NORMAL NUMBERS WITH RESPECT TO THE CANTOR SERIES EXPANSION

DISSERTATION

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Bill Mance, M.S.

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Dissertation Committee:

Dr. Vitaly Bergelson, Advisor

Dr. Gerald Edgar

Dr. Alexander Leibman

ABSTRACT

A sequence $Q = \{q_n\}_{n=1}^{\infty}$ is called a *basic sequence* if each q_n is an integer greater than or equal to 2. A basic sequence is *infinite in limit* if $q_n \to \infty$. The *Q-Cantor series expansion*, first studied by G. Cantor, is a generalization of the b-ary expansion where every real number in $[0, 1)$ is expressed in the form $\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$ with $E_n \in [0, q_n - 1] \cap \mathbb{Z}$ and $E_n \neq q_n - 1$ infinitely often. A real number x is *normal in base b* if every block of digits of length k occurs with frequency b^{-k} in its b-ary expansion; equivalently, the sequence $\{b^n x\}_{n=0}^{\infty}$ is *uniformly distributed mod* 1.

The notion of normality is extended to the Q-Cantor series expansion. We primarily consider three distinct notions of normality that are equivalent in the case of the b-ary expansion: *Q-normality*, *Q-ratio normality*, and *Q-distribution normality.* All Q-normal numbers are Q-ratio normal, but there is no inclusion between Q-normal numbers and Q-distribution normal numbers. Thus, the fundamental equivalence between notions of normality that holds for the b-ary expansion will no longer hold for the Q-Cantor series expansion, depending on the basic sequence Q.

We prove theorems that may be used to construct Q-normal and Q-distribution normal numbers for a restricted class of basic sequences Q. Using these theorems, we construct a number that is simultaneously Q-normal and Q-distribution normal for a certain Q. We also use the same theorems to provide an example of a basic

sequence Q and a number that is Q-normal, yet fails to be Q-distribution normal in a particularly strong manner. Many constructions of numbers that are Q-distribution normal, yet not Q-ratio normal are also provided.

In [24], P. Laffer asked for a construction of a Q-distribution normal number given an arbitrary Q. We provide a partial answer by constructing an uncountable family of Q-distribution normal numbers, provided that $Q = \{q_n\}_{n=1}^{\infty}$ satisfies the condition that it is infinite in limit. This set of Q-distribution normal numbers that we construct has the additional property that it is perfect and nowhere dense. Additionally, none of these numbers will be Q-ratio normal.

Also studied are questions of typicality for different notions of normality. We show that under certain conditions on the basic sequence Q, almost every real number is Q-normal. If Q is infinite in limit, then the set of Q-ratio normal numbers will be dense in $[0, 1)$, but may or may not have full measure. Almost every real number will be Q-distribution normal no matter our choice of Q. The set of Q-ratio normal and the set of Q-distribution normal numbers are small in the topological sense; they are both sets of the first category. We also study topological properties of other sets relating to digits of the Q-Cantor series expansion.

We define potentially stronger notions of normality: *strong Q-normality*, *strong Q-ratio normality*, and *strong Q-distribution normality* that are equivalent to normality in the case of the b-ary expansion. We show that the set of strongly Q-distribution normal numbers always has full measure, but the set of strongly Q-normal numbers will only under certain conditions. We study winning sets, in the sense of *Schmidt games* and show that the set of non-strongly Q-ratio normal numbers and the set

of non-strongly Q-distribution normal numbers are $1/2$-*winning sets* and thus have full Hausdorff dimension. We also examine the property of being a winning set as it applies to other sets associated with the Q-Cantor series expansion.

A number normal in base b is never rational. We study how well this notion transfers to the Q-Cantor series expansion. In particular, it will remain consistent for Q-distribution normal numbers, but fail in unusual ways for other notions of normality.

Dedicated to Abigail.

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VITA

April 6, 1981	Born - Royal Oak, MI
1999-2003	Undergraduate, Carnegie Mellon University
2003	B.S. in Mathematics, Carnegie Mellon University
2003-Present	Graduate Teaching Associate, The Ohio State University
2004	M.S. in Mathematics, The Ohio State University

PUBLICATIONS

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FIELDS OF STUDY

Major Field: Mathematics

Specialization: Normal Numbers and Uniform Distribution of Sequences

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CHAPTER 1

INTRODUCTION

1.1 E. Borel and Early Developments in the Theory of Normal Numbers

Suppose that $x = 0.d_1d_2d_3\ldots$ is the decimal expansion of some real number $x \in [0, 1)$. It is natural to ask, for example, if the digit 1 occurs infinitely often in the decimal expansion of $\sqrt{2} - 1 = 0.41421356237\ldots$. Since $\sqrt{2} - 1$ is irrational, we know that there are at least two digits among $0, 1, \ldots, 9$ that occur infinitely often.

An even harder question is, in what real numbers decimal expansion does the digit 1 occur with frequency 1/10? If the decimal expansion of a real number is "random", then even more should happen; in particular, the digit 9 followed by the digit 7 should occur with frequency 1/100. Intuitively, we say that a real number x is normal in base 10 if every block of k digits occurs with frequency 10^{-k}. We make the following definitions:

Definition 1.1.1. *Given an integer $b \geq 2$, the b-ary expansion of a real x in $[0, 1)$ is the (unique) expansion of the form*

$$x = \sum_{n=1}^{\infty} \frac{d_n}{b^n} = 0.d_1 d_2 d_3 \ldots \tag{1.1}$$

such that d_n is in $\{0, 1, \ldots, b-1\}$ for all n with $d_n \neq b-1$ infinitely often.

Definition 1.1.2. *Let b and k be positive integers. A block of length k in base b is an ordered k-tuple of integers in $\{0, 1, \ldots, b-1\}$. A block of length k is a block of length k in some base b. A block is a block of length k in base b for some integers k and b.*

Denote by $N_n^b(B, x)$ the number of times a block B occurs with its starting position no greater than n in the b-ary expansion of x.

Definition 1.1.3. *A real number x in $[0, 1)$ is normal in base b if for all k and blocks B in base b of length k, one has*

$$\lim_{n \to \infty} \frac{N_n^b(B, x)}{n} = b^{-k}. \tag{1.2}$$

A number x is simply normal in base b if (1.2) holds for $k = 1$.

We will use the notation $\mathrm{P}(A)$ to stand for the probability of an event A. If X is a random variable, we will denote its expected value and variance by $\mathrm{E}[X]$ and $\mathrm{Var}[X]$, respectively.

E. Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0, 1)$ are normal in all bases. In our terminology, he defined a number to be *normal in base b* if it is simply normal in the bases $b, b^2, b^3 \ldots$. He used the following:

Theorem 1.1.4. *(The Strong Law of Large Numbers) Suppose that $X_1, X_2, \ldots$ are pairwise independent identically distributed random variables with $E[X_i] < \infty$ for all i. Let $\mu = E[X_i]$ and $S_n = X_1 + X_2 + \ldots + X_n$. Then for almost every x,*

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \mu. \tag{1.3}$$

It is not immediately obvious that E. Borel's definition of normality is equivalent to Definition 1.1.3. The Strong Law of Large Numbers cannot directly be easily used to prove that almost every real number is normal in the sense of Definition 1.1.3. The Birkhoff's Ergodic Theorem, a generalization of the Strong Law of Large Numbers that we will encounter later, will allow us to directly prove this result.

The first constructions of normal numbers were due to H. Lebesgue [27] and W. Sierpiński [49]. Both of these constructions are simultaneously normal in every base. The following are more well known:

Example 1.1.5. *The best known example of a number that is normal in base 10 is due to Champernowne [8]. The number*

$$0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12 \ldots, \tag{1.4}$$

formed by concatenating the digits of every natural number written in increasing order in base 10*, is normal in base* 10*.*

Example 1.1.6. *A. Copeland and P. Erdős proved in [9] that the number*

$$0.2\ 3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\ldots, \tag{1.5}$$

formed by concatenating the digits of the prime numbers expressed in base 10*, is normal in base* 10*.*

Example 1.1.7. *Suppose that* $p(n)$ *is a positive increasing polynomial on the natural numbers. H. Davenport and P. Erdős proved in [13] that the number*

$$0.p(1)p(2)p(3)p(4)p(5)p(6)\ldots, \tag{1.6}$$

formed by concatenating the digits of the values $p(1), p(2), p(3), \ldots$ *written in base* 10*, is normal in base* 10*.*

Since then, many examples of numbers that are normal in at least one base have been given, including constructions that generalize Example 1.1.6 and Example 1.1.7. One can find a more thorough literature review in [10] and [23].

1.2 Uniformly Distributed Sequences

We will now examine uniformly distributed sequences, which are an indispensable facet of the study of normality. Our notation will remain consistent with that of [23].[1]

For the rest of this thesis, we will let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling functions, respectively. Additionally, given a real number x, we will write $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x.

Definition 1.2.1. *Suppose that $X = \{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers. For a positive integer N and some $I \subset [0, 1)$, we define $A_N(I, X)$ to be the number of terms x_n with $1 \leq n \leq N$, for which $\{x_n\} \in I$. Thus, we may write*

$$A_N(I, X) = \#\{n \in [1, N] : \{x_n\} \in I\}. \tag{1.7}$$

If $X = \{x_n\}_{n=1}^{m}$ is a finite sequence of real numbers, we write

$$A(I, X) = A_N(I, X) = \#\{n \in [1, N] : \{x_n\} \in I\}. \tag{1.8}$$

Definition 1.2.2. *The sequence $X = \{x_n\}_{n=1}^{\infty}$ is said to be uniformly distributed mod 1 if for every pair a, b of real numbers with*

[1] See [10] and [23] for a more thorough look at uniformly distributed sequences.

$$0 \leq a < b \leq 1, \tag{1.9}$$

we have

$$\lim_{N\to\infty} \frac{A_N([a,b], X)}{N} = b - a. \tag{1.10}$$

We note the following equivalent statement of normality in base b:

Proposition 1.2.3. *The sequence $x_1, x_2, x_3, \ldots$ is uniformly distributed mod 1 if and only if for every real-valued continuous function f defined on the closed unit interval $[0, 1]$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx. \tag{1.11}$$

1.2.1 Connection to Normality

The most fundamental and important result in the theory of normal numbers is the following theorem:

Theorem 1.2.4. *A real number $x \in [0, 1)$ is normal in base b if and only if the sequence $\{b^n x\}_{n=0}^{\infty}$ is uniformly distributed mod 1.*

We observe that if $x = 0.d_1d_2d_3\ldots$, then

$$bx = d_1.d_2d_3\ldots, \tag{1.12}$$

$$b^2x = d_1d_2.d_3d_4\ldots, \tag{1.13}$$

$$b^3x = d_1d_2d_3.d_4d_5\ldots, \tag{1.14}$$

and so on. Suppose $B = (b_1, b_2, \ldots, b_k)$. The essential idea behind Theorem 1.2.4 is that if $x = 0.d_1d_2d_3\ldots$ is the b-ary expansion of some $x \in [0, 1)$, then the block B occurs at position n of the b-ary expansion of x if and only if

$$b^n x \in \left[\frac{b_1}{b} + \frac{b_2}{b^2} + \ldots + \frac{b_k}{b^k}, \frac{b_1}{b} + \frac{b_2}{b^2} + \ldots + \frac{b_k+1}{b^k}\right). \tag{1.15}$$

The important observation is that multiplication by b shifts the decimal point. This result will later motivate Definition 2.4.2.

1.2.2 Classical Results

We list the following theorems that are well known results on uniformly distributed sequences. The following is due to H. Weyl ([53] and [54]):

Theorem 1.2.5. *(Weyl Criterion) The sequence $x_1, x_2, x_3, \ldots$ is uniformly distributed mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \tag{1.16}$$

for all integers $h \neq 0$.

The remaining theorems can be found in [23]:

Theorem 1.2.6. *(Fejér's Theorem) Let $\{f(n)\}, n = 1, 2, \ldots,$ be a sequence of real numbers such that $\Delta f(n) = f(n+1) - f(n)$ is monotone as n increases. Suppose that*

$$\lim_{n\to\infty} \Delta f(n) = 0 \tag{1.17}$$

and

$$\lim_{n\to\infty} n|\Delta f(n)| = \infty. \tag{1.18}$$

Then the sequence $\{f(n)\}_n$ is uniformly distributed mod 1.

Theorem 1.2.7. *(Van der Corput's Difference Theorem) Let $\{x_n\}_{n=1}^{\infty}$ be a given sequence of real numbers. If for every positive integer h the sequence $\{x_{n+h} - x_n\}_{n=1}^{\infty}$ is uniformly distributed mod 1, then $\{x_n\}_{n=1}^{\infty}$ is uniformly distributed mod 1.*

Theorem 1.2.8. *If a sequence $\{x_n\}_{n=1}^{\infty}$ has the property*

$$\Delta x_n = x_{n+1} - x_n \to \theta \text{ (irrational) as } n \to \infty, \tag{1.19}$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is uniformly distributed mod 1.

1.2.3 Discrepancy

The *discrepancy* of a finite sequence can be thought of as a measure of how far it is from being uniformly distributed mod 1. Estimations of discrepancy combined with Theorem 1.2.4 allow us to study normal numbers in far greater detail than we otherwise could. We make the following definition:

Definition 1.2.9. *For a finite sequence $X = (x_1, \ldots, x_n)$, we define the discrepancy*

$$D_n = D_n(z_1, \ldots, z_n) \tag{1.20}$$

as

$$\sup_{0 \le \alpha < \beta \le 1} \left| \frac{A([\alpha, \beta), z)}{n} - \gamma \right|. \tag{1.21}$$

Given an infinite sequence $w = (w_1, w_2, \ldots)$, we define

$$D_n(w) = D_n(w_1, w_2, \ldots, w_n). \tag{1.22}$$

Theorem 1.2.10. *The sequence X is uniformly distributed mod 1 if and only if* $\lim_{n\to\infty} D_n(X) = 0$.

We note the following inequalities as examples of theorems that are often used to estimate discrepancy:

Theorem 1.2.11. *(LeVeque's Inequality) The discrepancy D_N of the finite sequence $x_1, x_2, \ldots, x_N$ in $[0, 1)$ satisfies*

$$D_N \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)^{1/3}. \tag{1.23}$$

Theorem 1.2.12. *(The Erdős-Turán Inequality) For any finite sequences of real numbers $x_1, x_2, \ldots, x_N$ and any positive integer m, we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|. \tag{1.24}$$

Both Theorem 1.2.11 and Theorem 1.2.12 allow us to study uniform distribution by using powerful inequalities involving estimation of trigonometric sums. Theorem 1.2.11 was proven by W. LeVeque in [28]. Theorem 1.2.12 was proven by P. Erdős and P. Turán in [14].

1.2.4 Examples of Uniformly Distributed Sequences

Example 1.2.13. *The sequence*

$$0, 1/2, 0, 1/3, 2/3, 0, 1/4, 2/4, 3/4, \ldots \tag{1.25}$$

is uniformly distributed mod 1. This can be proven directly by elementary means.

Example 1.2.14. *The sequence* $\{\log n\}_n$ *is not uniformly distributed mod* 1*. This may be shown by an application of Theorem 1.2.5. On a more intuitive level, this sequence is not uniformly distributed as the function* $\log n$ *grows so slowly that for every interval* I*, the limit*

$$\lim_{N\to\infty} \frac{A_N\left(I, \{\log n\}_{n=1}^{\infty}\right)}{N} \tag{1.26}$$

does not exist.

Example 1.2.15. *The sequence*

$$0, 1/2, 0, 1/4, 2/4, 3/4, 0, 1/8, 2/8, 3/8, \ldots, 7/8, 1/16, 2/16, \ldots \tag{1.27}$$

is not uniformly distributed mod 1*. This follows by reasons similar to those found in Example 1.2.14.*

Theorem 1.2.16. *The sequence*

$$\alpha, 2\alpha, 3\alpha, 4\alpha, \ldots \tag{1.28}$$

is uniformly distributed mod 1 *if and only if* α *is irrational. This may be proven directly through elementary means. This also follows from the Weyl Criterion and was originally proven in [53].*

H. Weyl went further and proved:

Theorem 1.2.17. *(Weyl's Theorem) Let*

$$p(x) = \alpha_m x^m + \alpha_{m-1}x^{m-1} + \ldots + \alpha_0, m \geq 1, \tag{1.29}$$

be a polynomial with real coefficients and let at least one of the coefficients α_j *with* $j > 0$ *be rational. Then the sequence* $\{p(n)\}_{n=1}^{\infty}$ *is uniformly distributed mod* 1.

Theorem 1.2.17 follows by induction and Theorem 1.2.7 and was originally proven to be uniformly distributed mod 1 by H. Weyl in [53] and [54].

1.3 Ergodic Theory and Common Series Expansions

1.3.1 Definitions and Basic Examples

We will use notation consistent with that of [12].[2] For the rest of this thesis, λ will denote Lebesgue measure.

Definition 1.3.1. *A probability space is a triple* $(X, \mathcal{F}, \mu)$*, where* X *is a nonempty set,* $\mathcal{F}$ *is a* σ*-algebra, and* μ *is a measure on* $(X, \mathcal{F})$ *with* $\mu(X) = 1$.

Definition 1.3.2. *Let* $(X, \mathcal{F}, \mu)$ *be a probability space. A measurable transformation* $T : X \to X$ *is measure preserving with respect to* μ *(equivalently:* μ *is* T*-invariant or* μ *is an invariant measure for* T*), if* $\mu(T^{-1}A) = \mu(A)$ *for all* $A \in \mathcal{F}$.

[2] For a thorough introduction to ergodic theory, see [35] and [52].

