CHAPTER 1

Logic and Proofs

Although mathematics is both a science and an art, special characteristics distinguish mathematics from the humanities and from other sciences. Particularly important is the kind of reasoning that typifies mathematics. The natural or social scientist generally makes observations of particular cases or phenomena and seeks a general theory that describes or explains the observations. This approach is called **inductive reasoning,** and it is tested by making further observations. If the results are incompatible with theoretical expectations, the scientist usually must reject or modify the theory.

Mathematicians, too, frequently use inductive reasoning as they attempt to describe patterns and relationships among quantities and structures. The characteristic thinking of the mathematician, however, is **deductive reasoning,** in which one uses logic to draw conclusions based on statements accepted as true. The conclusions of a mathematician are proved to be true, *provided that the assumptions are true.* If the results of a mathematical theory are deemed incompatible with some portion of reality, the fault lies not in the theory but with the assumptions about reality that make the theory inapplicable to that portion of reality. Indeed, the mathematician is not restricted to the study of observable phenomena, even though one can trace the development of mathematics back to the need to describe spatial relations (geometry) and motion (calculus) or to solve financial problems (algebra). Using logic, the mathematician can draw conclusions about any mathematical structure imaginable.

The goal of this chapter is to provide a working knowledge of the basics of logic and the idea of proof, which are fundamental to deductive reasoning. This knowledge is important in many areas other than mathematics. For example, the thought processes used to construct an algorithm for a computer program are much like those used to develop the proof of a theorem.

1.1 Propositions and Connectives

Natural languages such as English allow for many types of sentences. Some sentences are interrogatory (Where is my sweater?), others exclamatory (Oh, no!), and others have a definite sense of truth to them (Abe Lincoln was the first U.S. president). A **proposition** is a sentence that is either true or false. Thus a proposition has exactly one truth value: true, which we denote by T, or false, which we denote by F.

Some examples of propositions are:

(a) $\sqrt{2}$ is irrational.
(b) $1 + 1 = 5$.
(c) The elephant will become extinct on the planet Earth before the rhinoceros will.
(d) Julius Caesar had two eggs for breakfast on his tenth birthday.

We are not concerned here with the difficulty of establishing the actual truth value of a proposition. We readily see that proposition (a) has the value T while (b) has the value F. It may take many years to determine whether proposition (c) is true or false, but its truth value will certainly be established if either animal ever becomes extinct. If both species (and Earth) somehow survive forever, the statement is false. There may be no way ever to determine what value proposition (d) has. Nevertheless, each of the above is either true or false, hence is a proposition.

Here are some sentences that are not propositions:

(e) What did you say?
(f) $x^2 = 36$.
(g) This sentence is false.

Sentence (e) is an interrogative statement that has no truth value. Sentence (f) could be true or false depending on what value x is assigned. We shall study sentences of this type in section 1.3.

Statement (g) is an example of a sentence that is neither true nor false, and it is referred to as a **paradox.** If (g) is true, then by its meaning (g) must be false. On the other hand, if (g) is false, then what it purports is false, so (g) must be true. Thus, (g) can have neither T nor F for truth value. The study of paradoxes such as this has played a key role in the development of modern mathematical logic. A famous example of a paradox formulated by the English logician Bertrand Russell is discussed in section 2.1.

Propositions (a)–(d) are **simple** or **atomic** in the sense that they do not have any other propositions as components. **Compound** propositions can be formed by using logical connectives with simple propositions.

> DEFINITIONS Given propositions P and Q,
>
> The **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when *both* P and Q are true.
>
> The **disjunction** of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when *at least one* of P or Q is true.
>
> The **negation** of P, denoted $\sim P$, is the proposition "not P." $\sim P$ is true exactly when P is false.

If P is "$1 \neq 3$" and Q is "7 is odd," then

$P \wedge Q$ is "$1 \neq 3$ and 7 is odd."
$P \vee Q$ is "$1 \neq 3$ or 7 is odd."
$\sim Q$ is "It is not the case that 7 is odd."

Since in this example both P and Q are true, $P \wedge Q$ and $P \vee Q$ are true, while $\sim Q$ is false.

All of the following are true propositions:

"It is not the case that $\sqrt{10} > 4$."
"$\sqrt{2} < \sqrt{3}$ or chickens have lips."
"Venus is smaller than Earth or $1 + 4 = 5$."
"$6 < 7$ and $7 < 8$."

All of the following are false:

"Mozart was born in Salzburg and π is rational."
"It is not the case that 10 is divisible by 2."
"$2^4 = 16$ and a quart is larger than a liter."

Other connectives commonly used in English are *but, while,* and *although,* each of which would normally be translated symbolically with the conjunction connective. A variant of the connective *or* is discussed in the exercises.

Example. Let M be "Milk contains calcium" and I be "Italy is a continent." Since M has the value T and I has the value F,

"Italy is a continent and milk contains calcium," symbolized $I \wedge M$, is false;
"Italy is a continent or milk contains calcium," $I \vee M$, is true;
"It is not the case that Italy is a continent," $\sim I$, is true.

An important distinction must be made between a proposition and the form of a proposition. In the previous example, "Italy is a continent and milk contains calcium" is a proposition with a single truth value (F), while the propositional form $P \wedge Q$, which may be used to represent the sentence symbolically, has no truth value. The form $P \wedge Q$ is an expression that obtains a value T or F after specific propositions are designated for P and Q (when for instance, we let P be "Italy is a continent" and Q be "Milk contains calcium"), or when the symbols P and Q are given truth values.

