_P = 0.2p; \quad \Omega_S = 0.4p$ implies:

$$20.Log_{10} \mid H(0.2p) \mid = -2 \quad \Rightarrow \mid H(0.2p) \mid^2 = 10^{-0.2}$$

$$20.Log_{10} \mid H(0.4p) \mid = -10 \quad \Rightarrow \mid H(0.4p) \mid^2 = 10^{-1}$$

For impulse-invariance design we have the equivalent analog filter specifications for $w = \Omega T = \Omega$ for $T = 1$.

$$\mid H_a(0.2p) \mid^2 = 10^{-0.2} \quad \text{and} \quad \mid H_a(0.4p) \mid^2 = 10^{-1}$$

This results fro a Butterworth filter:

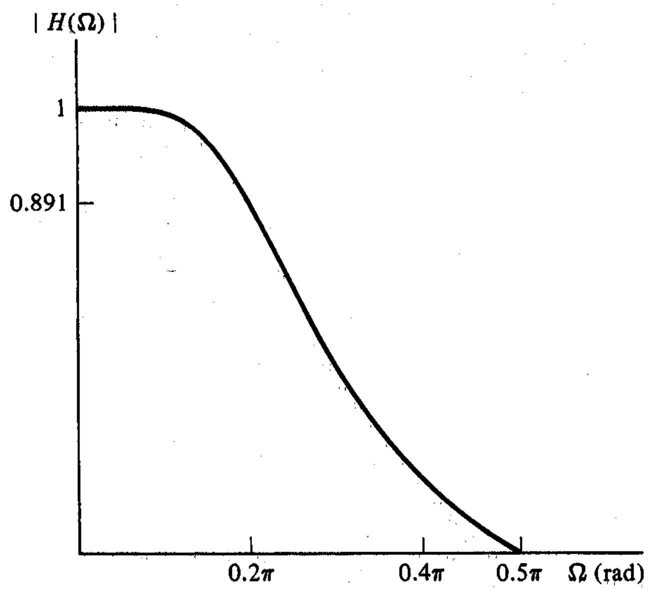
$$\mid H_a(jw) \mid^2 = \frac{1}{1 + (w/w_C)^{2N}} \qquad \Rightarrow 1 + (\frac{0.2p}{w_C})^{2N} = 10^{0.2} \quad and \quad 1 + (\frac{0.4p}{w_C})^{2N} = 10$$

Solution yields: $N = [1.9718] = 2 \quad and \quad w_C = 0.7185 \ rads/s$

$$H(s) = \frac{1}{(s/w_C)^2 + \sqrt{2} * (s/w_C) + 1} = \frac{0.5162}{s^2 + 1.01 * s + 0.5162}$$

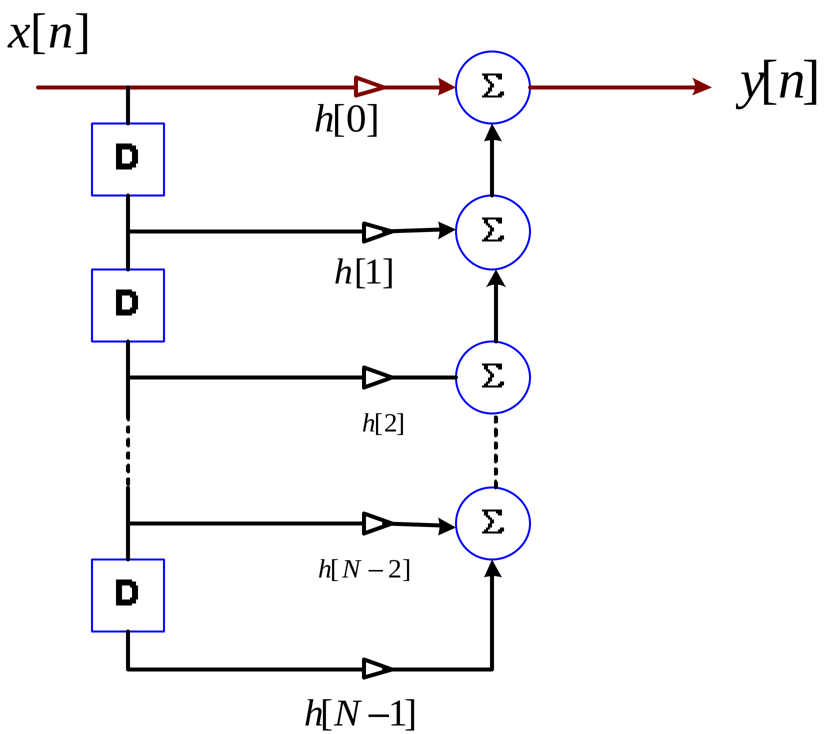
The corresponding z-transfer function:

$$H(z) = \frac{0.5854z}{z^2 - 1.051z + 0.362}$$



7.6 Digital FIR Filters

Digital finite-impulse response filters are also known as ”non-recursive filters are the most practical filters with absolute stability and no feedback. Canonical form of these filters are feedforward structures and consist of delay, multiply and accumulators and written by a difference equation of the form:



$$y[n] = \sum_{k=0}^{N-1} x[n-k].h[n] = \sum_{k=0}^{N-1} h[n-k].x[n] \qquad (7.27)$$

They have linear phase provided we have symmetrical coefficients:

$$h[n] = h[N-1-n] \qquad (7.28)$$

Their frequency responses are obtained from the term-by-term DTFT operation. However, filters of size odd and even exhibit different characteristics.

For $N: even$

$$H(\Omega) = \sum_{n=0}^{N-1} h[n].e^{-j\Omega n} = \sum_{n=0}^{N/2-1} h[n].e^{-j\Omega n} + \sum_{n=N/2}^{N-1} h[n].e^{-j\Omega n} \qquad (7.29)$$

Replace n by $N-n-1$ in the second sum and substitute (7.28):

$$H(\Omega)=\sum_{n=0}^{N/2-1} h[n].e^{-j\Omega n}+\sum_{n=0}^{N/2-1} h[n].e^{-j\Omega(N-1-n)}$$
$$=\left\{\sum_{n=0}^{N/2-1} 2h[n].Cos(\Omega(n-\frac{N-1}{2})\right\}.e^{-j\Omega\frac{N-1}{2}} \tag{7.30a}$$

Similarly, for $N:odd$

$$h(\Omega)=\left\{h(\frac{N-1}{2})+\sum_{n=0}^{(N-3)/2} 2.h[n].Cos(\Omega(n-\frac{N-1}{2})).\right\}.e^{-j\Omega\frac{N-1}{2}} \tag{7.30b}$$

In both cases, the term in braces is real, so that the phase of $H(\Omega)$ is given by the complex exponential at the end, which exhibit a linear phase shift or equivalently a delay of $\frac{N-1}{2}$ samples.

Given a desired frequency response $H_d(\Omega)$, such as an ideal low-pass filter symmetric about the origin, the corresponding impulse response $h_d[n]$ is symmetric about $n=0$.

- The most general procedure for obtaining FIR filter of length N is to obtain a finite sequence with a finite-length window sequence: $w[n]$.
- If $h_d[n]$ is symmetric then it has linear phase but
- the system is non-causal.
- Causal impulse response is obtained by shifting the sequence to the right by $\frac{N-1}{2}$ samples.