Definition 1.3.3. *A dynamical system is a quadruple* $(X, \mathcal{F}, \mu, T)$*, where* X *is a non-empty set,* $\mathcal{F}$ *is a* σ*-algebra on* X*,* μ *is a probability measure on* $(X, \mathcal{F})$ *and* $T : X \to X$ *is a surjective* μ*-preserving transformation.*

Definition 1.3.4. *Let* $(X, \mathcal{F}, \mu, T)$ *be a dynamical system. Then* T *is called ergodic if for every* μ*-measurable set* A *satisfying* $T^{-1}A = A$ *one has that* $\mu(A) \in \{0, 1\}$*.*

Theorem 1.3.5. *Let* $(X, \mathcal{F}, \mu)$ *be a probability space and let* $\mathcal{A}$ *be a generating semi-algebra. Let* $T : X \to X$ *be a measure preserving transformation; then* T *is ergodic if and only if for every* $A, B \in \mathcal{A}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B). \tag{1.30}$$

The following important theorem was proven by G. Birkhoff in 1931:

Theorem 1.3.6. *(The Pointwise Ergodic Theorem) Let* $(X, \mathcal{F}, \mu, T)$ *be a probability space and* $T : X \to X$ *a measure preserving transformation. Then, for any* $f \in L^1(\mu)$*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(T^i(x)\right) = f^*(x) \tag{1.31}$$

exists almost everywhere, is T*-invariant, and* $\int_X f d\mu = \int_X f^* d\mu$*. If, moreover,* T *is ergodic, then* f^* *is a constant almost everywhere and* $f^* = \int_X f d\mu$*.*

We also give the following property that is stronger than ergodicity:

Definition 1.3.7. *Suppose that T is a measure preserving transformation on a probability space $(X, \mathcal{F}, \mu)$. Then T is strongly mixing if for all $A, B \in \mathcal{F}$*

$$\lim_{n\to\infty} \mu\left(T^{-i}A \cap B\right) = \mu(A)\mu(B). \tag{1.32}$$

1.3.2 A Look at the b-ary Expansion Through Ergodic Theory

Using Theorem 1.3.6, we may now sketch a proof that for any integral $b \geq 2$, almost every real number in $[0, 1)$ is normal in base b. We use the following:

Lemma 1.3.8. *Define $T_b : [0, 1) \to [0, 1)$ by $Tx = bx \pmod 1$. Then T_b preserves Lebesgue measure and is ergodic.*

We let Y denote the set of points that have a unique b-ary expansion. Clearly, Y has full measure in $[0, 1)$. We see that if $x = 0.d_1d_2d_3\ldots$ is the b-ary expansion of x, then

$$T_b^i x = 0.b_{i+1}b_{i+2}b_{i+3}\ldots. \tag{1.33}$$

Let $B = (b_1, b_2, \ldots, b_k)$ be a block of length k and define

$$I_B = \left[\frac{b_1}{b} + \frac{b_2}{b^2} + \ldots + \frac{b_k}{b^k}, \frac{b_1}{b} + \frac{b_2}{b^2} + \ldots + \frac{b_k + 1}{b^k}\right). \tag{1.34}$$

Let $f(x) = \chi_{I_B}(x)$ be the characteristic function of I_B. Clearly,

$$N_n^b(B, x) = \sum_{i=0}^{n-1} f\left(T_b^i x\right). \tag{1.35}$$

Thus, by Theorem 1.3.6, for almost every $x \in [0, 1)$, we have

$$\lim_{n\to\infty} \frac{N_n^b(B, x)}{n} = \int \chi_{I_B} d\lambda = \lambda(I_B). \tag{1.36}$$

Since there are only countably many choices of the block B, almost every real number in $[0, 1)$ is normal in base b.

It should be noted that if $x = 0.d_1d_2d_3\dots$ is the b-ary expansion of x, then

$$d_i = k \text{ if and only if } b^{i-1}x \pmod 1 \in \left[\frac{k}{b}, \frac{k+1}{b}\right). \tag{1.37}$$

For many of the expansions we will study[3], we will fix a partition of the interval $[0, 1)$. The "digits" of some real number x will record which members of the partition that the orbit of x under some transformation T_b lands in. This idea should be considered fundamental for studying expansions connected to ergodic theory.

1.3.3 The Continued Fraction Expansion

The *continued fraction expansion* is of considerable importance in number theory and may be preferable to the b-ary expansion in many applications. If $x \in (0, 1)$ is an irrational number, we may write

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_1, a_2, a_3, \dots], \tag{1.38}$$

[3]See [12] for a more thorough treatment of the expansions covered in this section.

where the digits a_n are positive integers. If x is a rational number, we may write

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \frac{1}{a_n}}}} = [a_1, a_2, a_3, \ldots, a_n]. \tag{1.39}$$

A short computation shows that

$$[a_1, a_2, \ldots, a_{n-1}, a_n] = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1]. \tag{1.40}$$

The shorter of the two expansions in (1.40) will always be chosen when we consider the continued fraction expansion of a rational number.

Example 1.3.9.

$$\sqrt{2} - 1 = [2, 2, 2, 2, \ldots]. \tag{1.41}$$

To see why this is so, suppose that $x = [2, 2, 2, 2, \ldots]$*. Then*

$$x = \frac{1}{2 + x}, \tag{1.42}$$

so $x^2 + 2x - 1 = 0$*. Since* x *is positive, we have* $x = \sqrt{2} - 1$*.*

Example 1.3.10. *Through use of a calculator, one may compute the expansion*

$$\pi - 3 = [7, 15, 1, 292, 1, \ldots]. \tag{1.43}$$

It should be noted that there is no known pattern in the expansion (1.43).

For the remainder of this subsection, we define the shift transformation for $x \in (0,1)$:

$$Tx = \begin{cases} \{1/x\} & \text{for } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} . \tag{1.44}$$

By a simple computation, we see that

$$T[a_1, a_2, a_3, \ldots] = [a_2, a_3, a_4, \ldots] \tag{1.45}$$

and

$$T[a_1, a_2, a_3, \ldots, a_n] = [a_2, a_3, a_4, \ldots, a_n]. \tag{1.46}$$

For a positive integer n, set

$$I_n = \left[\frac{1}{n+1}, \frac{1}{n}\right) . \tag{1.47}$$

It can be shown through basic computation that $a_i(x) = n$ if and only if $T^{i-1}x \in I_n$.

One can show that T does not preserve Lebesgue measure. However, T is $\mu -$ *invariant*, where

$$\mu(I) = \frac{1}{\log 2} \int_I \frac{dx}{1+x} \tag{1.48}$$

for all intervals I. We note that μ is equivalent to Lebesgue measure.

Definition 1.3.11. *Suppose that* ν *is a probability measure on* $[0,1)$*. Then the sequence* $X = \{x_n\}_{n=1}^{\infty}$ *is said to be uniformly distributed mod* 1 *with respect to* ν *if for every pair* a, b *of real numbers with*

$$0 \leq a < b \leq 1, \tag{1.49}$$

we have

$$\lim_{N\to\infty} \frac{A_N([a,b],X)}{N} = \nu([a,b]). \tag{1.50}$$

Theorem 1.3.12. *The shift transformation* T *is strongly mixing with respect to* μ*.*

Thus, we may make the definition:

Definition 1.3.13. *A real number* $x \in [0,1)$ *is normal with respect to the continued fraction expansion if the sequence* $\{T^n x\}_{n=0}^{\infty}$ *is uniformly distributed mod* 1 *with respect to* μ*.*

Since μ is equivalent to Lebesgue measure, we may apply Theorem 1.3.6 and arrive at:

Theorem 1.3.14. *Almost every real number in* $[0,1)$ *is normal with respect to the continued fraction expansion.*

In particular, Theorem 1.3.14 implies that the frequency of the digit n in the continued fraction expansion of almost every real number is

$$\int_{I_n} \frac{dx}{1+x} = \frac{1}{\log 2} \log\left(1 + \frac{1}{n(n+2)}\right) \approx \frac{1}{\log 2} n^{-2}. \tag{1.51}$$

For example, the digit 1 occurs with frequency approximately 41.5% and the digit 2 occurs with frequency approximately 17% in the continued fraction expansion of almost every real number in $[0, 1)$.

Example 1.3.15. *Consider the number formed by concatenating the digits of the continued fraction expansion of the numbers*

$$1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \ldots. \tag{1.52}$$

This number was proven to be normal with respect to the continued fraction expansion by R. Adler, M. Keane, and M. Smorodinsky in [1].

1.3.4 The Lüroth Series Expansion

In this subsection, we partition the interval $[0, 1)$ into intervals of the form

$$I_n = \left[\frac{1}{n+1}, \frac{1}{n}\right). \tag{1.53}$$

We can write every $x \in [0, 1)$ in the form

$$x = [a_1(x), a_2(x), \ldots] = \frac{1}{a_1(x)} + \frac{1}{a_1(x)(a_1(x)-1)a_2(x)} + \ldots \tag{1.54}$$
$$+\frac{1}{a_1(x)(a_1(x)-1)\cdots a_{n-1}(x)(a_{n-1}(x)-1)a_n(x)} + \ldots,$$

where $a_n \geq 2$ are positive integers. (1.54) is called the *Lüroth series expansion* of x.

For the remainder of this subsection, define $T : [0,1) \to [0,1)$ by

$$Tx = \begin{cases} n(n+1)x - n & \text{for } x \in I_n \\ 0 & \text{if } x = 0 \end{cases} . \tag{1.55}$$

Then one may verify that

$$T[a_1, a_2, a_3, \ldots] = [a_2, a_3, a_4, \ldots] \tag{1.56}$$

and that $a_i(x) = n$ if and only if $T^{i-1}x \in I_{n-1}$.

Example 1.3.16.

$$\pi - 3 = [8, 2, 2, 2, 3, \ldots] = \frac{1}{8} + \frac{1}{8 \cdot 7 \cdot 2} \tag{1.57}$$

$$+\frac{1}{8 \cdot 7 \cdot 2 \cdot 1 \cdot 2} + \frac{1}{8 \cdot 7 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2} + \frac{1}{8 \cdot 7 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 3} + \ldots$$

The following may be found in [12]:

Theorem 1.3.17. *The transformation T is ergodic with respect to λ.*

Clearly, Theorem 1.3.17 suggests a reasonable definition of normality for the Lüroth series expansion that will hold for almost every $x \in [0,1)$:

Definition 1.3.18. *A real number $x \in [0,1)$ is normal with respect to the Lüroth series expansion if the sequence $\{T^n x\}_{n=0}^{\infty}$ is uniformly distributed mod 1 with respect to λ.*

Applying Theorem 1.3.6, we arrive at:

Theorem 1.3.19. *Almost every real number in $[0,1)$ is normal with respect to the Lüroth series expansion.*

1.3.5 The Generalized Lüroth Series Expansion

We will see that both the b-ary and Lüroth series expansions are special cases of a larger family of series expansions generated by piecewise linear transformations.

We consider any partition $\mathcal{I} = \{[l_n, r_n) : n \in \mathcal{D}\}$ of $[0,1)$ where $\mathcal{D} \subset \mathbb{N} \cup \{0\}$. We set $I_n = [l_n, r_n)$ for $n \in \mathcal{D}$. We also assume that if $i, j \in \mathcal{D}$ with $i > j$, then

$$0 < \lambda(L_i) \leq \lambda(L_j) < 1. \tag{1.58}$$

We call $\mathcal{D}$ the digit set. For the rest of this subsection, we define

$$Tx = \begin{cases} \frac{1}{r_n - l_n} x - \frac{l_n}{r_n - l_n}, & x \in I_n, n \in \mathcal{D} \\ 0, & x \in I_\infty = [0,1) \setminus \cup_{n \in \mathcal{D}} I_n \end{cases}. \tag{1.59}$$

For $x \in [l_n, r_n)$ and $n \in \mathcal{D}$, set

$$s(x) = \frac{1}{r_n - l_n} \text{ and } h(x) = \frac{l_n}{r_n - l_n}. \tag{1.60}$$

Thus, $Tx = xs(x) - h(x)$. Furthermore, we set

$$s_k(x) = \begin{cases} s(T^{k-1}x) & \text{if } T^{k-1}x \in \cup_{n\in\mathcal{D}} I_n \\ \infty & \text{if } T^{k-1}x \notin \cup_{n\in\mathcal{D}} I_n \end{cases} \tag{1.61}$$

and

$$h_k(x) = \begin{cases} h(T^{k-1}x) & \text{if } T^{k-1}x \in \cup_{n\in\mathcal{D}} I_n \\ 1 & \text{if } T^{k-1}x \notin \cup_{n\in\mathcal{D}} I_n \end{cases}. \tag{1.62}$$

So, if $x \in \cup_{n\in\mathcal{D}} I_n$ and $T^k x \in \cup_{n\in\mathcal{D}} I_n$ for all $k \geq 1$, we have that

$$x = \frac{h_1(x)}{s_1(x)} + \frac{h_2(x)}{s_1(x)s_2(x)} + \ldots + \frac{h_k(x)}{s_1(x)s_2(x)\cdots s_k(x)} + \ldots. \tag{1.63}$$

We define the digits $a_1(x), a_2(x), \ldots$ of the GLS($\mathcal{I}$) of a real $x \in [0,1)$ by

$$a_n(x) = k \text{ if } T^{n-1}x \in I_k, k \in \mathcal{D} \cup \{\infty\}. \tag{1.64}$$

The expansion (1.63) is called the *Generalized Lüroth series expansion.* The following are examples of Generalized Lüroth series expansions:

Example 1.3.20. *(b-ary expansion)*

Let $b \geq 2$ be a positive integer, $\mathcal{D} = \{0, 1, \ldots, b-1\}$, $I_k = \left[\frac{k}{b}, \frac{k+1}{b}\right)$ for $k \in \mathcal{D}$, and $s_n(x) = b$ for all x. Additionally, set $h_1(x) = k$ if $x \in I_k$.

Example 1.3.21. *(Lüroth series expansion)*

We set

$$\mathcal{D} = \{2, 3, 4, \ldots\}, I_k = \left[\frac{1}{k}, \frac{1}{k-1}\right), k \in \mathcal{D}, \tag{1.65}$$

$s_1(x) = k(k-1) = a_1(a_1 - 1)$ *if* $x \in I_k$, $s_k = a_k(a_k - 1)$, *and* $h_k = a_k - 1$.

The following may be found in [12]:

Theorem 1.3.22. *Let T be a GLS($\mathcal{I}$) transformation on $[0, 1)$. Then T is ergodic.*

1.3.6 β-expansions

We consider another generalization of the b-ary expansion. Let $\beta > 1$ be a real number and define

$$T_\beta x = \beta x \pmod 1. \tag{1.66}$$

For all $x \in [0, 1)$, we may write

$$x = \frac{a_1}{\beta} + \frac{d_2}{\beta^2} + \frac{d_3}{\beta^3} + \ldots, \tag{1.67}$$

where $a_n(x) = \lfloor \beta T_\beta^{n-1} x \rfloor$ for all positive integers n with $a_n(x) \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$. The expansion (1.67) is called the *β-expansion* of x.

Example 1.3.23. *Set*

$$\beta = \frac{\sqrt{5}+1}{2}. \tag{1.68}$$

Then $a_n(x) \in \{0,1\}$ *and* $a_n(x) = 0$ *if and only if* $T_\beta^{n-1}x \in [0, 1/\beta)$. *Additionally, the digit* 0 *can be followed by a* 0 *or* 1*, but the digit* 1 *may only be followed by a* 0*. For example, we have*

$$\frac{4}{5} = \frac{1}{\beta} + \frac{0}{\beta^2} + \frac{0}{\beta^3} + \frac{1}{\beta^4} + \frac{0}{\beta^5} + \frac{0}{\beta^6} + \frac{1}{\beta^7} + \frac{0}{\beta^8} + \dots. \tag{1.69}$$

Proposition 1.3.24. *If* β *is not an integer, then* T_β *is not invariant with respect to Lebesgue measure.*

The following is due to A. Rényi [39]:

Theorem 1.3.25. *For all* $\beta > 1$*, there exists a measure* ν_β *such that* T_β *is ergodic with respect to* ν_β*.*

A. Gelfond [18] and W. Parry [34] independently found an explicit formula for the measure ν_β in Theorem 1.3.25.[4]

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[4]Further work has been done. For example, V. Rochlin showed that the entropy of the map T_β is $\log\beta$.

1.4 f-expansions

In this section, we will see a general class of series expansions that includes the b-ary expansion, the continued fraction expansion, and many others.[5]

Definition 1.4.1. *Suppose that f is a monotone (increasing or decreasing) function and $f:(0,1)\to\mathbb{R}$. The f-expansion of a real number $x\in(0,1)$, if it exists, is the expansion of the form*

$$x = f(E_1(x) + f(E_2(x) + f(E_3(x) + \ldots))). \tag{1.70}$$

The digits in (1.70) are defined as follows. Let $\phi = f^{-1}$ and define

$$r_0(x) = f(x) \tag{1.71}$$

and

$$r_{n+1}(x) = \{\phi(r_n(x)\} \text{ for } n = 0, 1, 2, \ldots. \tag{1.72}$$

We set

$$E_{n+1}(x) = \lfloor \phi(r_n(x)) \rfloor. \tag{1.73}$$

A. Rényi proved the following two theorems in [39]:

[5]For more detail, see A. Rényi's survey paper *Probabilistic Methods in Number Theory [40].*

Theorem 1.4.2. *If $f(x)$ is a monotone function and satisfies at least one of the following conditions, then every $x \in (0,1)$ can be represented in the form (1.70) with digits $E_n(x)$ defined as in (1.72):*

1. *$f(t)$ is positive valued, continuous, and strictly decreasing for $1 \le t \le T$ where $2 < T \le \infty$, further, $f(1) = 1$; if $T < \infty$, then $f(t) = 0$ for $t \ge T$, and if $T = \infty$, then $\lim_{t\to\infty} f(t) = 0$. In addition,*

$$|f(t_2) - f(t_1)| \le |t_2 - t_1| \text{ for } 1 \le t_1 \le t_2, \tag{1.74}$$

   and

$$|f(t_2) - f(t_1)| \le |t_2 - t_1| \text{ for } \tau - \epsilon < t_1 < t_2, \tag{1.75}$$

   where τ is the solution of the equation

$$1 + f(\tau) = \tau \tag{1.76}$$

   and $0 < \epsilon < \tau$.