By the form of a compound proposition, we mean how the proposition is put together using logical connectives. For components P and Q, $P \wedge Q$ and $P \vee Q$ are two different propositional forms. Informally, a **propositional form** is an expression involving finitely many logical symbols (such as $\wedge$ and $\sim$) and letters. Expressions that are single letters or are formed correctly from the definitions of connectives are called **well-formed formulas.** For example, $(P \wedge (Q \vee \sim Q))$ is well-formed, whereas $(P \vee Q\sim)$, $(\sim P \sim Q)$, and $\vee Q$ are not. A more precise definition and study of well-formed formulas may be found in Elliot Mendelson's *An Introduction to Mathematical Logic* (Chapman & Hall/CRC, 1997).

The truth values of a compound propositional form are readily obtained by exhibiting all possible combinations of the truth values for its components in a truth table. Since the connectives $\wedge$ and $\vee$ involve two components, their truth tables must list the four possible combinations of the truth values of those components. The truth tables for $P \wedge Q$ and $P \vee Q$ are

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Since the value of $\sim P$ depends only on the two possible values for P, its truth table is

P	$\sim P$
T	F
F	T

Frequently you will encounter compound propositions with more than two simple components. The propositional form $(P \wedge Q) \vee \sim R$ has three simple components P, Q, and R; it follows that there are $2^3 = 8$ possible combinations of truth values. The two main components are $P \wedge Q$ and $\sim R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee \sim R$
T	T	T	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	F	F	T	T
F	F	F	F	T	T

The propositional form $(\sim Q \vee P) \wedge (R \vee S)$ has 16 possible combinations of values for P, Q, R, S. Two main components are $\sim Q \vee P$ and $R \vee S$. Its truth table is shown here:

P	Q	R	S	$\sim Q$	$\sim Q \vee P$	$R \vee S$	$(\sim Q \vee P) \wedge (R \vee S)$
T	T	T	T	F	T	T	T
F	T	T	T	F	F	T	F
T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	T	T	T
F	T	F	T	F	F	T	F
T	F	F	T	T	T	T	T
F	F	F	T	T	T	T	T
T	T	T	F	F	T	T	T
F	T	T	F	F	F	T	F
T	F	T	F	T	T	T	T
F	F	T	F	T	T	T	T
T	T	F	F	F	T	F	F
F	T	F	F	F	F	F	F
T	F	F	F	T	T	F	F
F	F	F	F	T	T	F	F

Two propositions P and Q are **equivalent** if and only if they have the same truth value. The propositions "$1 + 1 = 2$" and "$6 < 10$" are equivalent (even though they have nothing to do with each other) because both are true. The ability to write equivalent statements from a given statement is an important skill in writing proofs.

Of course, in a proof we expect some logical connection between such statements. This connection may be based on the form of the propositions.

> **DEFINITION** Two propositional forms are **equivalent** if and only if they have the same truth tables.

For example, the propositional forms $P \vee (Q \wedge P)$ and P are equivalent. To show this, we examine their truth tables.

P	Q	$Q \wedge P$	$P \vee (Q \wedge P)$
T	T	T	T
F	T	F	F
T	F	F	T
F	F	F	F

Since the P column and the $P \vee (Q \wedge P)$ column are identical, the propositional forms are equivalent. This means that, whatever propositions we choose to use for P and for Q, the results will be equivalent. If we let P be "91 is prime" and Q be "$1 + 1 = 2$," then "91 is prime" is equivalent to the proposition "91 is prime, or $1 + 1 = 2$ and 91 is prime." With these propositions for P and Q, Q is true and both P and $P \vee (Q \wedge P)$ are false. Thus, we have an instance of the second line of the truth table.

Notice that "Two propositions are equivalent" has a different meaning from "Two propositional forms are equivalent." We don't look at truth tables to decide the equivalence of propositions, because a proposition has only one truth value. This makes the question of equivalence of propositions rather easy: all true propositions are equivalent to each other and all false propositions are equivalent to each other. On the other hand, propositional forms are neither true nor false; generally they have the value true for some assignments of truth values to their components and the value false for other assignments. Thus to decide equivalence of propositional forms, we must compare truth tables. Another example of equivalent propositional forms is P and $\sim(\sim P)$. The truth tables for these two propositional forms are shown:

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

> **DEFINITION** A **denial** of a proposition S is any proposition equivalent to $\sim S$.

By definition, the negation $\sim P$ is a denial of the proposition P, but a denial need not be the negation. A proposition has only one negation but many different denials. The ability to rewrite the negation of a proposition into a useful denial will be very important for writing indirect proofs (see section 1.5).

Example. The proposition P: "π is rational" has negation $\sim P$: "It is not the case that π is rational." Some useful denials are

"π is irrational."
"π is not the quotient of two integers."
"The decimal expansion of π is not repeating or terminating."

Note that since P is false, all denials of P are true.

Example. The proposition "The water is cold and the soap is not here" has two components, C: "The water is cold" and H: "The soap is here." The negation, $\sim(C \wedge \sim H)$, is "It is not the case that the water is cold and the soap is not here." Some other denials are

"Either the water is not cold or the soap is here."
"It is not the case that the water is cold and the soap is not here and the water is cold."

