

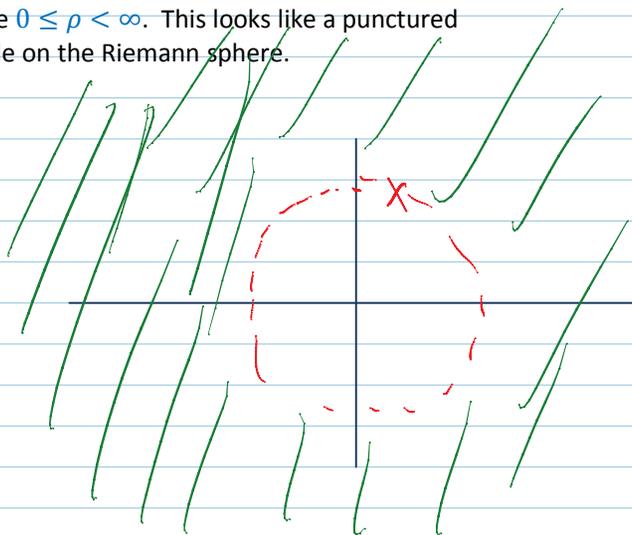
## Laurent Series Examples

Monday, November 11, 2013  
2:00 PM

Homework 3 due Friday, November 15 at 5 PM.

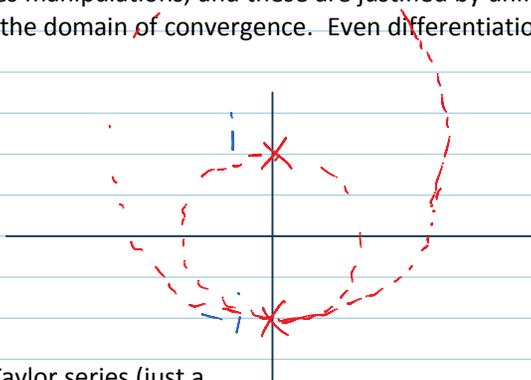
Most often, one is looking at Laurent series which are valid in a punctured neighborhood centered at a point  $z_0$ ; so they converge in a domain  $\{z \in \mathbb{C}: 0 < |z - z_0| < R\}$  for some  $R > 0$ . But recall that Laurent series more generally may only converge in some annulus, not necessarily a punctured neighborhood, but in any case the **annulus of convergence of a Laurent series is governed by the location of the singularities**. (Analogous theory to how the radius of convergence of a Taylor series is governed by the distance to the closest singularity).

It is often useful to talk about a Laurent series "in a punctured neighborhood of  $\infty$ ." This simply means a domain of the form  $\{z \in \mathbb{C}: \rho < |z - z_0| < \infty\}$ , for some  $0 \leq \rho < \infty$ . This looks like a punctured neighborhood of the north pole on the Riemann sphere.



**How does one develop Laurent series for concrete functions?** One can fall back to the definition given in the theorem about existence of Laurent series, but this is cumbersome. A more practical approach is simply to do formal power series manipulations, and these are justified by uniform convergence within any compact subset of the domain of convergence. Even differentiation is OK for complex analytic functions.

Example:  $f(z) = \frac{1}{1+z^2}$ .



About the point  $z_0 = 0$ , we can develop a Taylor series (just a special case of Laurent series expanded around non-singular points). As we saw, we can get it by just doing geometric series:

$$f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

converges in  $\{z \in \mathbb{C} \mid |z| < 1\}$

↑ no need for  
puncture; no  
singularity

How about a Laurent series about one of the singularities, say  $z = i$ .

$$f(z) = \frac{1}{1+z^2} = \frac{-\frac{1}{2}i}{z+i} + \frac{\frac{1}{2}i}{z-i}$$

↙ partial fractions

$$\frac{-\frac{1}{2}i}{z+i} = \frac{-\frac{1}{2}i}{(z-i) + 2i} \times \frac{\frac{1}{2i}}{\frac{1}{2i}}$$

$$= \frac{1/4}{1 + \frac{(z-i)}{2i}}$$

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n \quad \text{for } |r| < 1$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z-i}{2i}\right)^n$$

$$\frac{-\frac{1}{2}i}{z+i} = \sum_{n=0}^{\infty} 2^{-2n-2} i^{-n} (z-i)^n$$

The Laurent series expansion for  $f(z)$  about  $z=i$  can be written:

$$f(z) = \underbrace{\frac{1}{2i} (z-i)^{-1}}_{\text{principal part}} + \sum_{n=0}^{\infty} 2^{-2n-2} i^{-n} (z-i)^n$$

pole of order 1

This Laurent series converges for:

$$\{z \in \mathbb{C} \mid 0 < |z-i| < 2\}$$

either from the restriction on convergence of the geometric series, or more simply, the distance to the nearest singularity.