2. *$f(t)$ is continuous and strictly increasing for $0 \le t \le T$ where $1 < T \le \infty$ and $f(0) = 0$. If $T < \infty$, then $f(t) = 1$ for $t \ge T$; if $T = \infty$, then $\lim_{t\to\infty} f(t) = 1$; further, we have*

$$f(t_2) - f(t_1) < t_2 - t_1 \text{ for } 0 \le t_1 < t_2 \le T. \tag{1.77}$$

Given a function f, we define

$$f_n(x,t) = f(E_1(x) + f(E_2(x) + \ldots + f(E_n(x) + t)\ldots)). \tag{1.78}$$

Additionally, we set

$$H_n(x,t) = \frac{df_n(x,t)}{dt}. \tag{1.79}$$

Theorem 1.4.3. *Suppose that there exists a constant $C \geq 1$ for which*

$$\frac{\sup_{0<t<1} |H_n(x,t)|}{\inf_{0<t<1} |H_n(x,t)|} \leq C \ (0 < x < 1, n = 1, 2, \ldots) \tag{1.80}$$

and f(x) satisfies either condition 1. or 2. of Theorem 1.4.2 and T is an integer or $T = \infty$, then for any function $g(x)$ which is Lebesgue integrable in the interval $(0,1)$, we have that for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(r_k(x)) = M(g), \tag{1.81}$$

where $r_k(x)$ is defined by (1.72) and $M(g)$ is a constant independent of x and depending on $f(x)$ and $g(x)$ in the following way:

$$M(g) = \int_0^1 g(x)h(x)dx, \tag{1.82}$$

where $h(x)$ is a measurable function, depending only on $f(x)$ and satisfying the inequality

$$\frac{1}{C} \leq h(x) \leq C, \tag{1.83}$$

where C is the constant figuring in (1.79). The measure

$$\nu(E) = \int_E h(x)dx \tag{1.84}$$

is invariant under the transformation

$$Tx = \{\phi(x)\}, \tag{1.85}$$

where $y = \phi(x)$ is the inverse function of $x = f(y)$.

Example 1.4.4. *The continued fraction expansion is a special case of the f-expansion, where $f(x) = 1/x$.*

Example 1.4.5. *If $\beta > 1$ is a positive real number and*

$$f(x) = \begin{cases} \frac{x}{\beta} & \textit{for } 0 \leq x \leq \beta \\ 1 & \textit{if } x > \beta \end{cases}, \tag{1.86}$$

then the f-expansion coincides with the β-expansion.

Example 1.4.6. *Let*

$$f(x) = \begin{cases} \sqrt[m]{1+x} - 1 & \textit{for } 0 \leq x \leq 2^m - 1 \\ 1 & \textit{if } x > 2^m - 1 \end{cases}. \tag{1.87}$$

Then every $x \in (0,1)$ can be represented in the form

$$x = -1 + \sqrt[m]{E_1 + \sqrt[m]{E_2 + \sqrt[m]{E_3 + \ldots}}}, \tag{1.88}$$

where the digits E_n may take on the possible values $0, 1, \ldots, 2^m - 2$.

1.5 Other Common Expansions

In this section, we will look at three expansions that are not special cases of the f-expansions or generalized Lüroth series expansion.[6]

1.5.1 The Engel Series Expansion

Suppose that $x \in (0, 1)$. We define a sequence of positive integers $q_1, q_2, q_3, \ldots$ as follows. Suppose that q_1 satisfies

$$\frac{1}{q_1} \leq x < \frac{1}{q_1 - 1}. \tag{1.89}$$

Given $q_1, q_2, \ldots, q_{n-1}$, we determine q_n by the inequality

$$\frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n} \leq x < \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \ldots q_{n-1}(q_n - 1)}. \tag{1.90}$$

[6]For a different perspective than that covered in this thesis, see [16] where many of these expansions are developed as a special case of the *Oppenheim expansion*.

If

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n}, \tag{1.91}$$

then our expansion is finite and we do not need to determine $q_{n+1}, q_{n+2}, \ldots$. Otherwise, the *Engel series expansion* of x is

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n} + \ldots . \tag{1.92}$$

Example 1.5.1.

$$\frac{25}{29} = \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 2 \cdot 3 \cdot 3} + \frac{1}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 29}. \tag{1.93}$$

Example 1.5.2.

$$\pi - 3 = \frac{1}{8} + \frac{1}{8 \cdot 8} + \frac{1}{8 \cdot 8 \cdot 17} + \frac{1}{8 \cdot 8 \cdot 17 \cdot 19} + \frac{1}{8 \cdot 8 \cdot 17 \cdot 19 \cdot 300} + \ldots . \tag{1.94}$$

The following result was proven by P. Lévy in [29]:

Theorem 1.5.3. *For almost all $x \in (0, 1)$, we have*

$$\lim_{n \to \infty} \sqrt[n]{q_n} = e. \tag{1.95}$$

1.5.2 The Sylvester Series Expansion

The *Sylvester Series Expansion* is also known as the *greedy Egyptian expansion* and was used in ancienct Egypt as a way to represent rational numbers. Suppose that $x \in (0, 1)$. We define a sequence of positive integers $q_1, q_2, q_3, \ldots$ as follows. Suppose that q_1 is the smallest positive integer that satisfies

$$\frac{1}{q_1} \leq x. \tag{1.96}$$

Given $q_1, q_2, \ldots, q_{n-1}$, we let q_n be the smallest positive integer that satisfies

$$\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_n} \leq x. \tag{1.97}$$

If

$$x = \frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_n}, \tag{1.98}$$

then the Sylvester series expansion is finite and we do not need to continue the algorithm. Otherwise, the Sylvester series expansion of x is

$$x = \frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_n} + \ldots. \tag{1.99}$$

We can see that the sequence q_n grows fast in the following examples:

Example 1.5.4.

$$\pi - 3 = \frac{1}{8} + \frac{1}{5020} + \frac{1}{128541347} + \ldots. \tag{1.100}$$

Example 1.5.5.

$$\sqrt{2} - 1 = \frac{1}{3} + \frac{1}{13} + \frac{1}{253} + \frac{1}{218201} + \dots. \tag{1.101}$$

The following can be easily proven through elementary means:

Proposition 1.5.6. *For all $x \in (0,1)$ with infinite Sylvester series expansion, we have*

$$q_{n+1} \geq q_n(q_n - 1) + 1. \tag{1.102}$$

Additionally, P. Erdős, A. Rényi, and P. Szüsz proved in [11]:

Theorem 1.5.7. *The following limit exists for almost every $x \in (0,1)$:*

$$\lim_{n\to\infty} \sqrt[2^n]{q_n(x)} = l(x); \tag{1.103}$$

where $l(x)$ is a positive number which depends on x.

Theorem 1.5.8. *For almost all $x \in (0,1)$, we have*

$$\lim_{n\to\infty} \sqrt[n]{\frac{q_n(x)}{q_1(x)q_2(x)\cdots q_{n-1}(x)}} = e. \tag{1.104}$$

1.5.3 The Cantor Product

Suppose that $x > 1$. In [7], G. Cantor studied expansions of the form

$$x = \prod_{n=1}^{\infty} \left(1 + \frac{1}{q_n}\right), \tag{1.105}$$

where $q_1, q_2, \ldots$ is a sequence of positive integers.

Example 1.5.9. *The Cantor product expansion of* $\sqrt{2}$ *is*

$$\sqrt{2} = \left(1 + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{17}\right) \cdot \left(1 + \frac{1}{577}\right) \cdot \left(1 + \frac{1}{665857}\right) \cdots . \tag{1.106}$$

Example 1.5.10. *The Cantor product expansion of* π *is*

$$\pi = \left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{22}\right) \cdot \left(1 + \frac{1}{600}\right) \cdots . \tag{1.107}$$

Results similar to those appearing earlier in this section hold for this expansion. Namely, A. Rényi showed the following in [41]:

Theorem 1.5.11. *For almost all* x*, the limit*

$$\lim_{n \to \infty} \sqrt[2^n]{q_{n+1}(x)} = l(x) \tag{1.108}$$

exists and is finite and greater than 2*.*

Theorem 1.5.12. *For almost every x,*

$$\lim_{n\to\infty} \sqrt[n]{\frac{q_{n+1}(x)}{q_1(x)q_2(x)\cdots q_n(x)}} = e. \tag{1.109}$$

1.6 Fundamental Properties of Normal Numbers

The following properties hold for most notions of normality:

1. Normality of a real number x, in the sense of comparing the frequency of blocks of digits, is equivalent to some condition on the distribution of the orbit of x under some measure preserving transformation.

2. The set of normal numbers has full measure.

3. The set of non-normal numbers has full Hausdorff dimension. In fact, this set has the additional stronger property of being a winning set in the sense of Schmidt games. We will define both of these concepts later in this thesis.

4. The set of normal numbers is of the first category.

5. Normal numbers cannot be rational.[7]

[7] Sometimes even more may be true. For example, the continued fraction expansion of a quadratic irrational is eventually periodic so a quadratic irrational will never be normal with respect to the continued fraction expansion.

We will investigate each of these properties with respect to the Cantor series expansion in later chapters. We will see that while the fourth property holds and the third property almost holds, the first is no longer true and the second may only be true under certain conditions. The last property essentially holds, but there are some notable differences in the case of the Cantor series expansion depending on the notions of normality that one studies.

CHAPTER 2

NORMALITY WITH RESPECT TO THE CANTOR SERIES EXPANSION

We now turn our attention to a series expansion whose digits are not generated by any known ergodic transformation. For this reason, we will find many tools from probability theory to be more useful than those of ergodic theory in studying this expansion.

2.1 The Cantor Series Expansion

The *Q-Cantor series expansion*, first studied by G. Cantor in [6], is a natural generalization of the b-ary expansion.

Definition 2.1.1. $Q = \{q_n\}_{n=1}^{\infty}$ *is a basic sequence if each* q_n *is an integer greater than or equal to* 2*.*

We will say that a basic sequence Q is *non-trivial* if there do not exist positive integers N and b such that $q_n = b$ for all $n > N$.

Definition 2.1.2. *Given a basic sequence Q, the Q-Cantor series expansion of a real x in $[0,1)$ is the (unique) expansion of the form*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{2.1}$$

such that E_n is in $\{0, 1, \dots, q_n - 1\}$ for all n with $E_n \neq q_n - 1$ infinitely often.

We now provide a proof of the uniqueness of the Q-Cantor series expansion:

Proposition 2.1.3. *The Q-Cantor series expansion is unique.*

Proof. Suppose that some $x \in [0,1)$ has two distinct Q-cantor series expansions

$$\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} = \sum_{n=1}^{\infty} \frac{F_n}{q_1 q_2 \cdots q_n}. \tag{2.2}$$

Let j be the smallest integer such that $E_j \neq F_j$. Without loss of generality, we will assume that $E_j > F_j$. Then, multiplying both sides by $q_1 q_2 \cdots q_j$, (2.2) can be written as

$$E_j + \frac{E_{j+1}}{q_{j+1}} + \frac{E_{j+2}}{q_{j+1} q_{j+2}} + \dots = F_j + \frac{F_{j+1}}{q_{j+1}} + \frac{F_{j+2}}{q_{j+1} q_{j+2}} + \dots. \tag{2.3}$$

Subtracting

$$F_j + \left(\frac{E_{j+1}}{q_{j+1}} + \frac{E_{j+2}}{q_{j+1} q_{j+2}} + \dots \right) \tag{2.4}$$

from both sides of (2.3), we arrive at

$$E_j - F_j = \frac{F_{j+1} - E_{j+1}}{q_{j+1}} + \frac{F_{j+2} - E_{j+2}}{q_{j+1}q_{j+2}} + \frac{F_{j+3} - E_{j+3}}{q_{j+1}q_{j+2}q_{j+3}} + \dots . \tag{2.5}$$

However, since $0 \leq E_n \leq q_n - 1$ and $0 \leq F_n \leq q_n - 1$ for all n, we know that

$$-1 \leq \frac{F_{j+1} - E_{j+1}}{q_{j+1}} + \frac{F_{j+2} - E_{j+2}}{q_{j+1}q_{j+2}} + \frac{F_{j+3} - E_{j+3}}{q_{j+1}q_{j+2}q_{j+3}} + \dots \leq 1. \tag{2.6}$$

But, since $E_j > F_j$, (2.6) implies that $E_j = F_j + 1$, so

$$\frac{F_{j+1} - E_{j+1}}{q_{j+1}} + \frac{F_{j+2} - E_{j+2}}{q_{j+1}q_{j+2}} + \frac{F_{j+3} - E_{j+3}}{q_{j+1}q_{j+2}q_{j+3}} + \dots = 1. \tag{2.7}$$

Thus, we may conclude that $F_n - E_n = q_n - 1$ for all $n \geq j+1$. However, this implies that

$$E_n = 0 \text{ and } F_n = q_n - 1 \ \forall n \geq j+1. \tag{2.8}$$

So, the Q-Cantor series expansion is unique as long as we do not allow $E_n = q_n - 1$ for all large enough n.

□

Clearly, the b-ary expansion is a special case of (2.1) where $q_n = b$ for all n. If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair b-sided die, then a Q-Cantor series expansion may be thought of as representing an outcome of rolling a fair q_1 sided die, followed by a fair q_2 sided die and so on.

Example 2.1.4. *If $q_n = n+1$ for all n, then the Q-Cantor series expansion of $e-2$ is*

$$e - 2 = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \dots. \tag{2.9}$$

Example 2.1.5. *If $q_n = 10$ for all n, then the Q-Cantor series expansion for $1/4$ is*

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \dots. \tag{2.10}$$

We will primarily be concerned with the Q-Cantor series expansion of a real number $x \in [0,1)$. However, we will sometimes need to consider real numbers not contained in $[0,1)$. If Q is a basic sequence, E_0 is an integer, we say that

$$x = E_0.E_1E_2E_3\dots \text{ w.r.t. } Q \tag{2.11}$$

if $x = E_0 + y$, where

$$y = \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2\dots q_n} \tag{2.12}$$

is the Q-Cantor series expansion of $y \in [0,1)$.

2.2 Basic Definitions Relating to Normality

We will need the following definitions frequently throughout the rest of this thesis.

Definition 2.2.1. *For a given basic sequence Q, let $N_n^Q(B,x)$ denote the number of times a block B occurs starting at a position no greater than n in the Q-Cantor series expansion of x.*

Definition 2.2.2. *Given a basic sequence Q, we define*

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \dots q_{j+k-1}}. \tag{2.13}$$

Definition 2.2.3. *A basic sequence Q is k-divergent if*

$$\lim_{n\to\infty} Q_n^{(k)} = \infty. \tag{2.14}$$

Q is fully divergent if Q is k-divergent for all k.

Definition 2.2.4. *A basic sequence Q is k-convergent if*

$$\lim_{n\to\infty} Q_n^{(k)} < \infty. \tag{2.15}$$

Definition 2.2.5. *A basic sequence Q is infinite in limit if $q_n \to \infty$.*

We remark that a k-divergent basic sequence need not be infinite in limit, but a k-convergent basic sequence must always be infinite in limit.

Example 2.2.6. *Let* $q_n = \max(2, \log n)$*. Then* Q *is fully convergent and infinite in limit.*

Example 2.2.7. *Suppose that* p *is a positive integer and that* $q_n = \max(2, n^{1/p})$*. Then* Q *is* k*-divergent for all* $k \leq p$ *and* k*-convergent for all* $k > p$*.*

Definition 2.2.8. *Suppose that* Q *is a basic sequence. Then a* Q*-adic interval is an interval of the form*

$$\left[\frac{F_1}{q_1} + \frac{F_2}{q_1 q_2} + \ldots + \frac{F_n}{q_1 q_2 \cdots q_n}, \frac{F_1}{q_1} + \frac{F_2}{q_1 q_2} + \ldots + \frac{F_n + 1}{q_1 q_2 \cdots q_n}\right) \tag{2.16}$$

for some integer n *and positive integers* $F_1, F_2, \ldots, F_n$ *with* $F_i \in [0, q_i - 1)$ *for all* i*.*

We remark that a real number $x \in [0, 1)$ has $x = 0.F_1 F_2 \ldots F_n \ldots$ w.r.t. Q if and only if x is in the interval in (2.16). We will repeatedly use this fact without mention.

Given a block B, $|B|$ will represent the length of B. Given non-negative integers $l_1, l_2, \ldots, l_n$, at least one of which is positive, and blocks $B_1, B_2, \ldots, B_n$, the block

$$B = l_1 B_1 l_2 B_2 \ldots l_n B_n \tag{2.17}$$

will be the block of length $l_1|B_1| + \ldots + l_n|B_n|$ formed by concatenating l_1 copies of B_1, l_2 copies of B_2, through l_n copies of B_n. For example, if $B_1 = (2, 3, 5)$ and $B_2 = (0, 8)$, then $2B_1 1B_2 0B_2 = (2, 3, 5, 2, 3, 5, 0, 8)$.

Definition 2.2.9. *Given a block $B = (b_1, b_2, \ldots, b_k)$, we define the maximum and minimum values of the block B as follows:*

$$\max(B) = \max(b_1, b_2, \ldots, b_k) \tag{2.18}$$

and

$$\min(B) = \min(b_1, b_2, \ldots, b_k). \tag{2.19}$$

Definition 2.2.10. *Suppose that Q is a basic sequence and $E = (E_1, E_2, \ldots)$. Then we define the blocks*

$$Q_{n,k} = (q_n, q_{n+1}, \ldots, q_{n+k-1}) \tag{2.20}$$

and

$$E_{n,k} = (E_n, E_{n+1}, \ldots, E_{n+k-1}). \tag{2.21}$$

Definition 2.2.11. *Suppose that $B = (b_1, \ldots, b_k)$ and $B' = (b'_1, \ldots, b'_k)$ are two blocks of length k. Then we say that $B < B'$ if $b_j < b'_j$ for all $j \in [1, k]$, $B \le B'$ if $b_j \le b'_j$ for all $j \in [1, k]$, and $B = B'$ if $b_j = b'_j$ for all $j \in [1, k]$.*

Definition 2.2.12. *A block B of length k is Q-admissable if there exists a positive integer N such that*

$$B < Q_{n,k} \text{ for all } n \geq N. \tag{2.22}$$

2.3 Q-Normal Numbers

A. Rényi [38] defined a real number x to be *normal with respect to* Q if for all blocks B of length 1,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(1)}} = 1. \tag{2.23}$$

If $q_n = b$ for all n, then (2.23) is equivalent to simple normality in base b, but not equivalent to normality in base b. Thus, we want to generalize normality in a way that is equivalent to normality in base b when all $q_n = b$.

