COT 4501

Solutions for Exam Review Problems (Chapter 8 and 9)

8.12. Averaging the first-order accurate forward and backward difference formulas, we obtain

$$\frac{1}{2}\left(\frac{f(x+h)-f(x)}{h}+\frac{f(x)-f(x-h)}{h}\right)=\frac{f(x+h)-f(x-h)}{2h},$$

which is the centered difference formula we already know to be second-order accurate from the analysis in Section 8.6.1.

8.13. In the Taylor series expansion

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2}h^2+\frac{f'''(x)}{6}h^3+\cdots,$$

we solve for $f'(x)$ to obtain

$$f'(x)=\frac{f(x+h)-f(x)}{h}-\frac{f''(x)}{2}h-\frac{f'''(x)}{6}h^2+\cdots.$$

Similarly, in the Taylor series expansion

$$\begin{aligned} f(x+2h) &= f(x)+f'(x)(2h)+\frac{f''(x)}{2}(2h)^2+\frac{f'''(x)}{6}(2h)^3+\cdots \\ &= f(x)+2f'(x)h+2f''(x)h^2+\frac{4f'''(x)}{3}h^3+\cdots, \end{aligned}$$

we solve for $f'(x)$ to obtain

$$f'(x)=\frac{f(x+2h)-f(x)}{2h}-f''(x)h-\frac{2f'''(x)}{3}h^2+\cdots.$$

If we now subtract the second of these series for $f'(x)$ from twice the first, we obtain

$$f'(x)=\frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}+\frac{f'''(x)}{3}h^2+\cdots,$$

so that

$$f'(x)\approx\frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$$

is a second-order accurate, one-sided difference approximation to $f'(x)$.

8.14.

$$F(0) = F(h) + \frac{F(h) - F(h/2)}{2^{-1} - 1} = -0.8333 + \frac{-0.8333 + 0.9091}{0.5 - 1} = -0.9849.$$

8.15. (a) The power series expansion for the sine function is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

so

$$\begin{aligned} p_n &= n\left(\frac{\pi}{n} - \frac{\pi^3}{n^3 3!} + \frac{\pi^5}{n^5 5!} - \frac{\pi^7}{n^7 7!} + \cdots\right) = \pi - \frac{\pi^3}{n^2 3!} + \frac{\pi^5}{n^4 5!} - \frac{\pi^7}{n^6 7!} + \cdots \\ &= \pi - \frac{\pi^3}{3!}h^2 + \frac{\pi^5}{5!}h^4 - \frac{\pi^7}{7!}h^6 + \cdots, \end{aligned}$$

where $h = 1/n$, and hence we have $a_0 = \pi$. Similarly, the power series expansion for the tangent function is

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

so

$$\begin{aligned} q_n &= n\left(\frac{\pi}{n} + \frac{\pi^3}{n^3 3} + \frac{2\pi^5}{n^5 15} + \frac{17\pi^7}{n^7 315} + \cdots\right) = \pi + \frac{\pi^3}{n^2 3} + \frac{2\pi^5}{n^4 15} + \frac{17\pi^7}{n^6 315} + \cdots \\ &= \pi + \frac{\pi^3}{3}h^2 + \frac{2\pi^5}{15}n^4 + \frac{17\pi^7}{315}n^6 + \cdots, \end{aligned}$$

and hence we have $b_0 = \pi$. (b) Using Richardson extrapolation with $p = 2$ and $q = 2$, we have

$$a_0 = F(1/6) + \frac{F(1/6) - F(1/12)}{2^{-2} - 1} = 3.0000 + \frac{3.0000 - 3.1058}{0.25 - 1} = 3.1411.$$

$$b_0 = F(1/6) + \frac{F(1/6) - F(1/12)}{2^{-2} - 1} = 3.4641 + \frac{3.4641 - 3.2154}{0.25 - 1} = 3.1325.$$

9.4. (a) Yes, because the eigenvalue, -5, is negative, and hence the solutions decay exponentially. (b) No, since $|1 + h\lambda| = |1 + 0.5\,(-5)| = 1.5 > 1$. ($c$) $y_1 = 1 + 0.5(-5) = -1.5$. (d) Yes, since backward Euler is unconditionally stable. (e) $y_1 = 1 + 0.5\,(-5y_1) \Rightarrow y_1 = 1/3.5 = 0.2857$.

9.1. (*a*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= t + u_1 + u_2. \end{aligned}$$

(*b*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ u_3' &= u_3 + tu_1. \end{aligned}$$

(*c*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ u_3' &= u_3 - 2u_2 + u_1 - t + 1. \end{aligned}$$

9.2. (*a*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_2(1 - u_1^2) - u_1. \end{aligned}$$

(*b*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ u_3' &= -u_1 u_3. \end{aligned}$$

(*c*)

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -GMu_1/(u_1^2 + u_3^2)^{3/2}, \\ u_3' &= u_4, \\ u_4' &= -GMu_3/(u_1^2 + u_3^2)^{3/2}. \end{aligned}$$

9.3. Yes, since this first-order, linear, homogeneous system with constant coefficients can be written

$$\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boldsymbol{Ay},$$

and the eigenvalues of $\boldsymbol{A}$, -1 and -2, are both negative.