Design Procedure:

1. Obtain $H_d(\Omega)$ from $h_d[n]$ via DFT.
2. Multiply $h_d[n]$ by the window function $w[n]$..
3. Find the impulse response of the digital filter:

$$h[n]=h_d[n-(N-1)/2].w[n] \tag{7.31}$$

4. Determine $H(z)$ via z-transform or alternatively find the z-transform $H'(z)$ of the sequence $h_d[n].w[n]$:

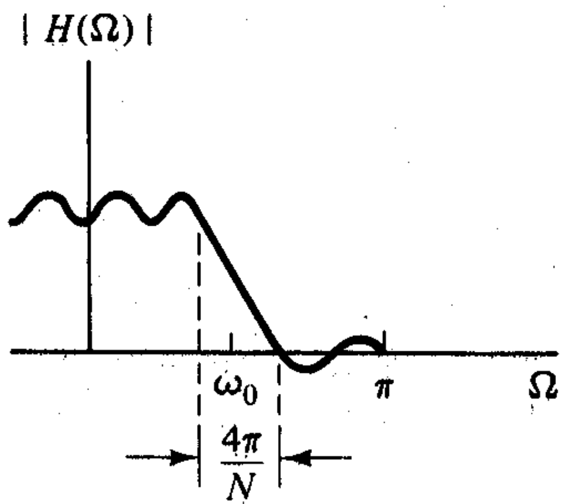
$$H(z)=z^{-(N-1)/2}.H'(z) \tag{7.32}$$

7.6 Design of Windowed FIR Filters

Depending upon the selection of a specific window function, one gets different filtering behavior.

1. Rectangular window filter:

$$w_R[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & Otherwise \end{cases} \tag{7.33}$$



2. Bartlett window filter:

$$w_B[n] = \begin{cases} 2n/N-1 & 0 \le n \le N-1 \\ 2-[2n/N-1] & (N-1)/2 \le n \le N-1 \\ 0 & Otherwise \end{cases} \tag{7.40}$$

3. Hanning window filter:

$$w_{Han}[n] = \begin{cases} 0.5*[1-Cos(2n\pi/(N-1))] & 0 \le n \le N-1 \\ 0 & Otherwise \end{cases} \tag{7.41}$$

4. Hamming window filter:

$$w_{Ham}[n] = \begin{cases} 0.54 - 0.46 * Cos(2n.p\ /(N-1)) & 0 \le n \le N-1 \\ 0 & Otherwise \end{cases} \quad (7.42)$$

5. Blackman window filter:

$$w_{Bl}[n] = \begin{cases} 0.42 - 0.5 * Cos(2n.p\ /(N-1)) + 0.08 * Cos(4n.p\ /(N-1)) & 0 \le n \le N-1 \\ 0 & Otherwise \end{cases} \quad (7.43)$$

6. Kaiser window filter:

$$w_K[n] = \begin{cases} \dfrac{I_0(a[(\frac{N-1}{2})^2 - (n - \frac{N-1}{2})^2]^{1/2})}{I_0[a(\frac{N-1}{2})]} & 0 \le n \le N-1 \\ 0 & Otherwise \end{cases} \quad (7.44)$$

where $I_0(x)$ is the modified zero-order Bessel function of the first kind usually studied with FM modulation.

Example 7.7: Let us design a 9-point LP FIR filter with a cutoff frequency $\Omega_C = 0.2p$.

$$h_d[n] = \frac{1}{2p} \int_{-2p}^{2p} e^{j\Omega.n} d\Omega = \frac{Sin(0.2n.p)}{n.p} \quad (7.45)$$

For a 9-point window, we evaluate $h_d[n]$ for $-4 \le n \le 4$:

Rectangular: $h_d[n] = \{\frac{0.147}{p}, \frac{0.317}{p}, \frac{0.475}{p}, \frac{0.588}{p}, 1, \frac{0.588}{p}, \frac{0.475}{p}, \frac{0.317}{p}, \frac{0.147}{p}\}$

$$H'(z) = \frac{0.147}{p} z^4 + \frac{0.317}{p} z^3 + \frac{0.475}{p} z^2 + \frac{0.588}{p} z^1 + 1 + \frac{0.588}{p} z^{-1} + \frac{0.475}{p} z^{-2} + \frac{0.317}{p} z^{-3} + \frac{0.147}{p} z^{-4}$$

$$H(z) = z^{-4} H'(z) = \frac{0.147}{p}(1 + z^{-8}) + \frac{0.317}{p}(z^{-1} + z^{-7}) + \frac{0.475}{p}(z^{-2} + z^{-6}) + \frac{0.588}{p}(z^{-3} + z^{-5}) + z^{-4}$$