We wish to extend A. Rényi's notion of normality to be more consistent with our current notions of normality for the b-ary expansion. In this section, we examine the first notion of normality that we will study. This notion is closest to comparing the frequency of digits in the b-ary expansion.

Definition 2.3.1. *A real number x is Q-normal of order k if for all Q-admissable blocks B of length k, we have*

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1. \tag{2.24}$$

We say that x is Q-normal if it is Q-normal of order k for all k.

We will see that for Q that are infinite in limit, the set of all x in $[0,1)$ that are Q-normal of order k has full Lebesgue measure if and only if Q is k-divergent. Therefore, if Q is infinite in limit, then the set of all x in $[0,1)$ that are Q-normal has full Lebesgue measure if and only if Q is fully divergent. Additionally, given an arbitrary non-negative integer a, F. Schweiger [48] proved that for almost every x with $\epsilon > 0$, one has

$$N_n((a),x) = Q_n^{(1)} + O\left(\sqrt{Q_n^{(1)}} \cdot \log^{3/2+\epsilon} Q_n^{(1)}\right). \tag{2.25}$$

We will improve upon these asymptotics with Theorem 7.3.10.

It is more difficult to construct specific examples of Q-normal numbers than it is to show that the typical real number is Q-normal. This is similar to the case of the b-ary expansion. The situation is more complicated when Q is infinite in limit as we need to consider blocks whose digits come from an infinite set. We will be able to construct examples with Theorem 3.3.13.

2.4 Q-Distribution Normal Numbers

Definition 2.4.1. *Let x be a number in $[0,1)$ and let Q be a basic sequence, then $T_{Q,n}(x)$ is defined as*

$$q_1 \cdots q_n x \pmod 1.$$

Definition 2.4.2. *A number x in $[0,1)$ is Q-distribution normal if the sequence $\{T_{Q,n}(x)\}_{n=0}^{\infty}$ is uniformly distributed in $[0,1)$.*

Note that in base b, where $q_n = b$ for all n, the notions of Q-normality and Q-distribution normality are equivalent. It might be surprising that this equivalence breaks down in the more general context of Q-Cantor series for general Q.

Definition 2.4.3. *A basic sequence Q is almost infinite in limit if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{q_k} = 0. \tag{2.26}$$

We note the well known characterization[1] of sequences satisfying (2.26) that motivates the definition of almost infinite in limit basic sequences:

Proposition 2.4.4. *If $\{a_n\}$ is a bounded sequence of real numbers, then the following are equivalent:*

1. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{a_k} = 0$.
2. *There exists $J \subset \mathbb{N}$ of density zero such that $\lim_{n\to\infty} a_n = \infty$ provided $n \notin J$.*
3. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{a_k^2} = 0$.

We will use the following theorem proven by T. Šalát in [45]:

[1] See, for example, [52]

Theorem 2.4.5. *Given a basic sequence Q and a real number x with Q-Cantor series expansion $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 \cdots q_n}$; if Q is almost infinite in limit, then x is Q-distribution normal if and only if*

$$\left\{ \frac{E_n}{q_n} \right\}_{n=1}^{\infty} \tag{2.27}$$

is uniformly distributed mod 1.

In most applications, it will suffice to use the following weaker result, originally proven by N. Korobov in [22]:

Theorem 2.4.6. *Given a basic sequence Q and a real number x with Q-Cantor series expansion $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 \cdots q_n}$; if Q is infinite in limit, then x is Q-distribution normal if and only if*

$$\left\{ \frac{E_n}{q_n} \right\}_{n=1}^{\infty} \tag{2.28}$$

is uniformly distributed mod 1.

Both Theorem 2.4.5 and Theorem 2.4.6 should be considered fundamental in our study of Q-distribution normal numbers.

We recall the following standard definition that will be useful in studying distribution normality:

Definition 2.4.7. *For a finite sequence $z = (z_1, \ldots, z_n)$, we define the star discrepancy $D_n^* = D_n^*(z_1, \ldots, z_n)$ as*

$$\sup_{0<\gamma\leq 1} \left| \frac{A([0,\gamma), z)}{n} - \gamma \right|. \tag{2.29}$$

Given an infinite sequence $w = (w_1, w_2, \ldots)$, we define

$$D_n^*(w) = D_n^*(w_1, w_2, \ldots, w_n). \tag{2.30}$$

For convenience, set $D^*(z_1, \ldots, z_n) = D_n^*(z_1, \ldots, z_n)$.

The star discrepancy of a sequence $z = (z_1, \ldots, z_n)$ is related to the discrepancy of the same sequence by the following theorem:

Theorem 2.4.8. *For any finite sequence $z = (z_1, \ldots, z_n)$,*

$$D_n^* \leq D_n \leq 2D_n. \tag{2.31}$$

Theorem 2.4.8 immediately suggests the following corollary that we will use frequently and without mention:

Corollary 2.4.9. *The sequence $w = (w_1, w_2, \ldots)$ is uniformly distributed mod 1 if and only if*

$$\lim_{n\to\infty} D_n^*(w) = 0. \tag{2.32}$$

The faster the sequence $\{q_n\}$ grows, the faster one can think of the sequence $\{T_{Q,n}(x)\}$ as getting closer to being uniformly distributed mod 1. Intuitively, as the value of q_n increases, we can approximate a real number in $[0,1)$ by E_n/q_n with better accuracy. We will see with Theorem 7.3.12 that the exact opposite is true with Q-normal numbers. This notion is formalized in the following theorem of J. Galambos [17]:

Theorem 2.4.10. *Let Q be a 1-divergent basic sequence. Let E_k be the digits of the Q-cantor series expansion of x and put $\theta_k = \theta_k(x) = E_k/q_k$. Then, for almost all x in $[0,1)$,*

$$D_n^*(\theta) \geq \frac{1}{2n}\sum_{k=1}^{n}\frac{1}{q_k} \tag{2.33}$$

for sufficiently large n.

We will need the following result pertaining to uniformly distributed sequences:

Proposition 2.4.11. *Suppose that $X = \{x_n\}$ is a sequence in $[0,1)$ and L and R are countable dense subsets of $[0,1)$. If for all $l \in L$ and $r \in R$ with $l < r$, we have*

$$\lim_{n\to\infty}\frac{A_n([l,r),X)}{n} = \lambda([l,r)) = r - l, \tag{2.34}$$

then X is uniformly distributed mod 1.

Proof. Let $\epsilon > 0$ and an arbitrary interval $I \subset [0,1)$ be given. Let I_1 and I_2 be intervals contained in $[0,1)$ with left endpoints in L and right endpoints in R such that

$$I_1 \subset I \subset I_2, \tag{2.35}$$

$$\lambda(I_1) > \lambda(I) - \epsilon/2, \tag{2.36}$$

and

$$\lambda(I_2) < \lambda(I) + \epsilon/2. \tag{2.37}$$

Suppose that M_ϵ is large enough so that for $n > M_\epsilon$ and $k = 1, 2$

$$\left| \frac{A_n(I_k, X)}{n} - \lambda(I_k) \right| < \frac{\epsilon}{2}. \tag{2.38}$$

We know that since $I_1 \subset I \subset I_2$,

$$\frac{A_n(I_1, X)}{n} \leq \frac{A_n(I, X)}{n} \leq \frac{A_n(I_2, X)}{n}. \tag{2.39}$$

By (2.38) and (2.39) we see that

$$\lambda(I_1) - \frac{\epsilon}{2} < \frac{A_n(I, X)}{n} < \lambda(I_2) + \frac{\epsilon}{2}. \tag{2.40}$$

Combining (2.40) with (2.36) and (2.37), we conclude that

$$\lambda(I) - \epsilon < \frac{A_n(I, X)}{n} < \lambda(I) + \epsilon. \tag{2.41}$$

Therefore,

$$\left|\frac{A_n(I,X)}{n} - \lambda(I)\right| < \epsilon, \tag{2.42}$$

so

$$\lim_{n\to\infty} \frac{A_n(I,X)}{n} = \lambda(I). \tag{2.43}$$

Since I was arbitrary, X is uniformly distrubuted mod 1.

□

We may now prove the following lemma which may be used to check for distribution normality:

Theorem 2.4.12. *If Q is a basic sequence, then x is Q-distribution normal if and only if for all intervals I with rational endpoints, we have*

$$\lim_{n\to\infty} \frac{A_n(I, \{T_{Q,m}(x)\}_{m=0}^{n-1})}{n} = \lambda(I). \tag{2.44}$$

Proof. We let $L = R = [0,1) \cap \mathbb{Q}$ and apply Proposition 2.4.11 to $X = \{T_{Q,n}(x)\}_{n=0}^{\infty}$. Clearly, x is not distribution normal if there is an interval where (2.44) does not hold.

□

2.5 Q-Ratio Normal Numbers

We will sometimes encounter a third notion of normality for the Q-Cantor series expansion that is strictly weaker than normality for Q that are infinite in limit.

Definition 2.5.1. *Suppose that Q is a basic sequence and that k is a positive integer. Then a real number x is Q-ratio of order k if for all Q-admissable blocks B and B' of length k, we have*

$$\lim_{n \to \infty} \frac{N_n^Q(B,x)}{N_n^Q(B',x)} = 1. \tag{2.45}$$

We say that x is Q-ratio normal if it is Q-ratio normal of order k for all positive integers k.[2]

Theorem 2.5.2. *If Q is a basic sequence and a real number x is Q-normal of order k, then x is also Q-ratio normal of order k.*

Proof. We know that for all $m \leq k$ and Q-admissable blocks B of length m

$$\lim_{n \to \infty} \frac{N_n^Q(B,x)}{Q_n^{(m)}} = 1. \tag{2.46}$$

Thus, if B_1 and B_2 are two Q-admissable blocks of length m, then

[2] This thesis was started by investigation of Q-ratio normal numbers. V. Bergelson suggested that this concept may be related to ergodic theory of infinite invariant measures. The possibility of this connection remains an open problem.

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = \lim_{n\to\infty} \frac{N_n^Q(B_1,x)/Q_n^{(m)}}{N_n^Q(B_2,x)/Q_n^{(m)}} = \frac{1}{1} = 1. \tag{2.47}$$

□

Corollary 2.5.3. *If Q is a basic sequence and a real number x is Q-normal, then x is also Q-ratio normal.*

We will start by defining operations on the basic sequence Q.

Definition 2.5.4. *Given any basic sequence Q and a function $f : \{2, 3, \ldots\} \to \mathbb{R}$, let $f(Q)$ be the basic sequence $Q' = \{q'_n\}$ where $q'_n = \max(\lfloor f(q_n) \rfloor, 2)$.*

Example 2.5.5. *If Q is a basic sequence, then $2Q = \{2q_n\}$, $Q^2 = \{q_n^2\}$, and*

$$\log Q = \{\max(\lfloor \log q_n \rfloor, 2)\}. \tag{2.48}$$

In the proofs of some of the theorems in this thesis, we will want to consider the digits of the Q-Cantor series expansion of some real number x and form a new number by modifying the base while keeping the digits unchanged. This motivates the following definition:

Definition 2.5.6. *If $Q = \{q_n\}$ and $Q' = \{q'_n\}$ are two basic sequences such that $Q \leq Q'$ and*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{2.49}$$

is the Q-cantor series expansion of some $x \in [0, 1)$, then define

$$\Phi_Q^{Q'}(x) = \sum_{n=1}^{\infty} \frac{E_n}{q'_1 q'_2 \dots q'_n} \tag{2.50}$$

and

$$\pi_Q(x) = (E_1, E_2, \dots) \tag{2.51}$$

Thus, $\Phi_Q^{Q'} : [0, 1) \to [0, 1)$ is a non-increasing function that maps a real number in base Q to a real number whose Q'-cantor series expansion has the same digits. The function π_Q maps a real number to the digits of its Q-cantor series expansion.

We define the following function, which will be useful in proving some theorems:

Definition 2.5.7. *Given a basic sequence Q, we will define the k-normality index of some $x \in [0, 1)$ that is Q-ratio normal of order k as follows:*

$$I_Q^{(k)}(x) = \lim_{n\to\infty} \frac{N_n(B, x)}{Q_n^{(k)}} \tag{2.52}$$

where B is any block of length k. Since x is Q-ratio normal of order k, the choice of the block B is unimportant.

Lemma 2.5.8. *A real number x is Q-normal of order k if and only if for all $m \leq k$, we have*

$$I_Q^{(m)}(x) = 1. \tag{2.53}$$

2.6 Basic Examples

We now turn our attention to providing examples that demonstrate the notions of normality that we discussed. These examples only make use of Theorem 2.4.6. In later chapters, we will be able to consider more sophisticated constructions once we have proven Theorem 3.2.10 and Theorem 3.3.13.

2.6.1 Q-Distribution Normality Without Simple Q-Normality

It should first be noted that it is easier to construct a basic sequence Q and a real number x that is Q-distribution normal but not Q-normal than it is to construct an example of a real number that is Q-normal but not Q-distribution normal. [3] For a simple example, we set

$$(E_1, E_2, \ldots) = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots) \tag{2.54}$$

[3] We will work towards constructing an example of a number that is Q-normal but not Q-distribution normal in chapter 3 and chapter 5.

and

$$(q_1, q_2, \ldots) = (2, 3, 3, 4, 4, 4, 5, 5, 5, 5, \ldots). \tag{2.55}$$

Thus, the number $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 \ldots q_n}$ is not Q-normal since none of the digits $\{E_n\}$ are equal to 0. However, x is Q-distribution normal by Theorem 2.4.6 since the sequence

$$1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, \ldots \tag{2.56}$$

is uniformly distributed mod 1.

2.6.2 A Simply Q-Normal Number for a 1-Convergent Q

We let the digits E_n be given by

$$E = (0, 0, 1, 1, 2, 2, 3, 3, \ldots). \tag{2.57}$$

We set $q_n = \max(2, n(n-1))$, so

$$Q = (2, 2, 6, 12, 20, 30, 42, 56, \ldots). \tag{2.58}$$

Then, clearly,

$$\lim_{n \to \infty} Q_n^{(1)} = 1 + \sum_{n=2}^{\infty} \frac{1}{n(n+1)} = 2. \tag{2.59}$$

However, each digit occurs exactly twice. So,

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{2.60}$$

is simply Q-normal. We note that x is not Q-distribution normal as

$$\lim_{n \to \infty} T_{Q,n}(x) \leq \lim_{n \to \infty} \frac{n/2 + 1}{n(n-1)} = 0. \tag{2.61}$$

2.6.3 Example of a Number that is Simply Q-Normal and Q-Distribution Normal for a Non-Trivial Basic Sequence Q

Lemma 2.6.1. *The n^{th} digit of the sequence*

$$2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots \tag{2.62}$$

consisting of two 2s, followed by three 3s, four 4s, five 5s, etc. is described by

$$d_n = \lceil (-1 + \sqrt{9 + 8n})/2 \rceil \tag{2.63}$$

for $n = 1, 2, 3, \dots$.

In particular, $d_n = n$ if and only if

$$\left(\sum_{j=1}^{m} j \right) - 1 < n \leq \sum_{j=1}^{m+1} j. \tag{2.64}$$

Proof. For positive integers m, we set

$$Z_m = \left(\sum_{j=1}^{m} j\right) - 1 = \frac{m(m+1)}{2} - 1 = \frac{m^2+m-2}{2}. \tag{2.65}$$

We note that

$$d_{Z_m} = \lceil(-1 + \sqrt{9 + 4(m^2 + m - 2)}\rceil \tag{2.66}$$

$$= \lceil(-1 + \sqrt{(2m+1)^2})/2\rceil = \lceil(-1 + 2m + 1)/2\rceil = m. \tag{2.67}$$

Let $k < m + 1$ be a positive integer. We see that if $n = Z_m + k$, then

$$\frac{-1 + \sqrt{9 + 8n}}{2} = \frac{-1 + \sqrt{9 + 8(Z_m + k)}}{2} = \tag{2.68}$$

$$= \frac{-1 + \sqrt{(2m+1)^2 + 8k}}{2} > m.$$

But $d_{Z_m} = m$ and $d_{Z_{m+1}} = m + 1$, so

$$m = d_{Z_m} < d_{Z_m+k} \leq d_{Z_{m+1}} = m + 1. \tag{2.69}$$

Therefore, $d_{Z_m+k} = m + 1$. So if $Z_{m-1} + 1 \leq n \leq Z_m$, then $d_n = m$.

□

Theorem 2.6.2. *If*

$$E = (0, 1, 0, 1, 2, 0, 1, 2, 3, \ldots) \tag{2.70}$$

and

$$Q = (2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots), \tag{2.71}$$

then

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n} \tag{2.72}$$

is simply Q-normal and Q-distribution normal.