Let's expand this function as a Laurent series about  $z = \infty$ :

The systematic way to develop such Laurent series is to develop power series in terms of  $\frac{1}{z}$  which are valid when  $\frac{1}{z}$  is small. Let's try to write our function in a way that has a sensible expansion for  $1/z$  small.

$$f(z) = \frac{1}{z^2+1} \times \frac{1/z^2}{1/z^2}$$
$$= \frac{1/z^2}{1+1/z^2}$$

Why this manipulation? When you have denominators where one term is expected to dominate the other within the domain of convergence of the Laurent series, then it's cleanest to multiply/divide out by a factor that will make the main term = 1.

Now do geometric series w.r.t. term that is small in the neighborhood of the point  $z_0$  of interest.

$$f(z) = \frac{1}{z^2} \times \frac{1}{1+1/z^2} \quad \rightarrow \text{geometric series}$$
$$= \frac{1}{z^2} \left( \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z^2}\right)^n \right)$$
$$= \sum_{n=0}^{\infty} (-1)^n z^{-2n-2}$$

Domain of convergence of this Laurent series is  $\{z \in \mathbb{C}: |z| > 1\}$  either by restriction of convergence of geometric series or by looking at the closest singularity to infinity.

How **classify a singularity at infinity**? One classifies the singularity of  $f(z)$  at  $z = \infty$  by how the function  $\zeta^{-2}f(\zeta^{-1})$  has its singularity at  $\zeta=0$ . The prefactor will be better explained when we do residue calculus; it arises from the fact that the Jacobian of the change of variables  $z = \zeta^{-1}$  in integration is  $\zeta^{-2}$ .

Note the inclusion of the factor  $\zeta^{-2}$  in the classification is not conventional; but without doing this, we can have "residues at infinity" for functions "analytic at infinity," which breaks the harmony of the language.

Making this change of variable in our Laurent series about infinity, we have:

$$\zeta^{-2} f(\zeta^{-1}) = \zeta^{-2} \sum_{n=0}^{\infty} (-1)^n (\zeta^{-1})^{-2n-2}$$

$$\begin{aligned}
 &= \rho^{-2} \sum_{n=0}^{\infty} (-1)^n \rho^{2n+2} \\
 &= \sum_{n=0}^{\infty} (-1)^n \rho^{2n}
 \end{aligned}$$

No principal part, so one can say that  $\infty$  is a **removable singularity** of  $f(z)$ . Or, a bit more boldly but justifiably, one can even say that  $f(z)$  is "**analytic at infinity**."

Example:  $f(z) = e^z$ .

Here we can just write the Taylor series about  $z=0$ , using the fact that  $f'(z) = f(z)$ , so all derivatives evaluated at  $z=0$  are 1.

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{converges for } z \in \mathbb{C}$$

(no singularities in the finite complex plane).

Since this series converges in a punctured neighborhood of infinity, it is also interpretable as a Laurent series about  $z = \infty$ . What kind of singularity is  $z = \infty$ ? It's an **essential singularity**, as is evident from having infinitely many positive powers.

$$\begin{aligned}
 \rho^{-2} f(\rho^{-1}) &= \rho^{-2} \sum_{n=0}^{\infty} \frac{(\rho^{-1})^n}{n!} \\
 &= \sum_{n=0}^{\infty} \frac{\rho^{-2n-2}}{n!}
 \end{aligned}$$

Example:  $f(z) = e^{\frac{1}{z}}$

There's an apparent singularity at  $z=0$ ; let's try a Laurent series in a neighborhood of  $z=0$ .