Similarly, for a 9-point Hamming window filter (7.42) results in a sequence:

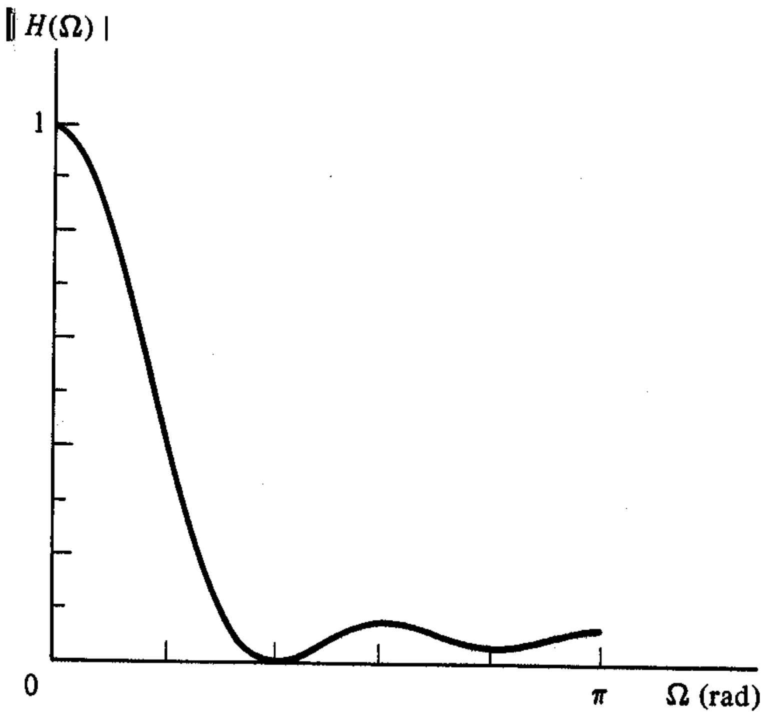
$$w[n] = \{0.081, 0.215, 0.541, 0.865, \underline{1}, 0.865, 0.541, 0.215, 0.081\}$$

$$h_d[n].w[n] = \{\frac{0.012}{p}, \frac{0.068}{p}, \frac{0.257}{p}, \frac{0.508}{p}, \underline{1}, \frac{0.508}{p}, \frac{0.257}{p}, \frac{0.068}{p}, \frac{0.012}{p}\}$$

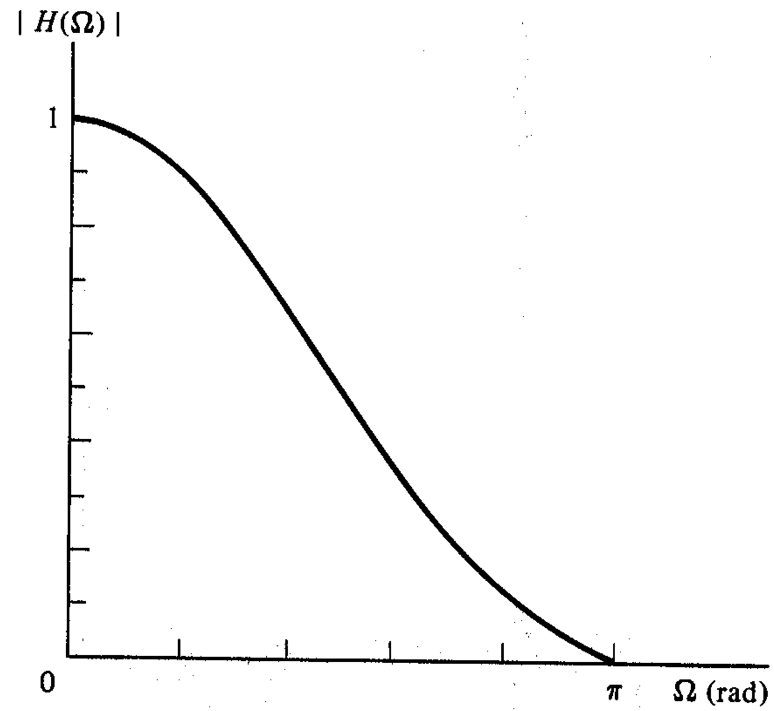
$$H'(z) = \frac{0.012}{p} z^{4} + \frac{0.068}{p} z^{3} + \frac{0.257}{p} z^{2} + \frac{0.508}{p} z + 1 + \frac{0.508}{p} z^{-1} + \frac{0.257}{p} z^{-2} + \frac{0.068}{p} z^{-3} + \frac{0.012}{p} z^{-4}$$