It may be verified by truth tables that the propositional forms $(\sim C) \vee H$ and $\sim[(C \wedge \sim H) \wedge C]$ are equivalent to $\sim(C \wedge \sim H)$.

Note that the negation in the last example is ambiguous when written in English. Does the "It is not the case" refer to the entire sentence or just to the component "The water is cold"? Ambiguities such as this are allowable in conversational English but can cause trouble in mathematics. To avoid ambiguities, we introduce delimiters such as parentheses ( ), square brackets [ ], and braces { }. The negation above may be written symbolically as $\sim(C \wedge \sim H)$.

To avoid writing large numbers of parentheses, we use the rule that, first, $\sim$ applies to the smallest proposition following it, then $\wedge$ connects the smallest propositions surrounding it, and, finally, $\vee$ connects the smallest propositions surrounding it. Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$. The negation of the disjunction $P \vee Q$ must be written with parentheses $\sim(P \vee Q)$. The propositional form $P \wedge \sim Q \vee R$ abbreviates $[P \wedge (\sim Q)] \vee R$. As further examples,

$P \vee Q \wedge R$ abbreviates $P \vee [Q \wedge R]$.
$P \wedge \sim Q \vee \sim R$ abbreviates $[P \wedge (\sim Q)] \vee (\sim R)$.
$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$.
$\sim P \wedge \sim R \vee \sim P \wedge R$ abbreviates $[(\sim P) \wedge (\sim R)] \vee [(\sim P) \wedge R]$.

When the same connective is used several times in succession, parentheses may also be omitted. We reinsert parentheses from the left, so that $P \vee Q \vee R$ is really $(P \vee Q) \vee R$. For example, $R \wedge P \wedge \sim P \wedge Q$ abbreviates $[(R \wedge P) \wedge (\sim P)] \wedge Q$, whereas $R \vee P \wedge \sim P \vee Q$, which does not use the same connective consecutively, abbreviates $(R \vee [P \wedge (\sim P)]) \vee Q$. Leaving out parentheses is not required; some propositional forms are easier to read with a few well-chosen "unnecessary" parentheses.

Some compound propositional forms always yield the value true just because of the nature of their form. **Tautologies** are propositional forms that are true for every assignment of truth values to their components. Thus a tautology will have the value true regardless of what proposition(s) we select for the components. For example, the Law of Excluded Middle, $P \vee \sim P$, is a tautology. Its truth table is

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

We know that "the ball is red or the ball is not red" is true because it has the form of the Law of Excluded Middle.

Example. Show that $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology. We see that the truth table for the propositional form is

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \vee Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
F	T	T	T	F	F	T
T	F	T	F	T	F	T
F	F	F	T	T	T	T

Thus $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

A **contradiction** is the negation of a tautology. Thus $\sim(P \vee \sim P)$ is a contradiction. The negation of a contradiction is, of course, a tautology.

Conjunction, disjunction, and negation are very important in mathematics. Two other important connectives, the conditional and biconditional, will be studied in the next section. Other connectives having two components are not as useful in mathematics, but some are extremely important in digital computer circuit design.

Exercises 1.1

1. Which of the following are propositions?
   (a) Where are my car keys?
   (b) Christopher Columbus wore red boots at least once.
   ★ (c) The national debt of Poland in 1938 was $2,473,596.38.
   (d) $x^2 \geq 20$
   ★ (e) Between January 1, 2205 and January 1, 2215, the population of the world will double.
   (f) There are no zeros in the decimal expansion of π.
   ★ (g) She works in New York City.
   (h) Keep your elbows off the table!
   (i) There are more than 3 false statements in this book and this statement is one of them.
   (j) There are more than 3 false statements in this book and this statement is not one of them.

2. Make truth tables for each of the following propositional forms.
 ★ (a) $P \wedge \sim P$ (b) $P \vee \sim P$
 ★ (c) $P \wedge (Q \vee R)$ (d) $(P \wedge Q) \vee (P \wedge R)$
 ★ (e) $P \wedge \sim Q$ (f) $P \wedge (Q \vee \sim Q)$
 ★ (g) $(P \wedge Q) \vee \sim Q$ (h) $\sim (P \wedge Q)$
 (i) $(P \vee \sim Q) \wedge R$ (j) $\sim P \wedge \sim Q$
 (k) $P \wedge P$ (l) $(P \wedge Q) \vee (R \wedge \sim S)$

3. Which of the following pairs of propositional forms are equivalent?
 ★ (a) $P \wedge P, P$ (b) $P \vee P, P$
 ★ (c) $P \wedge Q, Q \wedge P$ (d) $P \vee Q, Q \vee \sim P$
 ★ (e) $(P \wedge Q) \wedge R, P \wedge (Q \wedge R)$ (f) $\sim (P \wedge Q), \sim P \wedge \sim Q$
 ★ (g) $\sim P \wedge \sim Q, \sim (P \wedge \sim Q)$ (h) $(P \vee Q) \vee R, P \vee (Q \vee R)$
 ★ (i) $(P \wedge Q) \vee R, P \wedge (Q \vee R)$ (j) $\sim (P \vee Q), (\sim P) \wedge (\sim Q)$
 ★ (k) $\sim (P \wedge Q), (\sim P) \vee (\sim Q)$ (l) $(P \wedge Q) \vee R, P \vee (Q \wedge R)$
 ★ (m) $P \wedge (Q \vee R), (P \wedge Q) \vee (P \wedge R)$ (n) $(\sim P) \vee (\sim Q), \sim (P \vee \sim Q)$