One can obtain the series expansion by just using the composition of the functions  $e^w$  (known Laurent series about  $w = \infty$ ) and  $w = \frac{1}{z}$ .

$$\begin{aligned}
 e^w &= \sum_{n=0}^{\infty} \frac{w^n}{n!} \quad \text{for } w \in \mathbb{C} \\
 e^{1/z} &= \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} \quad \text{for } 1/z \in \mathbb{C}
 \end{aligned}$$

$$e^{-z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \quad \text{for } |z| < \infty$$

$z \neq 0$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

Principal part has infinitely many terms, so  $z = 0$  is an **essential singularity** of  $f(z)$ . And one can similarly check that  $z = \infty$  is a **removable singularity**, and we can say that  $f(z)$  is analytic at infinity.

Example:  $f(z) = \cos z \equiv \frac{e^{iz} + e^{-iz}}{2}$

One can obtain the Taylor series about  $z=0$  from the Taylor series for exponential by direct manipulation:

$$\begin{aligned} \cos z &= \frac{1}{2} (e^{iz} + e^{-iz}) \\ &= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right) \quad \text{for } z \in \mathbb{C} \\ &= \sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{2} \frac{z^n}{n!} \\ &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{(-1)^{n/2}}{n!} z^n = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j} \end{aligned}$$

$n = 2j$

converges for all  $z \in \mathbb{C}$ .

Similarly for  $f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$\begin{aligned} \sin z &= \frac{1}{2i} (e^{iz} - e^{-iz}) \\ &= \frac{1}{2i} \left( \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right) \quad \text{for all } z \in \mathbb{C} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n!} z^n \end{aligned}$$

$$= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{z^n}{n!} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

$n = 2j+1$

converges for  $z \in \mathbb{C}$

Example:  $f(z) = \frac{\sin z}{z}$

Laurent series about the apparent singularity at  $z=0$  (which we've before discussed should be a **removable singularity**). Simply divide the previous Laurent series by  $z$ .

$$\begin{aligned} f(z) = \frac{\sin z}{z} &= \frac{1}{z} \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j+1)!} \end{aligned}$$

Converges for all  $\{z \in \mathbb{C} : 0 < |z| < \infty\}$ .

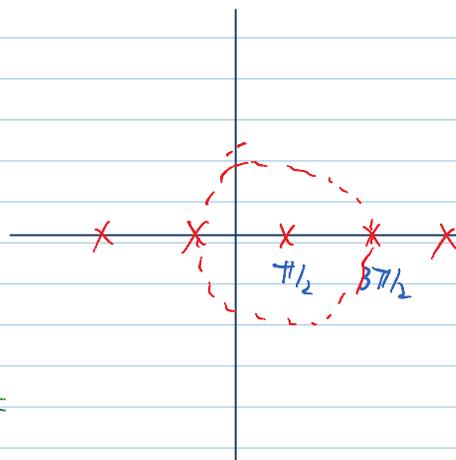
No principal part, so  $z=0$  is a **removable singularity**. Therefore, one can treat  $f(z)$  as analytic at  $z=0$ , if one defines  $f(0) = 1$ .

Example:  $f(z) = \tan z = \frac{\sin z}{\cos z}$

Let's compute a Laurent series about the singularity at  $z = \frac{\pi}{2}$ .

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} = \frac{\cos(\pi/2 - z)}{\sin(\pi/2 - z)} \\ &= \frac{\cos(w)}{\sin w} \quad w = \pi/2 - z \end{aligned}$$

$$\begin{aligned} &= \frac{1 - \frac{1}{2}w^2 + \frac{1}{24}w^4 + \dots}{w - \frac{1}{6}w^3 + \dots} \end{aligned}$$



$$= \frac{1}{w} \frac{1 - \frac{1}{2}w^2 + \dots}{1 - \frac{1}{6}w^2 + \dots}$$

(expand by geo series  $\frac{1}{1-r} = 1+r+\dots$ )

$$= \frac{1}{w} \left(1 - \frac{1}{2}w^2 + \dots\right) \left(1 + \frac{1}{6}w^2 + \dots\right)$$

$$= \frac{1}{w} \left(1 - \frac{1}{3}w^2 + \dots\right)$$

$$= w^{-1} - \frac{1}{3}w + \dots$$

$$w = \pi/2 - z$$

$$f(z) = \tan z = \underbrace{-\left(z - \frac{\pi}{2}\right)^{-1}}_{\text{Principal part}} + \frac{1}{3} \left(z - \frac{\pi}{2}\right) + O\left(\left(z - \frac{\pi}{2}\right)^3\right)$$

$\tan z$  has a pole of order one at  $z = \frac{\pi}{2}$ .