Proof. By Lemma 2.6.1, we see that

$$q_n = \lceil (-1 + \sqrt{9 + 8n})/2 \rceil. \tag{2.73}$$

Let b be any non-negative integer and let $B = (b)$. For $d \geq 0$, we let

$$C_d = (0, 1, \ldots, d) \tag{2.74}$$

and note that

$$E = 1C_1 1C_2 1C_3 \ldots. \tag{2.75}$$

We also note that the first occurence of the block C_d appears at the n^{th} digit of E where

$$n = 1 + \sum_{j=1}^{d-1} |C_j| = 1 + \sum_{j=1}^{d-1} (j+1) = \frac{d(d+1)}{2}. \tag{2.76}$$

Clearly, the first occurence of the block B in E is in C_b. The starting position of this block is, thus, $b(b+1)/2$ and ending position is at

$$\frac{b(b+1)}{2} + (b+1) - 1 = \frac{b(b+3)}{2}. \tag{2.77}$$

Thus, the first occurance of the block B will be between $E_{b(b+1)/2}$ and $E_{b(b+3)/2}$ with an additional occurrence in each C_d for $d > b$. It follows that if $m > b$ is a positive integer and $n = \frac{m(m+1)}{2}$, then

$$N_n(B, x) = m - b. \tag{2.78}$$

Therefore, if

$$\frac{m(m+1)}{2} \leq n \leq \frac{m(m+3)}{2}, \tag{2.79}$$

then

$$m - b \leq N_n(B, x) \leq m - b + 1. \tag{2.80}$$

We note that for some positive integer $m = m(n)$, we have

$$\sum_{k=1}^{n} \frac{1}{q_k} = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \ldots + \left(\frac{1}{m} + \ldots + \frac{1}{m}\right) + \left(\frac{1}{m+1} + \ldots + \frac{1}{m+1}\right), \tag{2.81}$$

where the last term has between 1 and $m + 1$ terms. Thus,

$$(m-1) + \frac{1}{m+1} \leq \sum_{k=1}^{n} \frac{1}{q_k} \leq (m-1) + 1, \tag{2.82}$$

so

$$m - 1 < \sum_{k=1}^{n} \frac{1}{q_k} \leq m. \tag{2.83}$$

Combining (2.80) and (2.83), we see that

$$\frac{m-b}{m} \leq \frac{N_n(B,x)}{\sum_{k=1}^{n} \frac{1}{q_k}} < \frac{m-b+1}{m-1}, \tag{2.84}$$

so

$$1 - \frac{b}{m} \leq \frac{N_n(B,x)}{\sum_{k=1}^{n} \frac{1}{q_k}} \leq 1 - \frac{b-2}{m-1}. \tag{2.85}$$

However, since

$$\frac{m(m+1)}{2} \leq n \leq \frac{m(m+3)}{2}, \tag{2.86}$$

we see that by similar reasoning to Lemma 2.6.1, we have

$$m = \lfloor (-1 + \sqrt{1+8n})/2 \rfloor. \tag{2.87}$$

Substituting (2.87) into (2.85), we see that

$$1 - \frac{b}{\lfloor (-1+\sqrt{1+8n})/2 \rfloor} \leq \frac{N_n(B,x)}{\sum_{k=1}^{n} \frac{1}{q_k}} \leq 1 - \frac{b-2}{\lfloor (-1+\sqrt{1+8n})/2 \rfloor - 1}. \tag{2.88}$$

Letting $n \to \infty$, we arrive at

$$\lim_{n \to \infty} \frac{N_n(B,x)}{\sum_{k=1}^{n} \frac{1}{q_k}} = 1, \tag{2.89}$$

so x is simply Q-normal.

To see that x is Q-distribution normal, we use Theorem 2.4.6 and note that the sequence $\{E_n/q_n\}_{n=1}^{\infty}$ is given by

$$(0/2, 1/2, 0/3, 1/3, 2/3, 0/4, 1/4, 3/4, \dots), \tag{2.90}$$

which is well known[4] to be uniformly distributed mod 1.

□

2.6.4 A Simply Q-Ratio Normal Number that is not Simply Q-Normal or Q-Distribution Normal

Example 2.6.3. *If*

$$E = (0, 1, 0, 1, 2, 0, 1, 2, 3, \dots) \tag{2.91}$$

and

$$Q = (4, 4, 6, 6, 6, 8, 8, 8, 8, \dots), \tag{2.92}$$

then

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{2.93}$$

[4] See Example 1.2.13.

is simply Q-ratio normal but not simply Q-normal or Q-distribution normal.

Proof. This follows directly from Lemma 2.5.8 and Theorem 2.6.2.

□

CHAPTER 3

GENERAL CONSTRUCTION THEOREMS FOR Q-NORMAL AND Q-DISTRIBUTION NORMAL NUMBERS FOR CERTAIN NON-TRIVIAL Q

In this chapter, we will prove two theorems that allow us to construct Q-normal and Q-distribution normal numbers for certain basic sequences Q. It should be noted that the primary use of Theorem 3.2.10 is to assist in constructing examples of numbers that are simultaneously Q-normal and Q-distribution normal. We will later prove Theorem 6.3.1, which will be far more powerful than Theorem 3.2.10 in the sense that it applies to a much larger class of basic sequences although the numbers it produces won't be Q-normal.

3.1 Basic Definitions and Conventions

Definition 3.1.1. [1] *A weighting* μ *is a collection of functions* $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \ldots$ *such that for all* k*,*

[1] [36] discusses normality in base 2 with respect to different weightings.

$$\sum_{j=0}^{\infty} \mu^{(1)}(j) = 1, \tag{3.1}$$

$$\mu^{(k)} : \{0, 1, 2, \ldots\}^k \to [0, 1], \tag{3.2}$$

and

$$\mu^{(k)}(b_1, b_2, \ldots, b_k) = \sum_{j=0}^{\infty} \mu^{(k+1)}(b_1, b_2, \ldots, b_k, j). \tag{3.3}$$

Definition 3.1.2. *The uniform weighting in base b is the collection λ_b of functions $\lambda_b^{(1)}, \lambda_b^{(2)}, \lambda_b^{(3)}, \ldots$ such that for all k and blocks B of length k in base b*

$$\lambda_b^{(k)}(B) = b^{-k}. \tag{3.4}$$

Definition 3.1.3. *Let p and b be positive integers such that $1 \leq p \leq b$. A weighting μ is (p, b)-uniform if for all k and blocks B of length k in base p, we have*

$$\mu^{(k)}(B) = \lambda_b^{(k)}(B) = b^{-k}. \tag{3.5}$$

Given blocks B and y, let $N(B, y)$ be the number of occurrences of the block B in the block y.

Definition 3.1.4. *Let ϵ be a real number such that $0 < \epsilon < 1$ and let k be a positive integer. Assume that μ is a weighting. A block of digits y is (ϵ, k, μ)-normal* [2] *if for all blocks B of length $m \leq k$, we have*

$$\mu^{(m)}(B)|y|(1-\epsilon) \leq N(B, y) \leq \mu^{(m)}(B)|y|(1+\epsilon). \tag{3.6}$$

For the rest of this chapter we use the following conventions freely and without comment. Given sequences of non-negative integers $\{l_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ with each $b_i \geq 2$ and a sequence of blocks $\{x_i\}_{i=1}^{\infty}$, we set

$$L_i = |l_1 x_1 \ldots l_i x_i| = \sum_{j=1}^{i} l_j |x_j|, \tag{3.7}$$

$$q_n = b_i \text{ for } L_{i-1} < n \leq L_i, \tag{3.8}$$

and

$$Q = \{q_n\}_{n=1}^{\infty}. \tag{3.9}$$

Moreover, if $(E_1, E_2, \ldots) = l_1 x_1 l_2 x_2 \ldots$, we set

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n}. \tag{3.10}$$

---

[2] Definition 3.1.4 is a generalization of the concept of (ϵ, k)-normality, originally due to Besicovitch [3].

Given $\{q_n\}_{n=1}^{\infty}$ and $\{l_i\}_{i=1}^{\infty}$, it is always assumed that x and Q are given by the formulas above.

Throughout the rest of this section, for a given n, the letter $i = i(n)$ is the unique integer satisfying

$$L_i < n \leq L_{i+1}. \tag{3.11}$$

3.2 Modular Friendly Families and Construction of Q-Distribution Normal Numbers

In this section, we will prove a theorem that allows us to construct a basic sequence Q such that the concatenation of strings of digits with a certain property will determine the digits of a real number that is Q-distribution normal.[3]

3.2.1 MFFs

Definition 3.2.1. *We say that* $V = \{(l_i, b_i, \epsilon_i)\}_{i=1}^{\infty}$ *is a modular friendly family (MFF) if* $\{l_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *are non-decreasing sequences of non-negative integers with* $b_i \geq 2$ *such that* $\{\epsilon_i\}_{i=1}^{\infty}$ *is a decreasing sequence of real numbers in* $(0, 1)$ *with* $\lim_{i\to\infty} \epsilon_i = 0$.

[3]This section appears in a joint work with C. Altomare [2].

Definition 3.2.2. *Let* $V = \{(l_i, b_i, \epsilon_i)\}_{i=1}^{\infty}$ *be an* MFF. *A sequence* $\{x_i\}_{i=1}^{\infty}$ *of* $(\epsilon_i, 1, \lambda_{b_i})$*-normal blocks of non-decreasing length with* $\lim_{i\to\infty} |x_i| = \infty$ *is said to be* V*-nice if the following two conditions hold:*

$$\frac{l_{i-1}}{l_i} \cdot \frac{|x_{i-1}|}{|x_i|} = o(1/i); \tag{3.12}$$

$$\frac{1}{l_i} \cdot \frac{|x_{i+1}|}{|x_i|} = o(1). \tag{3.13}$$

Throughout this section, we fix an MFF $V = \{(l_i, b_i, \epsilon_i)\}$ and a V-nice sequence of blocks $\{x_i\}$. Moreover, if $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,|x_i|})$, then y_i will be understood to stand for the sequence

$$\left\{\frac{x_{i,j}}{b_i}\right\}_{j=1}^{|x_i|}. \tag{3.14}$$

Given finite sequences $y_1, \ldots, y_t$ and non-negative integers $l_1, \ldots, l_t$, the notation $l_i y_i$ denotes the concatenation of l_i copies of y_i and the notation $l_1 y_1 \ldots l_t y_t$ denotes the concatenation of the sequences $l_1 y_1, \ldots, l_t y_t$.

Given a sequence $z = (z_1, \ldots, z_n)$ in $[0, 1)$ and $0 < \gamma \leq 1$, we define $A([0, \gamma), z)$ as

$$|\{i : 1 \leq i \leq n \text{ and } z_i \in [0, \gamma)\}|. \tag{3.15}$$

3.2.2 Discrepancy and V-Nice Sequences

Obviously, Definition 2.4.7 does not depend on the order that the z_i's are chosen in forming z. We will use this fact to reorder a sequence into an increasing sequence so that we may compute its star discrepancy with the following lemma from [23]:

Lemma 3.2.3. *If $0 \le z_1 \le \cdots \le z_n < 1$, then an upper bound for the star discrepancy $D_n^*(z_1, \ldots, z_n)$ is given by*

$$\frac{1}{2n} + \max_{1 \le i \le n} \left| z_i - \frac{2i-1}{2n} \right|. \tag{3.16}$$

We note that by Lemma 3.2.3,

$$\frac{1}{2n} \le D_n^*(z) \le 1 \tag{3.17}$$

for all sequences $z = (z_1, z_2, \ldots, z_n)$ with z_j in $[0, 1)$ for all j. It is well known that an infinite sequence $z = (z_1, \ldots, z_n, \ldots)$ is uniformly distributed mod 1 iff $\lim_{n\to\infty} D_n^*(z_1, \ldots, z_n) = 0$. This fact and Lemma 3.2.3 will allow us to prove Q-distribution normality of a well chosen Q and x by computing upper bounds on star discrepancies.

We recall the following lemma from [23]:

Lemma 3.2.4. *If t is a positive integer and for $1 \le j \le t$, z_j is a finite sequence in $[0, 1)$ with star discrepancy at most ϵ_j, then*

$$D^*(z_1 z_2 \cdots z_t) \leq \frac{\sum_{j=1}^t |z_j|\epsilon_j}{\sum_{j=1}^t |z_j|}. \tag{3.18}$$

Corollary 3.2.5. *If t is a positive integer and for $1 \leq j \leq t$, z_j is a finite sequence in $[0,1)$ with star discrepancy at most ϵ_j, then*

$$D^*(l_1 z_1 \cdots l_t z_t) \leq \frac{\sum_{j=1}^t l_j|z_j|\epsilon_j}{\sum_{j=1}^t l_j|z_j|}. \tag{3.19}$$

We note the following simple lemma:

Lemma 3.2.6. *Let U and U' be subsets of $\mathbb{R}$ such that U has a maximum M and a minimum*[4] *m. If $f : U \to U'$ is a monotone function, then $|f|$ has a maximum on U, which is either $f(m)$ or $f(M)$.*

Proof. Without loss of generality we may assume that f is increasing. Therefore f has a minimum at m and a maximum at M. If $f(m) \geq 0$, then $f(x) \geq 0$ for all x in U. This means that $|f| = f$ is increasing on U. Therefore $|f|$ attains a maximum at M. Similarly, if $f(M) \leq 0$, then $f(x) \leq 0$ for all x in U. This implies that $|f| = -f$ is decreasing on U. Therefore $|f|$ attains a maximum at m.

The remaining case is that $f(m) < 0 < f(M)$. Let U_A be the set of all x in U such that $f(x) \leq 0$ and let U_B be the set of all x in U such that $f(x) \geq 0$. Note that

[4] We say that a subset U of $\mathbb{R}$ has a maximum M if $M = \sup U \in U$. Similarly, U has a minimum m if $m = \inf U \in U$.

$|f|$ is decreasing on U_A and therefore $f|U_A$ has a maximum at m. Similarly, $|f|$ is increasing on U_B and therefore $f|U_B$ has a maximum at M. Since $U = U_A \cup U_B$, it follows that $|f|$ has a maximum at m or M.

□

Lemma 3.2.7. *Let $x = (E_1, \ldots, E_n)$ be an $(\epsilon, 1, \lambda_b)$-normal block in base b. If $y = (E_1/b, \ldots, E_n/b)$, then*

$$D^*(y) \leq \frac{1}{b} + \epsilon + \frac{1}{|x|}. \tag{3.20}$$

Proof. We wish to apply Lemma 3.2.3 to bound $D^*(y)$. However, Lemma 3.2.3 only applies to increasing sequences in $[0, 1)$, so we must first reorder the sequence y. Let $z = (z_1, \ldots, z_n)$ be the sequence of values $E_1/b, \ldots, E_n/b$ written in increasing order. We note that each z_t has the form j/b for some j in the set $\{0, 1, \ldots, b-1\}$. Since z is an increasing sequence, we may partition the integers from 1 to n into intervals $U_0, \ldots, U_{b-1}$ such that $z_t = j/b$ for t in U_j. We let m_j and M_j be the least and greatest elements of U_j, respectively.

By Lemma 3.2.3, we know that $D^*(z)$ is bounded above by

$$\frac{1}{2n} + \max_{1 \leq t \leq n} \left| z_t - \frac{2t-1}{2n} \right|. \tag{3.21}$$

Fix j. Note that $\frac{2t-1}{2n}$ is an increasing function of t on U_j and z_t is a constant function of t on U_j. Therefore $z_t - \frac{2t-1}{2n}$ is a decreasing function of t on U_j. So, for each j, Lemma 3.2.6 shows that the expression $\left|z_t - \frac{2t-1}{2n}\right|$ is maximized for $t = m_j$ or $t = M_j$.

By Definition 3.1.4, we know that x is $(\epsilon, 1, \lambda_b)$-normal iff for all j in $0, 1, \ldots, \ b-1$, we have

$$(1-\epsilon)\frac{1}{b}n \leq N((j), x) \leq (1+\epsilon)\frac{1}{b}n. \tag{3.22}$$

Thus,

$$m_j = \left(\sum_{t=0}^{j-1} N((t), x)\right) + 1 \geq \left(\sum_{t=0}^{j-1}(1-\epsilon)\frac{1}{b}n)\right) + 1 \tag{3.23}$$

$$= j(1-\epsilon)\frac{1}{b}n + 1 := \bar{m}_j$$

and

$$M_j = \sum_{t=0}^{j} N((t), x) \leq \sum_{t=0}^{j}(1+\epsilon)\frac{1}{b}n = (j+1)(1+\epsilon)\frac{1}{b}n := \bar{M}_j. \tag{3.24}$$

Letting

$$f_j(x) = \left(\frac{j}{b} - \frac{2x-1}{2n}\right), \tag{3.25}$$

we see that

$$D^*(y) \leq \frac{1}{2n} + \max_{1\leq t\leq n}\left|z_t - \frac{2t-1}{2n}\right| \tag{3.26}$$

$$= \frac{1}{2n} + \max_{0\leq j\leq b-1} \max\left(\left|f_j(m_j)\right|, \left|f_j(M_j)\right|\right).$$

Obviously, f is a monotone function. Note that

$$\bar{m}_j \leq m_j \leq M_j \leq \bar{M}_j. \tag{3.27}$$

By Lemma 3.2.6, the maximum of $|f_j(x)|$ on $[\bar{m}_j, \bar{M}_j]$ occurs at $\bar{m}_j$ or $\bar{M}_j$. Therefore

$$\max\{|f_j(m_j)|, |f_j(M_j)|\} \leq \max\{|f_j(\bar{m}_j)|, |f_j(\bar{M}_j)|\}. \tag{3.28}$$

Note that

$$|f_j(\bar{m}_j)| = \left| \frac{j}{b} - \frac{2\left(j(1-\epsilon)\frac{1}{b}n + 1\right) - 1}{2n} \right| \tag{3.29}$$

$$= \left| \frac{2nj - 2j(1-\epsilon)n + b}{2nb} \right| = \left| \frac{2nj\epsilon + b}{2nb} \right| = \frac{j\epsilon}{b} + \frac{1}{2n}.$$

Similarly, note that

$$|f_j(\bar{M}_j)| = \left| \frac{j}{b} - \frac{2(j+1)(1+\epsilon)\frac{1}{b}n - 1}{2n} \right| \tag{3.30}$$

$$= \left| \frac{2nj - 2nj - 2nj\epsilon - 2n - 2n\epsilon + b}{2nb} \right|$$

$$\leq \left| \frac{-2nj\epsilon - 2n - 2n\epsilon}{2nb} \right| + \left| \frac{b}{2nb} \right| = \frac{j+1}{b}\epsilon + \frac{1}{b} + \frac{1}{2n}.$$

Thus

$$\max(|f_j(\bar{m}_j)|, |f_j(\bar{M}_j)|) \leq \frac{j+1}{b}\epsilon + \frac{1}{b} + \frac{1}{2n} \tag{3.31}$$

and we see that

$$D^*(y) \leq \frac{1}{2n} + \max_{0 \leq j \leq b-1} \left(\frac{j+1}{b} \epsilon + \frac{1}{b} + \frac{1}{2n} \right) \tag{3.32}$$

$$= \frac{1}{2n} + \left(\frac{b}{b} \epsilon + \frac{1}{b} + \frac{1}{2n} \right) = \epsilon + \frac{1}{b} + \frac{1}{|x|}.$$

□

By Lemma 3.2.7, we know that $D^*(y_i)$ is bounded above by

$$\epsilon_i' := \frac{1}{b_i} + \epsilon_i + \frac{1}{|x_i|}. \tag{3.33}$$

Given a positive integer n, let $m = n - L_i$. Note that m can be written uniquely as $\alpha |x_{i+1}| + \beta$ with $0 \leq \alpha \leq l_{i+1}$ and $0 \leq \beta < |x_{i+1}|$. We define α and β as the unique integers satisfying these conditions.