4. If P, Q, and R are true while S and T are false, which of the following are true?
 ★ (a) $Q \wedge (R \wedge S)$ (b) $Q \vee (R \wedge S)$
 ★ (c) $(P \vee Q) \wedge (R \vee S)$ (d) $((\sim P) \vee (\sim Q)) \vee ((\sim R) \vee (\sim S))$
 ★ (e) $(\sim P) \vee (Q \wedge \sim Q)$ (f) $(\sim P) \vee (\sim Q)$
 ★ (g) $((\sim Q) \vee S) \wedge (Q \vee S)$ (h) $(S \wedge R) \vee (S \wedge T)$
 ★ (i) $(P \vee S) \wedge (P \vee T)$ (j) $((\sim T) \wedge P) \vee (T \wedge P)$
 ★ (k) $(\sim P) \wedge (Q \vee \sim Q)$ (l) $(\sim R) \wedge (\sim S)$

5. Give a useful denial of each statement.
 ★ (a) x is a positive integer. (Assume that x is some fixed integer.)
 (b) We will win the first game or the second one.
 ★ (c) $5 \geq 3$
 (d) 641,371 is a composite integer.
 ★ (e) Roses are red and violets are blue.
 (f) $x < y$ or $m^2 < 1$ (Assume that x, y, and m are fixed real numbers.)
 ★ (g) T is not green or T is yellow.
 (h) Sue will choose yogurt but will not choose ice cream.
 (i) n is even and n is not a multiple of 5. (Assume that n is a fixed integer.)

6. P, Q, and R are propositional forms, and P is equivalent to Q, and Q is equivalent to R. Prove that
 ★ (a) Q is equivalent to P. (b) P is equivalent to R.
 (c) $\sim Q$ is equivalent to $\sim P$. (d) $P \wedge Q$ is equivalent to $Q \wedge R$.
 (e) $P \vee Q$ is equivalent to $Q \vee R$.

7. Use A: "Horses have four legs," B: "17 is prime," and C: "Three quarters equal one dollar" to write the propositional form of each of the following. Decide which are true.
 (a) Either horses have four legs or 17 is not prime.
 (b) Neither *do* three quarters equal a dollar nor *do* horses have four legs.
 ★ (c) 17 is prime and three quarters do not equal one dollar.
 (d) Horses have four legs but three quarters do not equal one dollar.

8. Let P be the sentence "Q is true" and Q be the sentence "P is false." Is P a proposition? Explain.

1.2

9. The word *or* is used in two different ways in English. We have presented the truth table for $\vee$, the **inclusive or,** whose meaning is "one or the other or both." The **exclusive or,** meaning "one or the other but not both" and denoted $\textcircled{\vee}$, has its uses in English, as in "She will marry Heckle or she will marry Jeckle." The "inclusive or" is much more useful in mathematics and is the accepted meaning unless there is a statement to the contrary.
   ★ **(a)** Make a truth table for the "exclusive or" connective, $\textcircled{\vee}$
   (b) Show that $A \textcircled{\vee} B$ is equivalent to $(A \vee B) \wedge \sim(A \wedge B)$.

10. Restore parentheses to these abbreviated propositional forms.
   (a) $\sim\sim P \vee \sim Q \wedge \sim S$
   (b) $Q \wedge \sim S \vee \sim(\sim P \wedge Q)$
   (c) $P \wedge \sim Q \vee \sim P \wedge \sim R \vee \sim P \wedge S$

11. Determine whether each of the following is a tautology, a contradiction, or neither. Prove your answers.
   (a) $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$
   (b) $\sim(P \wedge \sim P)$
   ★ **(c)** $(P \wedge Q) \vee ((\sim P) \vee (\sim Q))$
   (d) $(A \wedge B) \vee (A \wedge \sim B) \vee ((\sim A) \wedge B) \vee ((\sim A) \wedge (\sim B))$
   (e) $(Q \wedge \sim P) \wedge \sim(P \wedge R)$
   (f) $P \vee ((\sim Q) \wedge P) \wedge (R \vee Q)$

1.2 Conditionals and Biconditionals

The most important kind of proposition in mathematics is a sentence of the form "If P, then Q." Examples include "If a natural number is written in two ways as a product of primes, then the two factorizations are identical except for the order in which the prime factors are written"; "If two lines in a plane have the same slope, then they are parallel"; and "If f is differentiable at x_0 and $f(x_0)$ is a relative minimum for f, then $f'(x_0) = 0$."

DEFINITIONS Given propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ (read "P implies Q") is the proposition "If P, then Q." The proposition P is the **antecedent** and Q is the **consequent.**

The conditional sentence $P \Rightarrow Q$ is true whenever the antecedent is false or the consequent is true. Thus, $P \Rightarrow Q$ is defined to be equivalent to $(\sim P) \vee Q$.