$$H(z) = z^{-4}.H'(z) = \frac{0.012}{p}(1 + z^{-8}) + \frac{0.068}{p}(z^{-1} + z^{-7}) + \frac{0.257}{p}(z^{-2} + z^{-6}) + \frac{0.508}{p}(z^{-3} + z^{-5}) + z^{-4}$$

The frequency responses clearly indicate differences in behavior.



9-point Rectangular Window LP Filter



9-point Hamming Window LP Filter

Example 7.8: Let us design a differentiator (Hilbert Transformer), which cannot be implemented in the analog domain. Hilbert transformer is used for obtaining single-side band (SSB) communication signal to generate signals that are in phase quadrature to a sinusoidal input. The ideal differentiator:

$$H(w) = jw \tag{7.46}$$

Hilbert transformer:

$$H(w) = -jSgn(w) \tag{7.47}$$

Desired response in the frequency-domain:

$$\begin{aligned} H_d(w) &= H(w) \quad -w_S/2 \le w \le w_S/2 \\ H_d(\Omega) &= H(\Omega T) \quad -p \le w \le p \end{aligned} \tag{7.48}$$

for some sampling period T. Expand the desired spectrum in a Fourier series since it was periodic:

$$H_d(\Omega) = \sum_{n=-\infty}^{\infty} h_d[n].e^{-jn\Omega} \tag{7.49}$$

$$h_d[n] = \frac{1}{2p}\int_{-p}^{p} H_d(\Omega).e^{jn\Omega}d\Omega \tag{7.50}$$

- If $H_d(\Omega)$ is purely real, the impulse response exhibits even symmetry; $h_d[n] = h_d[-n]$.
- If $H_d(\Omega)$ is purely imaginary, the impulse response exhibits odd symmetry; $h_d[n] = -h_d[-n]$.

If the input to a Hilbert transformer is $x_a(t) = Cos(w_0 t)$ then we get the output $y_a(t) = Sin(w_0 t)$. The impulse response of this transformer is given by:

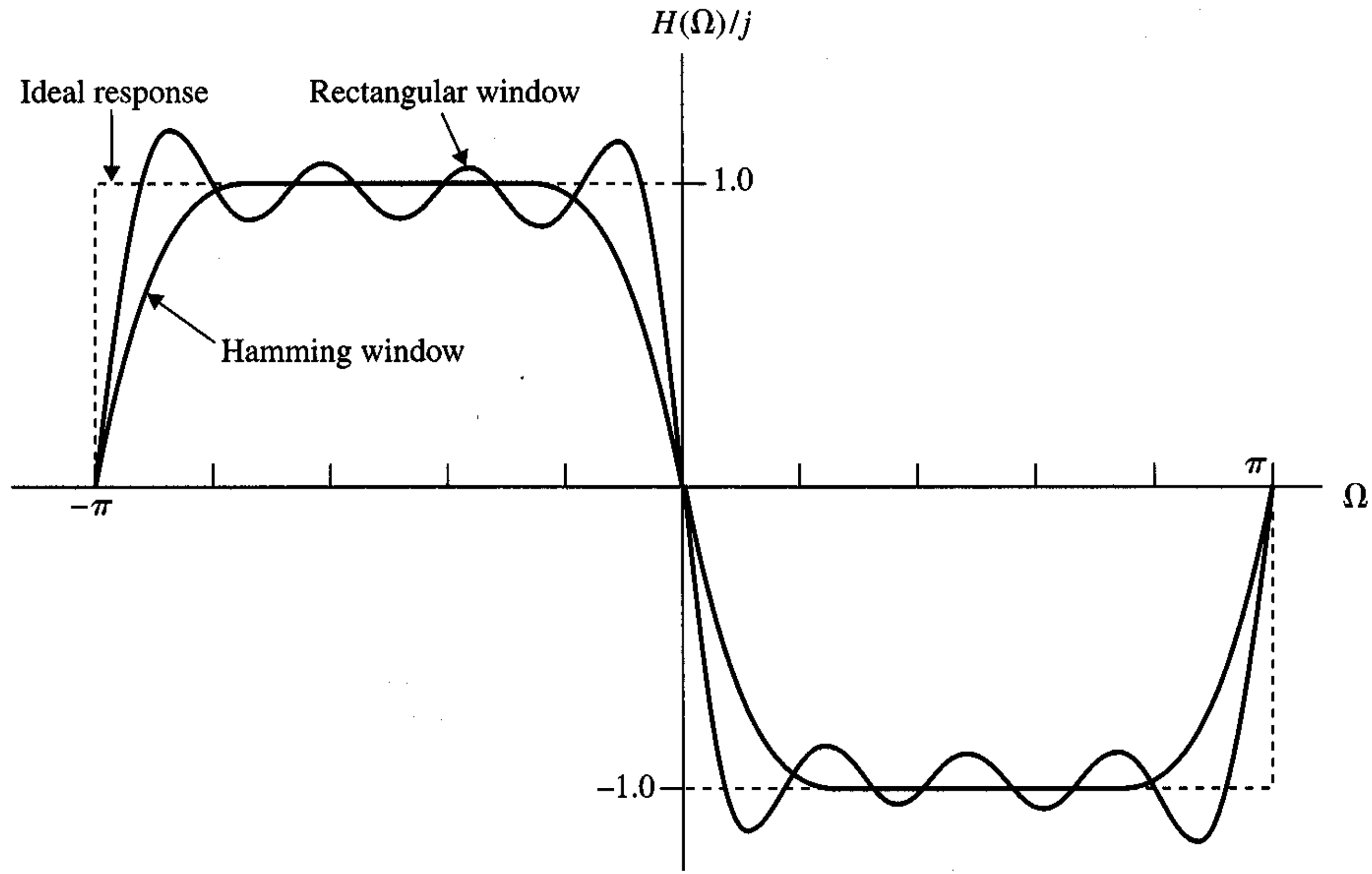
$$h_d[n] = \frac{1}{2p}\int_{-\infty}^{\infty} -j.Sgn(\Omega).e^{jn\Omega}.d\Omega = \begin{cases} 0 & n \ even \\ \dfrac{2}{np} & n \ odd \end{cases} \tag{7.51}$$

For a rectangular window of length 15 the impulse sequence from (7.51) becomes:

$$h_d[n] = \{-\frac{2}{7p},0,-\frac{2}{5p},0,-\frac{2}{3p},0,-\frac{2}{p},0,\frac{2}{p},0,\frac{2}{3p},0,\frac{2}{5p},0,\frac{2}{7p}\}$$

$$H(z) = \frac{2}{p}[-\frac{1}{7}-\frac{1}{5}z^{-2}-\frac{1}{3}z^{-4}-z^{-6}+z^{-8}+\frac{1}{3}z^{-10}+\frac{1}{5}z^{-12}+\frac{1}{7}z^{-14}]$$

Frequency response of this rectangular and similarly Hamming window versions of 15 tap filter are displayed.


$H(\Omega)/j$
Ideal response
Rectangular window
Hamming window
1.0
−1.0
−π
π
Ω