One can compute higher order terms through more effort; involves something known as Bernoulli coefficients.

Whatever it is...it converges for:  $\{z \in \mathbb{C}: 0 < |z - \frac{\pi}{2}| < \pi\}$

either by considering the convergence of the geometric series (not so clear how to do this) or more simply the distance to the nearest singularity.

Example: Laurent series for  $f(z) = \tan^{-1} z$  about  $z=0$ . (inverse tangent, principal branch).

$$\frac{d}{dz} \tan^{-1} z = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad \text{for } |z| < 1$$

Integrate term by term:

$$\tan^{-1} z = q_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

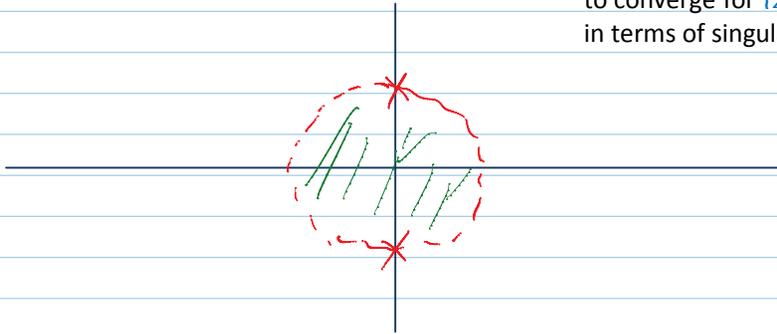
Principal branch:  $0 = \tan^{-1} 0 \Rightarrow 0 = q_0$

principal branch:  $U = \tan^{-1} U \rightarrow U = \gamma_0$

$$\tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

on the principal branch.

This Taylor series can be seen, by the root test to converge for  $z \in \mathbb{C}: |z| < 1$ . How see this in terms of singularities?



Example: Fourier series is a special case of a Laurent series!

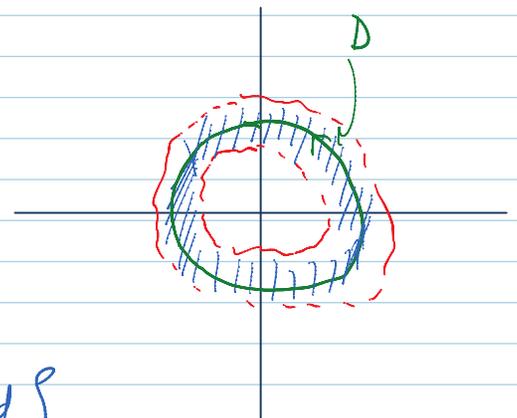
Suppose that we have a function  $f(z)$  which is analytic over some annulus  $D$  containing the circle  $C(0,1)$ . Then we can use Laurent series about  $z_0 = 0$  to write:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{in } D$$

where:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - 0)^{n+1}} d\zeta$$

$\zeta_0$



Let's focus on the behavior of  $f(z)$  on the circle  $C(0,1)$ . Then the above expression is entirely related to the behavior of the function  $g(\theta) = f(e^{i\theta})$ , which we can think of as a complex function on the real axis which is periodic with period  $2\pi$ . Let's rewrite the above Laurent series in terms of this periodic function:



$$g(\theta) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n (e^{i\theta})^n = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(\theta')}{(e^{i\theta'})^{n+1}} i e^{i\theta'} d\theta'$$

$= f(e^{i\theta'}) = f(\zeta)$

$$\zeta = e^{i\theta'}, \quad 0 \leq \theta' < 2\pi$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} g(\theta') e^{-in\theta'} d\theta'$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta') e^{-in\theta'} d\theta'$$

which is the Euler-Fourier formula for Fourier coefficients.

What this has shown is that if one is given a complex or real periodic function with period  $2\pi$ , and if one can find an **analytic extension** of that given function over an open annulus containing the circle of radius one under the mapping  $z = e^{i\theta}$ , then the Fourier series can be derived from the Laurent series results. What kinds of functions have such an "analytic extension"? **Real analytic** functions, which mean that real-valued Taylor series exist about each point, and actually they converge over some finite radius when extended to the complex plane. There are plenty of examples of real-valued functions that are infinitely smooth but not real-analytic. Classical example:

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-1/x^2} & \text{for } x > 0 \end{cases}$$



$$f(x) \in C^\infty$$

Not real analytic at  $x=0$ !