The truth table for $P \Rightarrow Q$ is

P (antecedent)	Q (consequent)	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

This table gives $P \Rightarrow Q$ the value F only when P is true and Q is false, and thus it agrees with the meaning of "if . . . , then . . ." in promises. For example, the person who promises, "If Lincoln was the second U.S. president, I'll give you a dollar" would not be a liar for failing to give you a dollar. In fact, he could give you a dollar and still not be a liar. In both cases we say the statement is true because the antecedent is false.

One curious consequence of the truth table for $P \Rightarrow Q$ is that conditional sentences may be true even when there is no connection between the antecedent and the consequent. The reason for this is that the truth value of $P \Rightarrow Q$ depends *only* upon the truth value of components P and Q, not on their interpretation. For this reason all of the following are true:

$\sin 30° = \frac{1}{2} \Rightarrow 1 + 1 = 2.$
Mars has ten moons $\Rightarrow 1 + 1 = 2.$
Mars has ten moons $\Rightarrow$ Paul Revere made plastic spoons.

and both of the following are false:

$1 + 2 = 3 \Rightarrow 1 < 0.$
Ducks have webbed feet $\Rightarrow$ Canada lies south of the equator.

Our truth table definition of $\Rightarrow$ is not really unfamiliar; it captures the same meaning for "if . . . , then . . . " that you have always used in mathematics. We all agree that "If x is odd, then $x + 1$ is even" is a true statement about any integer x. It would be hopeless to protest that in the case where x is 6, then $x + 1$ is 7, which is not even. After all, the claim is only that *if* x is odd, *then* $x + 1$ is even.

We know that "If (1, 3) and (2, 5) are points on a line L, then the line L has slope 2" is true because $(5 - 3)/(2 - 1) = 2$. The truth values of the antecedent and consequent depend on what line L we are talking about, but in all cases the value of the conditional sentence is true:

In the case that the line L is $y = 2x + 1$, the antecedent and consequent are both true. This matches the first line of the truth table for $P \Rightarrow Q$, where $P \Rightarrow Q$ is true.

In some cases, such as the line $y = 2x + 4$, the antecedent is false and the consequent is true. This matches the second line of the truth table, where $P \Rightarrow Q$ is also true.

In other cases, such as $y = 3x + 1$, we find instances of the fourth line of the truth table, since both the antecedent and consequent are false. Again, $P \Rightarrow Q$ is true.

There is no example of the third line of the truth table for $P \Rightarrow Q$ for this sentence; that is why "If (1, 3) and (2, 5) are points on a line L, then the line L has slope 2" is true.

Note that in the truth table of $P \Rightarrow Q$ the only line in which both P and $P \Rightarrow Q$ are true is the first line, in which case Q is also true. In other words, if we know that

The

both P and $P \Rightarrow Q$ are true, then we know that Q must be true. This deduction, called **modus ponens,** is one of several we will discuss in section 1.4.

Two propositions closely related to the conditional sentence $P \Rightarrow Q$ are its converse and its contrapositive.

> **DEFINITIONS** For propositions P and Q, the **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$, and the **contrapositive** of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$.

For the conditional sentence "If a function f is differentiable at x_0, then f is continuous at x_0," its converse is "If f is continuous at x_0, then f is differentiable at x_0," whereas the contrapositive is "If f is not continuous at x_0, then f is not differentiable at x_0." Calculus students know that the converse is a false statement.

If P is the proposition "It is raining here" and Q is "It is cloudy overhead," then $P \Rightarrow Q$ is true. Its contrapositive is "If it is not cloudy overhead, then it is not raining here," which is also true. However, the converse "If it is cloudy overhead, then it is raining here" is not a true proposition. We describe the relationships between a conditional sentence and its contrapositive and converse in the following theorem.

Theorem 1.1

(a) The propositional form $P \Rightarrow Q$ is equivalent to its contrapositive $(\sim Q) \Rightarrow (\sim P)$.

(b) The propositional form $P \Rightarrow Q$ is not equivalent to its converse, $Q \Rightarrow P$.

Proof. A proof requires examining the truth tables:

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$(\sim Q) \Rightarrow (\sim P)$	$Q \Rightarrow P$
T	T	T	F	F	T	T
F	T	T	F	T	T	F
T	F	F	T	F	F	T
F	F	T	T	T	T	T

Comparing the third and sixth columns, we conclude that $P \Rightarrow Q$ is equivalent to $(\sim Q) \Rightarrow (\sim P)$. Comparing the third and seventh columns, we see they differ in the second and third lines. Thus, $P \Rightarrow Q$ and $Q \Rightarrow P$ are not equivalent. ■

The equivalence of a conditional sentence and its contrapositive will be the basis for an important proof technique developed in section 1.5 (proof by contraposition). However, no proof technique will be developed using the converse because the truth of a conditional sentence cannot be inferred from the truth of its converse. The converse cannot be used to prove a conditional sentence.

Closely related to the conditional sentence is the **biconditional sentence** $P \Leftrightarrow Q$. The double arrow $\Leftrightarrow$ reminds one of both $\Leftarrow$ and $\Rightarrow$, and this is no accident, for $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

> **DEFINITION** For propositions P and Q, the **biconditional sentence** $P \Leftrightarrow Q$ is the proposition "P if and only if Q." The sentence $P \Leftrightarrow Q$ is true exactly when P and Q have the *same truth values.*

The truth table for $P \Leftrightarrow Q$ is

P	Q	$P \Leftrightarrow Q$
T	T	T
F	T	F
T	F	F
F	F	T

As a form of shorthand, the words "if and only if" are frequently abbreviated to "iff" in mathematics. The statements

"A rectangle is a square iff the rectangle's diagonals are perpendicular"

and

"$1 + 7 = 6$ iff $\sqrt{2} + \sqrt{3} = \sqrt{5}$"

are both true biconditional sentences, while

"Lake Erie is in Peru iff π is an irrational number"

is a false biconditional sentence.