Let $y = l_1 y_1 l_2 y_2 \ldots$ and recall that $D^*(z)$ is bounded above by 1 for all finite sequences z of real numbers in $[0, 1)$. By Corollary 3.2.5,

$$D_n^*(y) \leq f_i(\alpha, \beta) := \frac{l_1 |x_1| \epsilon_1' + \ldots + l_i |x_i| \epsilon_i' + (|x_{i+1}| \epsilon_{i+1}') \alpha + \beta}{l_1 |x_1| + \ldots + l_i |x_i| + |x_{i+1}| \alpha + \beta}. \tag{3.34}$$

Note that $f_i(\alpha, \beta)$ is a rational function in α and β. We consider the domain of f_i to be $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ where $\mathbb{R}_0^+$ is the set of all non-negative real numbers. Now we give an upper bound for $D_n^*(y)$. Since $D_n^*(y)$ is at most $f_i(\alpha, \beta)$, it is enough to bound $f_i(\alpha, \beta)$ from above on $[0, l_{i+1}] \times [0, |x_{i+1}|]$.

Lemma 3.2.8. *If* $l_i > 0$, $|x_i| > 0$, $\epsilon'_{i+1} < 1$,

$$l_1|x_1| + \ldots + l_{i-1}|x_{i-1}| > l_1|x_1|\epsilon'_1 + \ldots + l_{i-1}|x_{i-1}|\epsilon'_{i-1}, \tag{3.35}$$

$$\frac{|x_{i+1}|}{l_i|x_i|} < \frac{1-\epsilon'_i}{\epsilon'_{i+1}}, \tag{3.36}$$

and

$$(w, z) \in \{0, \ldots, l_{i+1}\} \times \{0, \ldots, |x_{i+1}| - 1\}, \tag{3.37}$$

then

$$f_i(w, z) < f_i(0, |x_{i+1}|) = \frac{l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + |x_{i+1}|}{l_1|x_1| + \ldots + l_i|x_i| + |x_{i+1}|}. \tag{3.38}$$

Proof. To bound $f_i(w, z)$, we first compute its partial derivatives $\frac{\partial f_i}{\partial z}(w, z)$ and $\frac{\partial f_i}{\partial w}(w, z)$. We will show that $\frac{\partial f_i}{\partial w}(w, z)$ is always negative, while $\frac{\partial f_i}{\partial z}(w, z)$ is always positive. Note that this is enough to prove Lemma 3.2.8 since $0 \leq \alpha$ and $\beta < |x_{i+1}|$.

First, we note that $f_i(w, z)$ is a rational function of w and z of the form

$$f_i(w, z) = \frac{C + Dw + Ez}{F + Gw + Hz}, \tag{3.39}$$

where

$$C = l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i,\ D = |x_{i+1}|\epsilon'_{i+1},\ E = 1, \tag{3.40}$$

$$F = l_1|x_1| + \ldots + l_i|x_i|,\ G = |x_{i+1}|, \text{ and } H = 1. \tag{3.41}$$

Therefore,

$$\frac{\partial f_i}{\partial w}(w,z) = \frac{D(F + Gw + Hz) - G(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{3.42}$$

$$= \frac{D(F + Hz) - G(C + Ez)}{(F + Gw + Hz)^2};$$

$$\frac{\partial f_i}{\partial z}(w,z) = \frac{E(F + Gw + Hz) - H(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{3.43}$$

$$= \frac{E(F + Gw) - H(C + Dw)}{(F + Gw + Hz)^2}.$$

Thus, the sign of $\frac{\partial f_i}{\partial w}(w,z)$ does not depend on w and the sign of $\frac{\partial f_i}{\partial z}(w,z)$ does not depend on z. We will show that $f_i(w,z)$ is a decreasing function of w by proving that

$$D(F + Hz) < G(C + Ez). \tag{3.44}$$

Similarly, we show that $f_i(w,z)$ is an increasing function of z by verifying that

$$E(F + Gw) > H(C + Dw). \tag{3.45}$$

Substituting the values in (3.41) into (3.44), we see that

$$|x_{i+1}|\epsilon'_{i+1}(l_1|x_1| + \ldots + l_i|x_i| + z) < |x_{i+1}|(l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + z). \tag{3.46}$$

Since $|x_{i+1}| \geq |x_i| > 0$, we may divide both sides by $|x_{i+1}|$ to obtain

$$l_1|x_1|\epsilon'_{i+1} + \ldots + l_i|x_i|\epsilon'_{i+1} + z\epsilon'_{i+1} < l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + z. \tag{3.47}$$

So, we only have to show (3.47), which is true since

$$\epsilon'_i = \frac{1}{b_i} + \epsilon_i + \frac{1}{|x_i|} \tag{3.48}$$

is decreasing and $\epsilon'_{i+1} < 1$.

Also, by substituting the values in (3.41) into (3.45), we see that

$$(l_1|x_1| + \ldots + l_{i-1}|x_{i-1}|) + (l_i|x_i| + w|x_{i+1}|) \tag{3.49}$$

$$> (l_1|x_1|\epsilon'_1 + \ldots + l_{i-1}|x_{i-1}|\epsilon'_{i-1}) + (l_i|x_i|\epsilon'_i + |x_{i+1}|\epsilon'_{i+1}).$$

By condition (3.35), we know that

$$l_1|x_1| + \ldots + l_{i-1}|x_{i-1}| > l_1|x_1|\epsilon'_1 + \ldots + l_{i-1}|x_{i-1}|\epsilon'_{i-1}. \tag{3.50}$$

Therefore, it is enough to show

$$l_i|x_i| + w|x_{i+1}| > l_i|x_i|\epsilon'_i + |x_{i+1}|\epsilon'_{i+1}. \tag{3.51}$$

Since $l_i|x_i|$ is the smallest possible value of $l_i|x_i| + w|x_{i+1}|$ for non-negative w, we need only show that

$$l_i|x_i| > l_i|x_i|\epsilon'_i + |x_{i+1}|\epsilon'_{i+1}. \tag{3.52}$$

By routine algebra, this is equivalent to

$$\frac{|x_{i+1}|}{l_i|x_i|} < \frac{1-\epsilon'_i}{\epsilon'_{i+1}}, \tag{3.53}$$

which is true by (3.36).

∎

Set

$$\bar{\epsilon}_i = f_i(0, |x_{i+1}|) = \frac{l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + |x_{i+1}|}{l_1|x_1| + \ldots + l_i|x_i| + |x_{i+1}|}. \tag{3.54}$$

Lemma 3.2.9. $\lim_{n\to\infty} \bar{\epsilon}_{i(n)} = 0.$

Proof. We write i for $i(n)$ throughout. For i large enough, we have

$$\frac{l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + |x_{i+1}|}{l_1|x_1| + \ldots + l_i|x_i| + |x_{i+1}|} < \frac{l_1|x_1|\epsilon'_1 + \ldots + l_i|x_i|\epsilon'_i + |x_{i+1}|}{l_i|x_i|} \tag{3.55}$$

$$= \frac{l_1|x_1|\epsilon'_1 + \ldots + l_{i-1}|x_{i-1}|\epsilon'_{i-1}}{l_i|x_i|} + \epsilon'_i + \frac{|x_{i+1}|}{l_i|x_i|}$$

$$< \frac{l_{i-1}|x_{i-1}|}{l_i|x_i|} \cdot i \cdot \epsilon'_{i-1} + \epsilon'_i + \frac{|x_{i+1}|}{l_i|x_i|},$$

where the last inequality uses the fact that ϵ'_i is decreasing. Note that

$$\frac{l_{i-1}|x_{i-1}|}{l_i|x_i|} \cdot i \cdot \epsilon'_{i-1} \to 0 \tag{3.56}$$

by (3.12),

$$\epsilon'_i = \frac{1}{b_i} + \epsilon_i + \frac{1}{|x_i|} \to 0, \tag{3.57}$$

and

$$\frac{|x_{i+1}|}{l_i|x_i|} \to 0 \tag{3.58}$$

by (3.13). Therefore, $\lim_{i\to\infty} \bar{\epsilon}_i = 0$. Since i can be made arbitrarily large by choosing large enough n, the lemma follows. □

3.2.3 Main Theorem

Theorem 3.2.10. *If V is an MFF and $\{x_i\}_{i=1}^{\infty}$ is a V-nice sequence, then x is Q-distribution normal.*

Proof. By Theorem 2.4.6, it is enough to show that $D_n^*(y) \to 0$. Since x_i is $(\epsilon_i, 1, \lambda_{b_i})$-normal, we see that $D^*(y_i) \le \epsilon'_i$ by Lemma 3.2.7. We wish to apply Corollary 3.2.5 and for large enough i apply Lemma 3.2.8 as well. To apply Lemma 3.2.8 for large i, we need only prove several inequalities for large i. In applying these inequalities, we will have $i = i(n)$ as defined in (3.11), so it is worth noting that i may be chosen as large as one likes by choosing a large enough n.

For the first inequality, note that $\lim_{i\to\infty} l_i|x_i| = \infty$. For large enough i, the product $l_i|x_i|$ is nonzero. For the second, we have $|x_i| > 0$. For the third inequality, $\epsilon'_{i+1} < 1$ for large enough i as $\epsilon'_i \to 0$. Next, since $l_{i-1}|x_{i-1}|$ asymptotically dominates $l_{i-1}|x_{i-1}|\epsilon'_{i-1}$, it follows that $l_1|x_1|+\ldots+l_{i-1}|x_{i-1}|$ asymptotically dominates $l_1|x_1|\epsilon'_1+\ldots+l_{i-1}|x_{i-1}|\epsilon'_{i-1}$ as well. In particular, for large enough i, we have

$$l_1|x_1| + \ldots + l_{i-1}|x_{i-1}| > l_1|x_1|\epsilon'_1 + \ldots + l_{i-1}|x_{i-1}|\epsilon'_{i-1}. \tag{3.59}$$

Finally, for the fifth inequality, noting that

$$\lim_{i\to\infty} \frac{|x_{i+1}|}{l_i|x_i|} = 0 \tag{3.60}$$

and that

$$\lim_{i\to\infty} \frac{1-\epsilon'_i}{\epsilon'_{i+1}} = \infty \tag{3.61}$$

since $\lim_{i\to\infty} \epsilon'_i = 0$, we see that

$$\frac{|x_{i+1}|}{l_i|x_i|} < \frac{1-\epsilon'_i}{\epsilon'_{i+1}} \tag{3.62}$$

for large i.

So, for large enough i, $D^*_n(y) \le \bar{\epsilon}_i$ and $\lim_{i\to\infty} \bar{\epsilon}_i = 0$. Thus, $\lim_{n\to\infty} D^*_n(y) = 0$.

□

3.3 Block Friendly Families and Construction of Q-Normal Numbers

In this section, we will prove a theorem that will allow us to construct Q-normal numbers for a certain class of basic sequences Q where q_n grows slowly.[5]

3.3.1 BFFs

For convenience, we define the notion of a block friendly family (BFF):

Definition 3.3.1. *A BFF is a sequence of* 6*-tuples*

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty} \tag{3.63}$$

with non-decreasing sequences of non-negative integers $\{l_i\}_{i=1}^{\infty}$*,* $\{b_i\}_{i=1}^{\infty}$*,* $\{p_i\}_{i=1}^{\infty}$ *and* $\{k_i\}_{i=1}^{\infty}$ *for which* $b_i \geq 2$*,* $b_i \to \infty$ *and* $p_i \to \infty$*, such that* $\{\mu_i\}_{i=1}^{\infty}$ *is a sequence of* (p_i, b_i)*-uniform weightings and* $\{\epsilon_i\}_{i=1}^{\infty}$ *strictly decreases to* 0*.*

Definition 3.3.2. *Let*

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty} \tag{3.64}$$

[5]This section appears in [30].

be a BFF. If $\lim k_i = K < \infty$*, then let* $R(W) = \{0, 1, 2, \ldots, K\}$*. Otherwise, let* $R(W) = \{0, 1, 2, \ldots\}$*. If* $\{x_i\}_{i=1}^{\infty}$ *is a sequence of blocks such that* $|x_i|$ *is non-decreasing and* x_i *is* (ϵ_i, k_i, μ_i)*-normal, then* $\{x_i\}_{i=1}^{\infty}$ *is said to be* W*-good if for all* k *in* R*,*

$$\frac{b_i^k}{\epsilon_{i-1} - \epsilon_i} = o\left(|x_i|\right); \tag{3.65}$$

$$\frac{l_{i-1}}{l_i} \cdot \frac{|x_{i-1}|}{|x_i|} = o(i^{-1} b_i^{-k}); \tag{3.66}$$

$$\frac{1}{l_i} \cdot \frac{|x_{i+1}|}{|x_i|} = o(b_i^{-k}). \tag{3.67}$$

3.3.2 Technical Lemmas

For this section, we will fix a BFF W and a W-good sequence $\{x_i\}$. For a given n, the letter $i = i(n)$ will always be understood to be the positive integer that satisfies $L_{i-1} < n \leq L_i$. This usage of i will be made frequently and without comment. Let $m = n - L_i$, which allows m to be written in the form

$$m = \alpha |x_{i+1}| + \beta \tag{3.68}$$

where α and β satisfy

$$0 \leq \alpha \leq l_{i+1} \text{ and } 0 \leq \beta < |x_{i+1}|. \tag{3.69}$$

Thus, we can write the first n digits of x in the form

$$l_1 x_1 l_2 x_2 \dots l_{i-1} x_{i-1} \ l_i x_i \ \alpha x_{i+1} \ 1y, \tag{3.70}$$

where y is the block formed from the first β digits of x_{i+1}.

Given a block B of length k in $R(W)$, we will first get upper and lower bounds on $N_n^Q(B,x)$, which will hold for all n large enough that $k \leq k_i$. This will allow us to bound

$$\left| \frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1 \right| \tag{3.71}$$

and show that

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1. \tag{3.72}$$

We will arrive at upper and lower bounds for $N_n^Q(B,x)$ by breaking the first n digits of x into three parts: the initial block $l_1 x_1 l_2 x_2 \dots l_{i-1} x_{i-1}$, the middle block $l_i x_i$ and the last block $\alpha x_{i+1} \ 1y$.

Lemma 3.3.3. *If $k \leq k_i$ and B is a block of length k in base $b \leq p_i$, then the following bounds hold:*

$$(1-\epsilon_i) b_i^{-k} |x_i| \leq N_{|x_i|}(B, x_i) \leq (1+\epsilon_i) b_i^{-k} |x_i|; \tag{3.73}$$

$$(1-\epsilon_{i+1}) b_{i+1}^{-k} \alpha |x_{i+1}| \leq N_m(B, l_{i+1} x_{i+1}) \leq (1+\epsilon_{i+1}) b_{i+1}^{-k} \alpha |x_{i+1}| + \beta + k\alpha. \tag{3.74}$$

Proof. Since x_i is (ϵ_i, k_i, μ_i)-normal and μ_i is (p_i, b_i)-uniform, it immediately follows that

$$(1-\epsilon_i)b_i^{-k}|x_i| \leq N_{|x_i|}(B, x_i) \leq (1+\epsilon_i)b_i^{-k}|x_i|. \tag{3.75}$$

We can estimate $N_m(B, l_{i+1}x_{i+1})$ by using the fact that $k \leq k_{i+1}$ and x_{i+1} is $(\epsilon_{i+1}, k_{i+1}, \mu_{i+1})$-normal so that

$$(1-\epsilon_{i+1})b_{i+1}^{-k}|x_{i+1}| \leq N_{|x_{i+1}|}(B, x_{i+1}) \leq (1+\epsilon_{i+1})b_{i+1}^{-k}|x_{i+1}|. \tag{3.76}$$

The upper bound for $N_m(B, l_{i+1}x_{i+1})$ is determined by assuming that B occurs at every location in the initial substring of length β of a copy of x_{i+1} and k times on each of the α boundaries. The lower bound is attained by assuming B never occurs in these positions, so

$$(1-\epsilon_{i+1})b_{i+1}^{-k}\alpha|x_{i+1}| \leq N_m(B, l_{i+1}x_{i+1}) \leq (1+\epsilon_{i+1})b_{i+1}^{-k}\alpha|x_{i+1}| + \beta + k\alpha. \tag{3.77}$$

□

We define the following quantity, which simplifies the statement of Lemma 3.3.6 and proof of Lemma 3.3.8:

Definition 3.3.4. *Given a positive integer n, we define*

$$\kappa_n = \left(L_{i-1} + k(l_i+1) + (1+\epsilon_i)b_i^{-k}l_i|x_i|\right) + ((1+\epsilon_{i+1})b_{i+1}^{-k}|x_{i+1}| + k)\alpha + \beta. \tag{3.78}$$

Lemma 3.3.5. *If $k \leq k_i$ and B is a block of length k in base $b \leq p_i$, then*

$$N_n^Q(B,x) \geq (1-\epsilon_i)b_i^{-k}l_i|x_i| + (1-\epsilon_{i+1})b_{i+1}^{-k}\alpha|x_{i+1}|. \tag{3.79}$$

Proof. For the lower bound, we consider the case where B never occurs in any of the blocks x_j or on the borders for $j < i$. By combining this with our estimates for $N_{|x_i|}(B,x_i)$ and $N_m(B, l_{i+1}x_{i+1})$ in Lemma 3.3.3, we get

$$N_n^Q(B,x) \geq (1-\epsilon_i)b_i^{-k}l_i|x_i| + (1-\epsilon_{i+1})b_{i+1}^{-k}\alpha|x_{i+1}|. \tag{3.80}$$