Any properly stated definition is an example of a biconditional sentence. Although a definition might not include the iff wording, biconditionality does provide a good test of whether a statement could serve as a definition or just a description. The sentence "A diameter of a circle is a chord of maximum length" is a correct definition of diameter because "A chord is a diameter iff the chord has maximum length" is a true proposition. However, the sentences "A sundial is an instrument for measuring time" and "A square is a quadrilateral whose interior angles are right angles" can be recognized as descriptions rather than definitions.

Because the biconditional sentence $P \Leftrightarrow Q$ has the value T exactly when the values of P and Q are the same, we can use the biconditional connective to restate the meaning of equivalent propositional forms. That is,

> The propositional forms P and Q are equivalent precisely when $P \Leftrightarrow Q$ is a tautology.

One key to success in mathematics is the ability to replace a statement by a more useful or enlightening one. This is precisely what you do to "solve" the equation $x^2 - 7x = -12$ by the method of factoring:

$$\begin{aligned} x^2 - 7x = -12 &\Leftrightarrow x^2 - 7x + 12 = 0 \\ &\Leftrightarrow (x-3)(x-4) = 0 \\ &\Leftrightarrow x - 3 = 0 \quad \text{or} \quad x - 4 = 0 \\ &\Leftrightarrow x = 3 \quad \text{or} \quad x = 4 \end{aligned}$$

Each statement is simply an equivalent of its predecessor but is more illuminating as to the solution. The ability to write equivalents is crucial in writing proofs. The next theorem contains seven important pairs of equivalent propositional forms. They should be memorized.

Theorem 1.2 For propositions P, Q, and R,

(a) $P \Rightarrow Q$ is equivalent to $(\sim P) \vee Q$.
(b) $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
(c) $\sim(P \wedge Q)$ is equivalent to $(\sim P) \vee (\sim Q)$.
(d) $\sim(P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$.
(e) $\sim(P \Rightarrow Q)$ is equivalent to $P \wedge \sim Q$.
(f) $\sim(P \wedge Q)$ is equivalent to $P \Rightarrow \sim Q$.
(g) $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.
(h) $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$.

You will be asked to give a proof of this theorem in exercise 7. Before plunging into the truth table computations, you should think about the meaning behind each equivalence. For example, parts (c) and (d), later referred to as De Morgan's Laws, are quite similar in form. They distribute negation over conjunction (c) and over disjunction (d). In (c), $\sim(P \wedge Q)$ is true precisely when $P \wedge Q$ is false. This happens when one of P or Q is false, or, in other words, when one of $\sim P$ or $\sim Q$ is true. Thus, $\sim(P \wedge Q)$ is equivalent to $(\sim P) \vee (\sim Q)$. Another way to say this is, "If you do not have both P and Q, then either you do not have P or you do not have Q."

Your reasoning for part (e) should be something like this: If $\sim(P \Rightarrow Q)$ is true, then $P \Rightarrow Q$ is false, which forces P to be true and Q to be false. But this means that both P and $\sim Q$ are true, and so $P \wedge \sim Q$ is true. This reasoning can be reversed to show that if $P \wedge \sim Q$ is true, then $\sim(P \Rightarrow Q)$ is true. We conclude that $\sim(P \Rightarrow Q)$ is true precisely when $P \wedge \sim Q$ is true, and thus they are equivalent. For example, given any fixed triangle ABC, the statement "It is not the case that if triangle ABC has a right angle, then it is equilateral" is equivalent to "Triangle ABC has a right angle and is not equilateral."

Recognizing the structure of a sentence and translating the sentence into symbolic form using logical connectives is an aid in determining its truth or falsity. The translation of sentences into propositional symbols is sometimes very complicated because English is such a rich and powerful language, with many nuances, and because the ambiguities we tolerate in English would destroy structure and usefulness if we allowed them in mathematics.

Connectives in English that may be translated symbolically using the conditional or biconditional logical connectives present special problems. The word *unless* is variously used to mean a conditional or its converse or a biconditional. For example, consider the sentence "The Dolphins will not make the play-offs unless the Bears win all the rest of their games." In conversation an explanation can clarify the meaning. Lacking that explanation, here are three of the nonequivalent ways people translate the sentence, using the symbols D: "The Dolphins make the play-offs" and B: "The Bears win all the rest of their games." Dictionaries indicate that the conditional meaning of *unless* is preferred (the first translation), but the speaker may have meant any of the three:

$$(\sim B) \Rightarrow (\sim D)$$
$$(\sim D) \Rightarrow (\sim B)$$
$$(\sim B) \Leftrightarrow (\sim D)$$

Sometimes a sentence in English explicitly uses a conditional connective but the converse is understood, so that the meaning is biconditional. For example, "I will pay you if you fix my car" and "I will pay you only if you fix my car" both mean "I will pay you if, but only if, you fix my car." Contrast this with the situation in mathematics: "If $x = 2$, then x is a solution to $x^2 = 2x$" is not to be understood as a biconditional, because "x is a solution to $x^2 = 2x$" does not imply "$x = 2$."

Shown below are some phrases in English that are ordinarily translated by using the connectives $\Rightarrow$ and $\Leftrightarrow$, and an example of each.

Use $P \Rightarrow Q$ to translate:	Examples:
If P, then Q.	If $a > 5$, then $a > 3$.
P implies Q.	$a > 5$ implies $a > 3$.
P is sufficient for Q.	$a > 5$ is sufficient for $a > 3$.
P only if Q.	$a > 5$ only if $a > 3$.
Q, if P.	$a > 3$, if $a > 5$.
Q whenever P.	$a > 3$ whenever $a > 5$.
Q is necessary for P.	$a > 3$ is necessary for $a > 5$.
Q, when P.	$a > 3$, when $a > 5$.

Use $P \Leftrightarrow Q$ to translate:	Examples.
P if and only if Q.	$\lvert t \rvert = 2$ if and only if $t^2 = 4$.
P if, but only if, Q.	$\lvert t \rvert = 2$ if, but only if, $t^2 = 4$.
P is equivalent to Q.	$\lvert t \rvert = 2$ is equivalent to $t^2 = 4$.
P is necessary and sufficient for Q.	$\lvert t \rvert = 2$ is necessary and sufficient for $t^2 = 4$.

In the following examples of sentence translations, it is not essential to know the meaning of all the words because the logical connectives are what concern us.

Examples. Assume that S and G have been specified. The sentence

"S is compact is sufficient for S to be bounded"

is translated

S is compact $\Rightarrow$ S is bounded.

The sentence

"A necessary condition for a group G to be cyclic is that G is abelian"

is translated

G is cyclic $\Rightarrow$ G is abelian.

The sentence

"A set S is infinite if S has an uncountable subset"

is translated

S has an uncountable subset $\Rightarrow$ S is infinite.

If we let P denote the proposition "Roses are red" and Q denote the proposition "Violets are blue," we can translate the sentence "It is not the case that roses are red, nor that violets are blue" in at least two ways: $\sim(P \vee Q)$ or $(\sim P) \wedge (\sim Q)$. Fortunately, these are equivalent by Theorem 1.2(d). Note that the proposition "Violets are purple" requires a new symbol, say R, since it expresses a new idea that cannot be formed from the components P and Q.

The sentence "17 and 35 have no common divisors" shows that the meaning, and not just the form of the sentence, must be considered in translating; it cannot be broken up into the two propositions: "17 has no common divisors" and "35 has no common divisors." Compare this with the proposition "17 and 35 have digits totaling 8," which can be written as a conjunction.

Example. Suppose b is a real number. "If b is an integer, then b is either even or odd" may be translated into $P \Rightarrow (Q \vee R)$, where P is "b is an integer," Q is "b is even," and R is "b is odd."

Example. Suppose a, b, and p are integers. "If p is a prime number that divides ab, then p divides a or b" becomes $(P \wedge Q) \Rightarrow (R \vee S)$, when P is "p is prime," Q is "p divides ab," R is "p divides a," and S is "p divides b."

The convention governing use of parentheses, adopted at the end of section 1.1, can be extended to the connectives $\Rightarrow$ and $\Leftrightarrow$. The connectives $\sim$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are applied in the order listed. That is, $\sim$ applies to the smallest possible proposition, and so forth, and otherwise parentheses are added left to right. For example, $P \Rightarrow \sim Q \vee R \Leftrightarrow S$ is an abbreviation for $(P \Rightarrow [(\sim Q \vee R]) \Leftrightarrow S$, while $P \vee \sim Q \Leftrightarrow R \Rightarrow S$ abbreviates $[P \vee (\sim Q)] \Leftrightarrow (R \Rightarrow S)$, and $P \Rightarrow Q \Rightarrow R$ abbreviates $(P \Rightarrow Q) \Rightarrow R$.

Exercises 1.2

1. Identify the antecedent and the consequent for each of the following conditional sentences. Assume that a, b, and f represent some fixed sequence, integer, or function, respectively.
   - ★ **(a)** If squares have three sides, then triangles have four sides.
   - **(b)** If the moon is made of cheese, then 8 is an irrational number.
   - **(c)** b divides 3 only if b divides 9.

★ **(d)** The differentiability of f is sufficient for f to be continuous.
(e) A sequence a is bounded whenever a is convergent.
★ **(f)** A function f is bounded if f is integrable.
(g) $1 + 2 = 3$ is necessary for $1 + 1 = 2$.
(h) The fish bite only when the moon is full.
(i) A grade point average of 3.0 is sufficient to graduate with honors.

☆ **2.** Write the converse and contrapositive of each conditional sentence in exercise 1.