□

Lemma 3.3.6. *If $k \leq k_i$ and B is a block of length k in base $b \leq p_i$, then*

$$N_n^Q(B,x) \leq \kappa_n. \tag{3.81}$$

Proof. We can get an upper bound for $N_n^Q(B,x)$ by assuming that B occurs at every position in each of the x_j for $j < i$ and k times on each of the boundaries. Thus, we see that

$$\begin{aligned} N_n^Q(B,x) \leq (l_1|x_1| + \ldots + l_{i-1}|x_{i-1}|) + (1+\epsilon_i)b_i^{-k}l_i|x_i| \qquad (3.82) \\ +(1+\epsilon_{i+1})b_{i+1}^{-k}\alpha|x_{i+1}| + \beta + k(l_i+1+\alpha) \\ = \left(L_{i-1} + k(l_i+1) + (1+\epsilon_i)b_i^{-k}l_i|x_i|\right) + ((1+\epsilon_{i+1})b_{i+1}^{-k}|x_{i+1}| + k)\alpha + \beta. \end{aligned}$$

□

Due to the algebraic complexity of $Q_n^{(k)}$, it will be difficult to directly estimate (3.71). Thus, we will introduce a quantity close in value to $Q_n^{(k)}$ that will make this easier. Let

$$S_n^{(k)} = \sum_{j=1}^{i} b_j^{-k} l_j |x_j| + b_{i+1}^{-k} m \tag{3.83}$$

$$= b_1^{-k} l_1 |x_1| + b_2^{-k} l_2 |x_2| + \ldots + b_i^{-k} l_i |x_i| + b_{i+1}^{-k} m.$$

Lemma 3.3.7. $\lim_{n\to\infty} \frac{Q_n^{(k)}}{S_n^{(k)}} = 1.$

Proof. Let $s = \min\{t : k < |x_t|\}$. For $j \geq s$, define

$$\bar{Q}_j^{(k)} = \left(\sum_{t=1}^{l_j|x_j|-(k-1)} \frac{1}{b_j^k} \right) + \left(\frac{1}{b_j^{k-1} b_{j+1}} + \ldots + \frac{1}{b_j b_{j+1}^{k-1}} \right) \tag{3.84}$$

$$= \frac{l_j|x_j| - (k-1)}{b_j^k} + \sum_{t=1}^{k-1} \frac{1}{b_J^{k-1-t} b_{j+1}^t}.$$

Thus, by the definition of $Q_n^{(k)}$ and our choice of Q, we see that

$$Q_n^{(k)} = Q_{L_{s-1}}^{(k)} + \sum_{j=s}^{i} \bar{Q}_j^{(k)} + \sum_{t=L_i+1}^{n} \frac{1}{q_t q_{t+1} \cdots q_{t+k-1}}, \tag{3.85}$$

where the last summation will contain up to $l_{i+1}|x_{i+1}| - (k-1)$ terms identical to $\frac{1}{b_{j+1}^k}$ and up to $k-1$ terms of the form $\frac{1}{b_{i+1}^{k-1-t} b_{i+2}^t}$, depending on m.

Similarly to $\bar{Q}_j^{(k)}$, for $j \geq s$, define

$$\bar{S}_j^{(k)} = \sum_{u=1}^{l_j|x_j|} \frac{1}{b_j^k} = \frac{l_j|x_j|}{b_j^k}. \tag{3.86}$$

Thus,

$$S_n^{(k)} = S_{L_{s-1}}^{(k)} + \sum_{j=s}^{i} \bar{S}_j^{(k)} + \sum_{t=L_i+1}^{n} \frac{1}{q_t q_{t+1} \dots q_{t+k-1}}. \tag{3.87}$$

We note that almost all terms in $Q_n^{(k)}$ and $S_n^{(k)}$ are identical and are equal to $\frac{1}{b_j^k}$ for some j and will thus cancel out when we consider $S_n^{(k)} - Q_n^{(k)}$. The only corresponding terms that remain in the difference are thus of the form

$$\frac{1}{b_j^k} - \frac{1}{b_J^{k-1-t} b_{j+1}^t}. \tag{3.88}$$

However, each of these terms is non-negative as $\{b_i\}$ is a non-decreasing sequence. Therefore, $S_n^{(k)} - Q_n^{(k)}$ is non-decreasing in n and

$$S_n^{(k)} \geq Q_n^{(k)} \tag{3.89}$$

for all n. In particular, we arrive at the following bound:

$$S_n^{(k)} - Q_n^{(k)} \leq S_{L_{i+1}}^{(k)} - Q_{L_{i+1}}^{(k)} = \left(S_{L_{s-1}}^{(k)} - Q_{L_{s-1}}^{(k)}\right) + \sum_{j=s}^{i+1} \left(\bar{S}_j^{(k)} - \bar{Q}_j^{(k)}\right). \tag{3.90}$$

However,

$$\bar{S}_j^{(k)} - \bar{Q}_j^{(k)} = (l_j|x_j| - (k-1)) \left(\frac{1}{b_j^k} - \frac{1}{b_j^k}\right) + \sum_{t=1}^{k-1} \left(\frac{1}{b_j^k} - \frac{1}{b_J^{k-1-t} b_{j+1}^t}\right) \tag{3.91}$$

$$< (l_j|x_j| - (k-1)) \cdot 0 + \sum_{t=1}^{k}(1-0) = k.$$

If we let $r_s = \left(S^{(k)}_{L_{s-1}} - Q^{(k)}_{L_{s-1}}\right)$ and combine (3.90) and (3.91), then we find that

$$S_n^{(k)} - Q_n^{(k)} < r_s + \sum_{j=s}^{i+1} k = r_s + k(i+2-s). \tag{3.92}$$

Lastly, we note that

$$S_n^{(k)} = \sum_{j=1}^{i} b_j^{-k} l_j|x_j| + b_{i+1}^{-k} m \geq l_i|x_i|. \tag{3.93}$$

Using (3.92) and (3.93), we may now show that $\lim_{n\to\infty} \frac{Q_n^{(k)}}{S_n^{(k)}} = 1$:

$$\left|\frac{Q_n^{(k)}}{S_n^{(k)}} - 1\right| = \frac{S_n^{(k)} - Q_n^{(k)}}{S_n^{(k)}} < \frac{(r_s + k - ks) + ki}{l_i|x_i|} = \frac{r_s + k - ks}{l_i|x_i|} + k\frac{i}{l_i|x_i|}. \tag{3.94}$$

However, $r_s + k - ks$ is constant with respect to n and $|x_i| \to \infty$, so

$$\frac{r + k - ks}{l_i|x_i|} \to 0. \tag{3.95}$$

By (3.66),

$$k\frac{i}{l_i|x_i|} \to 0. \tag{3.96}$$

□

We will also use the following rational functions to estimate (3.71):

$$f_i(w,z) = \frac{\left(S^{(k)}_{L_{i-1}} + \epsilon_i b_i^{-k} l_i |x_i|\right) + \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}|\right) w + b_{i+1}^{-k} z}{S^{(k)}_{L_i} + (b_{i+1}^{-k}|x_{i+1}|) w + b_{i+1}^{-k} z}; \tag{3.97}$$

$$g_i(w,z) = \frac{\left(L_{i-1} + \epsilon_i b_i^{-k} l_i |x_i| + k(l_i+1)\right) + (\epsilon_{i+1} b_{i+1}^{-k}|x_{i+1}| + k) w + z}{S^{(k)}_{L_i} + (b_{i+1}^{-k}|x_{i+1}|) w + b_{i+1}^{-k} z}. \tag{3.98}$$

We consider the domain of f_i and g_i to be $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ where $\mathbb{R}_0^+$ is the set of all non-negative real numbers.

Lemma 3.3.8. *Let $k \in R(W)$ and let B be a block of length k in base b. If n is large enough so that $S_n^{(k)}/Q_n^{(k)} < 2$, $k \le k_i$ and $b \le p_i$, then*

$$\left| \frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1 \right| < 2 g_i(\alpha,\beta) + \frac{S_n^{(k)} - Q_n^{(k)}}{S_n^{(k)}}. \tag{3.99}$$

Proof. Using our lower bound from Lemma 3.3.5 on $N_n^Q(B,x)$,

$$\frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1 < 0. \tag{3.100}$$

We use (3.89) and arrive at the upper bound

$$\left| \frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1 \right| \le 1 - \frac{(1-\epsilon_i) b_i^{-k} l_i |x_i| + (1-\epsilon_{i+1}) b_{i+1}^{-k} \alpha |x_{i+1}|}{Q_n^{(k)}} \tag{3.101}$$

$$< \frac{S_n^{(k)} - ((1-\epsilon_i) b_i^{-k} l_i |x_i| + (1-\epsilon_{i+1}) b_{i+1}^{-k} \alpha |x_{i+1}|)}{Q_n^{(k)}} \cdot \frac{Q_n^{(k)}}{S_n^{(k)}} \cdot \frac{S_n^{(k)}}{Q_n^{(k)}}$$

$$< 2 \frac{S_n^{(k)} - ((1-\epsilon_i) b_i^{-k} l_i |x_i| + (1-\epsilon_{i+1}) b_{i+1}^{-k} \alpha |x_{i+1}|)}{S_n^{(k)}} = 2 f_i(\alpha,\beta).$$

Similarly to (3.101) and using our upper bound from Lemma 3.3.6 for $N_n^Q(B,x)$, we conclude that

$$\left|\frac{N_n^Q(B,x)}{Q_n^{(k)}}-1\right| \leq -1+\frac{\kappa_n}{Q_n^{(k)}}=\frac{\kappa_n-Q_n^{(k)}}{Q_n^{(k)}}=\frac{\kappa_n-S_n^{(k)}}{Q_n^{(k)}}+\frac{S_n^{(k)}-Q_n^{(k)}}{Q_n^{(k)}} \tag{3.102}$$

$$=\frac{\kappa_n-S_n^{(k)}}{Q_n^{(k)}}\cdot\frac{Q_n^{(k)}}{S_n^{(k)}}\cdot\frac{S_n^{(k)}}{Q_n^{(k)}}+\frac{S_n^{(k)}-Q_n^{(k)}}{Q_n^{(k)}}<2\frac{\kappa_n-S_n^{(k)}}{S_n^{(k)}}+\frac{S_n^{(k)}-Q_n^{(k)}}{Q_n^{(k)}}.$$

However,

$$\frac{\kappa_n-S_n^{(k)}}{S_n^{(k)}}=\frac{1}{S_n^{(k)}}\left(\left(\sum_{j=1}^{i-1}(1-j^{-k})l_j|x_j|+k(l_i+1)+\epsilon_i b_i^{-k}l_i|x_i|\right)\right. \tag{3.103}$$

$$\left.+(\epsilon_{i+1}b_{i+1}^{-k}|x_{i+1}|+k)\alpha+(1-b_{i+1}^{-k})\beta\right)$$

$$<\frac{\left(L_{i-1}+\epsilon_i b_i^{-k}l_i|x_i|+k(l_i+1)\right)+(\epsilon_{i+1}b_{i+1}^{-k}|x_{i+1}|+k)\alpha+\beta}{S_{L_i}+(b_{i+1}^{-k}|x_{i+1}|)\alpha+b_{i+1}^{-k}\beta}=g_i(\alpha,\beta).$$

So,

$$\left|\frac{N_n^Q(B,x)}{Q_n^{(k)}}-1\right|<\max\left(2f_i(\alpha,\beta),2g_i(\alpha,\beta)+\frac{S_n^{(k)}-Q_n^{(k)}}{S_n^{(k)}}\right). \tag{3.104}$$

Since the numerator of $g_i(\alpha,\beta)$ is clearly greater than the numerator of $f_i(\alpha,\beta)$ and their denominators are the same, we conclude that

$$f_i(\alpha,\beta)<g_i(\alpha,\beta). \tag{3.105}$$

Therefore,

$$\left|\frac{N_n^Q(B,x)}{Q_n^{(k)}}-1\right| < 2g_i(\alpha,\beta)+\frac{S_n^{(k)}-Q_n^{(k)}}{S_n^{(k)}}. \tag{3.106}$$

□

In light of Lemma 3.3.8, we will want to find a good bound for $g_i(w,z)$ where (w,z) ranges over values in $\{0,1,\ldots,l_{i+1}\}\times\{0,1,\ldots,|x_{i+1}|-1\}$.

Lemma 3.3.9. *If $k\in R(W)$, $l_i>0$, $|x_i|>4k$,*

$$|x_{i+1}|>\frac{kb_{i+1}^k}{\epsilon_i-\epsilon_{i+1}}, \tag{3.107}$$

and

$$(w,z)\in\{0,1,\ldots,l_{i+1}\}\times\{0,1,\ldots,|x_{i+1}|-1\}, \tag{3.108}$$

then

$$g_i(w,z)<g_i(0,|x_{i+1}|)=\frac{(L_{i-1}+\epsilon_i b_i^{-k}l_i|x_i|+k(l_i+1))+|x_{i+1}|}{S_{L_i}^{(k)}+b_{i+1}^{-k}|x_{i+1}|}. \tag{3.109}$$

Proof. We note that $g_i(w,z)$ is a rational function of w and z of the form

$$g_i(w,z)=\frac{C+Dw+Ez}{F+Gw+Hz}, \tag{3.110}$$

where

$$C = L_{i-1} + \epsilon_i b_i^{-k} l_i |x_i| + k(l_i + 1),\ D = \epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k,\ E = 1, \tag{3.111}$$

$$F = S_{L_i},\ G = b_{i+1}^{-k} |x_{i+1}|\ \text{, and}\ \ H = b_{i+1}^{-k}.$$

We will show that if we fix z, then $g_i(w, z)$ is a decreasing function of w and if we fix w, then $g_i(w, z)$ is an increasing function of z. To see this, we compute the partial derivatives:

$$\frac{\partial g_i}{\partial w}(w, z) = \frac{D(F + Gw + Hz) - G(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{3.112}$$

$$= \frac{D(F + Hz) - G(C + Ez)}{(F + Gw + Hz)^2};$$

$$\frac{\partial g_i}{\partial z}(w, z) = \frac{E(F + Gw + Hz) - H(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{3.113}$$

$$= \frac{E(F + Gw) - H(C + Dw)}{(F + Gw + Hz)^2}.$$

Thus, the sign of $\frac{\partial g_i}{\partial w}(w, z)$ does not depend on w and the sign of $\frac{\partial g_i}{\partial z}(w, z)$ does not depend on z. We will first show that $g_i(w, z)$ is an increasing function of z by verifying that

$$E(F + Gw) > H(C + Dw). \tag{3.114}$$

Let

$$S_i^* = b_{i+1}^{-k} L_{i-1} + \epsilon_i b_i^{-k} b_{i+1}^{-k} l_i |x_i| + b_{i+1}^{-k} k(l_i + 1). \tag{3.115}$$

Thus, (3.114) can be written as

$$S_{L_i} + \left[b_{i+1}^{-k} |x_{i+1}| w \right] > S_i^* + \left[b_{i+1}^{-k} (\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k) w \right]. \tag{3.116}$$

In order to show that $S_{L_i} > S_i^*$, we first note that

$$S_{L_i} = S_{L_{i-1}} + b_i^{-k} l_i |x_i|. \tag{3.117}$$

Since $S_{L_{i-1}} \geq b_{i+1}^{-k} L_{i-1}$, we need to show that

$$b_i^{-k} l_i |x_i| > b_{i+1}^{-k} (\epsilon_i b_i^{-k} l_i |x_i| + k(l_i + 1)). \tag{3.118}$$

However, by rearranging terms, (3.118) is equivalent to

$$|x_i| > \frac{l_i + 1}{l_i} \cdot \left(\frac{b_i}{b_{i+1}} \right)^k \cdot \frac{1}{1 - b_{i+1}^{-k} \epsilon_i} \cdot k. \tag{3.119}$$

Since $l_i > 0$, we know that $(l_i + 1)/l_i \leq 2$. Since $b_{i+1} \geq 2$ and $\epsilon_i < 1$, we know that $(1 - b_{i+1}^{-k} \epsilon_i)^{-1} < 2$. Additionally, $\{b_i\}$ non-decreasing implies $\left(\frac{b_i}{b_{i+1}} \right)^k \leq 1$. Therefore,

$$\frac{l_i + 1}{l_i} \cdot \left(\frac{b_i}{b_{i+1}} \right)^k \cdot \frac{1}{1 - b_{i+1}^{-k} \epsilon_i} \cdot k < 2 \cdot 1 \cdot 2 \cdot k = 4k. \tag{3.120}$$

However, $|x_i| > 4k$, so (3.119) is satisfied and $S_{L_i} > S_i^*$.

The last step to verifying (3.116) is to show that

$$b_{i+1}^{-k}|x_{i+1}|w \geq b_{i+1}^{-k}(\epsilon_{i+1}b_{i+1}^{-k}|x_{i+1}| + k)w. \tag{3.121}$$

However, this is equivalent to

$$|x_{i+1}|w \geq (\epsilon_{i+1}b_{i+1}^{-k}|x_{i+1}| + k)w. \tag{3.122}$$

Clearly, (3.122) is true if $w = 0$. If $w > 0$ we can cancel out the w term on each side and rewrite (3.122) as

$$|x_{i+1}| \geq \frac{1}{1 - b_{i+1}^{-k}\epsilon_{i+1}} \cdot k. \tag{3.123}$$

Similar to (3.119),

$$(1 - b_{i+1}^{-k}\epsilon_{i+1})^{-1}k \leq 2k < |x_i| < |x_{i+1}|. \tag{3.124}$$

Thus, (3.114) is satisfied and $g_i(w, z)$ is an increasing function of z.