3. Identify the antecedent and consequent for each conditional sentence in the following statements from this book.
(a) Theorem 1.3(a), section 1.3 **(b)** Exercise 5(a), section 1.5
(c) Theorem 2.4, section 2.1 **(d)** Theorem 2.12, section 2.5
(e) Theorem 2.20, section 2.6 **(f)** Theorem 3.8, section 3.4
(g) Theorem 4.3, section 4.2 **(h)** Corollary 5.7(a), section 5.1

4. Which of the following conditional sentences are true?
★ **(a)** If triangles have three sides, then squares have four sides.
(b) If a hexagon has six sides, then the moon is made of cheese.
★ **(c)** If $7 + 6 = 14$, then $5 + 5 = 10$.
(d) If $5 < 2$, then $10 < 7$.
★ **(e)** If one interior angle of a right triangle is $92°$, then the other interior angle is $88°$.
(f) If Euclid's birthday was April 2, then rectangles have four sides.
(g) 5 is prime if $\sqrt{2}$ is not irrational.
(h) $1 + 1 = 2$ is sufficient for $3 > 6$.
(i) Horses have four legs whenever September 15 falls on a Saturday.

5. Which of the following are true?
★ **(a)** Triangles have three sides iff squares have four sides.
(b) $7 + 5 = 12$ iff $1 + 1 = 2$.
★ **(c)** b is even iff $b + 1$ is odd. (Assume that b is some fixed integer.)
(d) $5 + 6 = 6 + 5$ iff $7 + 1 = 10$.
(e) A parallelogram has three sides iff 27 is prime.
(f) The Eiffel Tower is in Paris if and only if the chemical symbol for helium is H.
(g) $\sqrt{10} + \sqrt{13} < \sqrt{11} + \sqrt{12}$ iff $\sqrt{13} - \sqrt{12} < \sqrt{11} - \sqrt{10}$.

6. Make truth tables for these propositional forms.
(a) $P \Rightarrow (Q \wedge P)$.
★ **(b)** $((\sim P) \Rightarrow Q) \vee (Q \Leftrightarrow P)$.
★ **(c)** $(\sim Q) \Rightarrow (Q \Leftrightarrow P)$.
(d) $(P \vee Q) \Rightarrow (P \wedge Q)$.
(e) $(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$.
(f) $[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$.

☆ **7.** Prove Theorem 1.2 by constructing truth tables for each equivalence.

8. Rewrite each of the following sentences using logical connectives. Assume that each symbol f, n, x, S, $\mathbf{B}$ represents some fixed object.

★ **(a)** If f has a relative minimum at x_0 and if f is differentiable at x_0, then $f'(x_0) = 0$.
(b) If n is prime, then $n = 2$ or n is odd.
(c) A number x is real and not rational whenever x is irrational.
★ **(d)** If $x = 1$ or $x = -1$, then $|x| = 1$.
★ **(e)** f has a critical point at x_0 iff $f'(x_0) = 0$ or $f'(x_0)$ does not exist.
(f) S is compact iff S is closed and bounded.
(g) $\mathbf{B}$ is invertible is a necessary and sufficient condition for det $\mathbf{B} \neq 0$.
(h) $6 \geq n - 3$ only if $n > 4$ or $n > 10$.
(i) x is Cauchy implies x is convergent.

9. Show that the following pairs of statements are equivalent.
(a) $(P \vee Q) \Rightarrow R$ and $\sim R \Rightarrow (\sim P \wedge \sim Q)$
★ **(b)** $(P \wedge Q) \Rightarrow R$ and $(P \wedge \sim R) \Rightarrow \sim Q$
(c) $P \Rightarrow (Q \wedge R)$ and $(\sim Q \vee \sim R) \Rightarrow \sim P$
(d) $P \Rightarrow (Q \vee R)$ and $(P \wedge \sim R) \Rightarrow Q$
(e) $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \sim Q) \vee R$
(f) $P \Leftrightarrow Q$ and $(\sim P \vee Q) \wedge (\sim Q \vee P)$

10. Give, if possible, an example of a true conditional sentence for which
★ **(a)** the converse is true.
(b) the converse is false.
★ **(c)** the contrapositive is false.
(d) the contrapositive is true.

11. Give, if possible, an example of a false conditional sentence for which
(a) the converse is true.
(b) the converse is false.
(c) the contrapositive is true.
(d) the contrapositive is false.

12. Give the converse and contrapositive of each sentence of exercise 8(a), (b), (c), and (d). Tell whether each converse and contrapositive is true or false.

13. The **inverse,** or **opposite,** of the conditional sentence $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.
(a) Show that $P \Rightarrow Q$ and its inverse are not equivalent forms.
(b) For what values of the propositions P and Q are $P \Rightarrow Q$ and its inverse both true?
(c) Which is equivalent to the converse of a conditional sentence, the contrapositive of its inverse, or the inverse of its contrapositive?

14. Determine whether each of the following is a tautology, a contradiction, or neither.
★ **(a)** $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.
(b) $P \Leftrightarrow P \wedge (P \vee Q)$.
(c) $P \Rightarrow Q \Leftrightarrow P \wedge \sim Q$.
★ **(d)** $P \Rightarrow [P \Rightarrow (P \Rightarrow Q)]$.
(e) $P \wedge (Q \vee \sim Q) \Leftrightarrow P$.
(f) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.
(g) $(P \Leftrightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \vee (\sim P \wedge Q)$.
(h) $[P \Rightarrow (Q \vee R)] \Rightarrow [(Q \Rightarrow R) \vee (R \Rightarrow P)]$.
(i) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.
(j) $(P \vee Q) \Rightarrow Q \Rightarrow P$.
(k) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.
(l) $[P \Rightarrow (Q \wedge R)] \Rightarrow R \Rightarrow (P \Rightarrow Q)$.