Due to the difficulty of directly showing that $\frac{\partial g_i}{\partial w}(w, z) < 0$, we will proceed as follows: because the sign of $\frac{\partial g_i}{\partial w}(w, z)$ does not depend on w, we will know that $g_i(w, z)$ is decreasing in w if for each z

$$\lim_{w \to \infty} g_i(w, z) < g_i(0, z). \tag{3.125}$$

Since $g_i(w, z)$ is an increasing function of z, we know for all z that $g_i(0, 0) < g_i(0, z)$. Hence, it is enough to show that

$$\lim_{w \to \infty} g_i(w, z) < g_i(0, 0) \tag{3.126}$$

Since $\lim_{w \to \infty} g_i(w, z) = D/G$ and $g_i(0, 0) = C/F$, it is sufficient to show that $CG > DF$. We proceed as follows:

$$\left(L_{i-1} + \epsilon_i b_i^{-k} l_i |x_i| + k(l_i + 1)\right) b_{i+1}^{-k} |x_{i+1}| \tag{3.127}$$

$$> \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) S_{L_i} = \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) \left(S_{L_{i-1}} + b_i^{-k} l_i |x_i|\right)$$

$$\Leftrightarrow L_{i-1} b_{i+1}^{-k} |x_{i+1}| + \epsilon_i b_i^{-k} b_{i+1}^{-k} l_i |x_i| |x_{i+1}| + k b_{i+1}^{-k} (l_i + 1) |x_{i+1}| \tag{3.128}$$

$$> \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) S_{L_{i-1}} + \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) b_i^{-k} l_i |x_i|.$$

We will verify (3.128) by showing that

$$L_{i-1} b_{i+1}^{-k} |x_{i+1}| > \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) S_{L_{i+1}} \tag{3.129}$$

and

$$\epsilon_i b_i^{-k} b_{i+1}^{-k} l_i |x_i| |x_{i+1}| > \left(\epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k\right) b_i^{-k} l_i |x_i|. \tag{3.130}$$

Since $L_{i-1} > S_{L_{i-1}}$, in order to prove inequality (3.128), it is enough to show that

$$b_{i+1}^{-k} |x_{i+1}| > \epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k, \tag{3.131}$$

which is equivalent to

$$|x_{i+1}| > \frac{kb_{i+1}^k}{1-\epsilon_{i+1}}. \tag{3.132}$$

But $\epsilon_i < 1$, so

$$\frac{kb_{i+1}^k}{1-\epsilon_{i+1}} < \frac{kb_{i+1}^k}{\epsilon_i-\epsilon_{i+1}} < |x_{i+1}|. \tag{3.133}$$

To verify the second inequality we cancel the common term $b_i^{-k} l_i |x_i|$ on each side to get

$$\epsilon_i b_{i+1}^{-k} |x_{i+1}| > \epsilon_{i+1} b_{i+1}^{-k} |x_{i+1}| + k, \tag{3.134}$$

which is equivalent to

$$|x_{i+1}| > \frac{kb_{i+1}^k}{\epsilon_i-\epsilon_{i+1}}, \tag{3.135}$$

which is given in the hypotheses.

So, we may conclude that $g_i(w,z)$ is a decreasing function of w and an increasing function of z. We can thus achieve an upper bound on $g_i(w,z)$ by setting $w=0$ and $z=|x_{i+1}|$:

$$g_i(w,z) < g_i(0,|x_{i+1}|) = \frac{(L_{i-1} + \epsilon_i b_i^{-k} l_i |x_i| + k(l_i+1)) + |x_{i+1}|}{S_{L_i} + b_{i+1}^{-k}|x_{i+1}|}. \tag{3.136}$$

□

For convenience we will define

$$\epsilon_i' = \frac{(L_{i-1} + \epsilon_i b_i^{-k} l_i |x_i| + k(l_i+1)) + |x_{i+1}|}{S_{L_i} + b_{i+1}^{-k}|x_{i+1}|}. \tag{3.137}$$

Thus, under the conditions of Lemma 3.3.8 and Lemma 3.3.9,

$$\left| \frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1 \right| < 2\epsilon_i' + \frac{S_n^{(k)} - Q_n^{(k)}}{S_n^{(k)}}. \tag{3.138}$$

We will need to prove the following two lemmas in order to show that $\epsilon_i' \to 0$:

Lemma 3.3.10. *If $k \in R(W)$, then*

$$\lim_{i\to\infty} \frac{k(l_i+1)}{b_i^{-k} l_i |x_i|} = 0. \tag{3.139}$$

Proof.

$$\frac{k(l_i+1)}{b_i^{-k} l_i |x_i|} \le \frac{b_i^k 2k l_i}{l_i |x_i|} = \frac{b_i^k 2k}{|x_i|} \to 0, \tag{3.140}$$

by (3.65).

□

Lemma 3.3.11. *If $k \in R(W)$, then*

$$\lim_{i\to\infty} \frac{\sum_{j=1}^{i-2} l_j |x_j|}{b_i^{-k} l_i |x_i|} = 0. \tag{3.141}$$

Proof. Since $\{l_j\}$ and $\{|x_j|\}$ are non-decreasing sequences, we see that

$$\frac{\sum_{j=1}^{i-2} l_j |x_j|}{b_i^{-k} l_i |x_i|} < \frac{i l_{i-2}|x_{i-2}|}{b_i^{-k} l_i |x_i|} = \left(\frac{l_{i-2}|x_{i-2}|}{l_{i-1}|x_{i-1}|} \right) \cdot \left(i b_i^k \frac{l_{i-1}|x_{i-1}|}{l_i |x_i|} \right) \tag{3.142}$$

By (3.66), we see that

$$\frac{l_{i-2}|x_{i-2}|}{l_{i-1}|x_{i-1}|} \to 0 \tag{3.143}$$

and

$$ib_i^k \frac{l_{i-1}|x_{i-1}|}{l_i|x_i|} \to 0. \tag{3.144}$$

□

Lemma 3.3.12. *If $k \in R(W)$, then*

$$\lim_{i\to\infty} \epsilon_i' = 0. \tag{3.145}$$

Proof.

$$\epsilon_i' = \frac{\sum_{j=1}^{i-1} l_j|x_j| + \epsilon_i b_i^{-k} l_i|x_i| + |x_{i+1}| + k(l_i+1)}{\sum_{j=1}^{i-1} j^{-k} l_j|x_j| + b_i^{-k} l_i|x_i| + b_{i+1}^{-k}|x_{i+1}|} \tag{3.146}$$

$$< \frac{\sum_{j=1}^{i-1} l_j|x_j| + \epsilon_i b_i^{-k} l_i|x_i| + |x_{i+1}| + k(l_i+1)}{b_i^{-k} l_i|x_i|}$$

$$= \frac{\sum_{j=1}^{i-2} l_j|x_j|}{b_i^{-k} l_i|x_i|} + \frac{l_{i-1}|x_{i-1}|}{b_i^{-k} l_i|x_i|} + \epsilon_i + \frac{|x_{i+1}|}{b_i^{-k} l_i|x_i|} + \frac{k(l_i+1)}{b_i^{-k} l_i|x_i|}.$$

However, each of these terms converges to 0 by (3.66), (3.67), Lemma 3.3.10 and Lemma 3.3.11.

□

3.3.3 Main Theorem

Theorem 3.3.13. *Let W be a BFF and $\{x_i\}_{i=1}^{\infty}$ a W-good sequence. If $k \in R(W)$, then x is Q-normal of order k. If $k_i \to \infty$, then x is Q-normal.*

Proof. Let b be a positive integer, $k \in R(W)$, and let B be an arbitrary block of length k in base b. Since

$$|x_i| = \omega\left(\frac{b_i^k}{\epsilon_{i-1} - \epsilon_i}\right), \tag{3.147}$$

there exists n large enough so that $|x_i|$ and $|x_{i+1}|$ satisfy the hypotheses of Lemma 3.3.9. Additionally, assume that n is large enough so that $k \leq k_i$, $b \leq p_i$ and $S_n^{(k)}/Q_n^{(k)} < 2$. Thus, by Lemma 3.3.8 and Lemma 3.3.9,

$$\left|\frac{N_n^Q(B,x)}{Q_n^{(k)}} - 1\right| < 2\epsilon_i' + \frac{S_n^{(k)} - Q_n^{(k)}}{S_n^{(k)}}. \tag{3.148}$$

By Lemma 3.3.7, we have

$$\lim_{n\to\infty} \frac{S_n^{(k)} - Q_n^{(k)}}{S_n^{(k)}} = 0. \tag{3.149}$$

However, $\lim_{n\to\infty} i = \infty$. So, by Lemma 3.3.12,

$$\lim_{n\to\infty} \epsilon_i' = 0. \tag{3.150}$$

Thus, by (3.148), (3.149) and (3.150),

$$\lim_{n\to\infty}\left|\frac{N_n^Q(B,x)}{Q_n^{(k)}}-1\right|=0. \tag{3.151}$$

So,

$$\lim_{n\to\infty}\frac{N_n^Q(B,x)}{Q_n^{(k)}}=1 \tag{3.152}$$

and we may conclude that x is Q-normal of order k.

□

CHAPTER 4

CONSTRUCTION OF A NUMBER THAT IS Q-NORMAL AND Q-DISTRIBUTION NORMAL FOR A CERTAIN NON-TRIVIAL Q

In this chapter[1], we will construct a specific example of a number that is Q-normal and Q-distribution normal for a certain Q. This number is constructed somewhat similarly to the Champernowne number.

Let $C_{b,w}$ be the block formed by concatenating all the blocks of length w in base b in lexicographic order. For example,

$$C_{3,2} = 1(0,0)1(0,1)1(0,2)1(1,0)1(1,1)1(1,2)1(2,0)1(2,1)1(2,2) \tag{4.1}$$

$$= (0,0,0,1,0,2,1,0,1,1,1,2,2,0,2,1,2,2).$$

Lemma 4.0.14. *For all $b \geq 2$ and $w \geq 1$,*

$$|C_{b,w}| = wb^w. \tag{4.2}$$

[1]This chapter appears in [30].

Proof. Since there will be b^w such blocks and each is of length w, we arrive at $|C_{b,w}| = wb^w$.

□

We will show in Lemma 4.0.15 and Lemma 4.0.16 that $C_{b,w}$ is (ϵ, K, μ)-normal for appropriate choices of ϵ, K and μ. We will use this information to construct a good sequence and apply Theorem 3.3.13 to arrive at our Q-normal number.

Lemma 4.0.15. *Let* $n = |C_{b,w}|$.

1. *Suppose that* $1 \leq k \leq w$ *and* B *is a block of length* k *in base* b. *Then*

$$(w - k + 1)b^{w-k} \leq N_n(B, C_{b,w}) \leq wb^{w-k}. \tag{4.3}$$

2. *If* B *is a block in base* $b' > b$ *and* B *is not a block in base* b, *then*

$$N_n(B, C_{b,w}) = 0. \tag{4.4}$$

Proof. The second case is trivial as $C_{b,w}$ is a block in base b.

Suppose that B is a block of length k in base b. Let

$$C_1, C_2, \ldots, C_{b^w} \tag{4.5}$$

be the blocks of length w in base b written in lexicographic order. Thus,

$$C_{b,w} = 1C_1 1C_2 \ldots 1C_{b^w}. \tag{4.6}$$

We will achieve a lower bound for $N_n(B, C_{b,w})$ by counting the number of occurrences of B inside the blocks C_i. In other words, we will use the estimate

$$\sum_{i=1}^{b^w} N_w(B, C_i) \leq N_n(B, C_{b,w}). \tag{4.7}$$

For each j such that $1 \leq j \leq w-k+1$, we will count the number of i such that there is a copy of B at position j in C_i. Such j will correspond to copies of B that don't straddle the boundary between C_i and C_{i+1}. Since B is of length k and each C_i is of length w, there will be $w-k$ positions that are undetermined and can take on any of the values $0, 1, \ldots, b-1$. Thus, there are b^{w-k} values of i such that a copy of B is at position j of C_i. Since there are $w-k+1$ choices for j, we arrive at the estimate

$$(w-k+1)b^{w-k} \leq N_n(B, C_{b,w}). \tag{4.8}$$

In order to arrive at an upper bound for $N_n(B, C_{b,w})$, we will find an upper bound for the number of copies of B that straddle the boundaries between the blocks C_i and C_{i+1} and add this to the number of copies of B that occur inside each of the C_i. These will correspond to a copy of B starting at position j of C_i for $w-k+2 \leq j \leq w$ and finishing in C_{i+1}. Given a block $D = (d_1, d_2, \ldots, d_t)$ in base b, define

$$\phi(D) = d_1 b^{t-1} + d_2 b^{t-2} + \ldots + d_{t-1} b + d_t. \tag{4.9}$$

Thus,

$$\phi(C_{i+1}) = \phi(C_i) + 1. \tag{4.10}$$

If a copy of B is at position j of C_i, then the first $w-j+1$ digits of B are at the end of C_i and the last $k-(w-j+1)$ digits of B are at the beginning of C_{i+1}. However, the last $w-j+1$ digits of C_{i+1} are uniquely determined by B from (4.10). The first $k-(w-j+1)$ have already directly been determined by B so there are at most

$$w-(w-j+1)-(k-(w-j+1)) = w-k \tag{4.11}$$

undetermined digits of C_{i+1}, giving b^{w-k} ways to pick C_{i+1}. Additionally, there are $k-1$ positions j that straddle the boundaries giving an upper bound of $(k-1)b^{w-k}$ copies of B that lie on the boundaries. Thus,

$$N_n(B, C_{b,w}) \leq (w-k+1)b^{w-k} + (k-1)b^{w-k} = wb^{w-k}. \tag{4.12}$$

□

Lemma 4.0.16. *If $K < w$ and $\epsilon = \frac{K}{w}$, then $C_{b,w}$ is (ϵ, K, λ_b)-normal.*

Proof. Let $n = |C_{b,w}| = wb^w$ and let B be a block of length $k \leq K$ in base b. We first note that

$$(w-k+1)b^{w-k} = b^{-k}n\frac{(w-k+1)b^w}{n} \tag{4.13}$$

$$= \lambda_b^{(k)}(B)n\left(1-\frac{k-1}{w}\right) > \lambda_b^{(k)}(B)n\left(1-\frac{K}{w}\right).$$

We also note that

$$wb^{w-k} = b^{-k}n\frac{wb^w}{n} = \lambda_b^{(k)}(B)n(1+0) < \lambda_b^{(k)}(B)n\left(1+\frac{K}{w}\right). \tag{4.14}$$

Thus, by Lemma 4.0.15, (4.13), and (4.14),

$$\lambda_b^{(k)}(B)n\left(1-\frac{K}{w}\right) < N_n(B, C_{b,w}) < \lambda_b^{(k)}(B)n\left(1+\frac{K}{w}\right). \tag{4.15}$$

So, $C_{b,w}$ is (ϵ, K, λ_b)-normal. □

Theorem 4.0.17. *Let $x_1 = (0,1)$, $b_1 = 2$ and $l_1 = 0$. For $i \geq 2$, let $x_i = C_{i,i^2}$, $b_i = i$ and $l_i = i^{3i}$. If x and Q are defined as in Theorem 3.3.13, then x is Q-normal.*

Proof. Let $\epsilon_1 = 3/5$, $k_1 = 1$, $p_1 = 2$ and $\mu_1 = \lambda_2$. For $i \geq 2$, let $\epsilon_i = 1/i$, $k_i = i$, $p_i = b_i$, $\mu_i = \lambda_i$ and

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}. \tag{4.16}$$

Thus, since $x_i = C_{b,w}$ where $b = i$ and $w = i^2$, by Lemma 4.0.16, x_i is $(\epsilon_i, k_i, \lambda_{b_i})$-normal.

In order to show that $\{x_i\}$ is a W-good sequence we need to verify (3.65), (3.66) and (3.67). Since $k_i \to \infty$, we let k be an arbitrary positive integer. We will make repeated use of the fact that

$$|x_i| = i^2 \cdot i^{i^2}. \tag{4.17}$$

We first verify (3.65):

$$\lim_{i\to\infty}|x_i| \Bigg/ \left(\frac{i^k}{\frac{1}{i-1}-\frac{1}{i}}\right) = \lim_{i\to\infty}\frac{i^2\cdot i^{i^2}}{i^k\cdot i(i-1)} = \infty. \tag{4.18}$$

We next verify (3.66). Since $l_{i-1}/l_i < 1$, $(i-1)^2/i^2 < 1$ and $(1-1/i)^{i^2} < e^{-i}$,

$$\lim_{i\to\infty}\frac{\frac{l_{i-1}}{l_i}\cdot\frac{x_{i-1}}{x_i}}{i^{-1}i^{-k}} \leq \lim_{i\to\infty} i^{k+1}\cdot 1\cdot\frac{(i-1)^2}{i^2}\cdot\frac{(i-1)^{(i-1)^2}}{i^{i^2}} \tag{4.19}$$

$$\leq \lim_{i\to\infty} i^{k+1}\cdot 1\cdot(1-1/i)^{i^2}\cdot(i-1)^{-2i+1} \leq \lim_{i\to\infty} i^{k+1}e^{-i}(i-1)^{-2i+1} = 0.$$

Lastly, we will verify (3.67). Since $(i+1)^2/i^2 \leq 2$, $(1+1/i)^{2i} < e^2$, and $(1+1/i)^{i^2} < e^i$, we see that

$$\lim_{i\to\infty}\frac{\frac{1}{l_i}\cdot\frac{|x_{i+1}|}{|x_i|}}{i^{-k}} = \lim_{i\to\infty} i^{-3i+k}\cdot\frac{(i+1)^2}{i^2}\cdot\frac{(i+1)^{(i+1)^2}}{i^{i^2}} \tag{4.20}$$

$$\leq \lim_{i\to\infty} i^{-3i+k}\cdot 2\cdot(1+1/i)^{i^2}\cdot(i+1)^{(2i+1)}$$

$$\leq \lim_{i\to\infty} 2e^i(1+1/i)^{2i}i^{-i+k}(i+1) \leq \lim_{i\to\infty} 2(i+1)e^{i+2}\cdot i^{-i+k} = 0.$$

Since λ_{b_i} is (p_i, b_i)-uniform, $\{x_i\}$ is a W-good sequence and by Theorem 3.3.13 x is Q-normal. □

Theorem 4.0.18. *Let $x_1 = (0,1)$, $b_1 = 2$ and $l_1 = 0$. If for $i \geq 2$, we let $x_i = C_{i,i^2}$, $b_i = i$, and $l_i = i^{3i}$, then x is Q-distribution normal.*

Proof. We let $\epsilon_1 = 3/5$. For $i \geq 2$, we let $\epsilon_i = 1/i$. By [30], we know that x_i is $(\epsilon_i, 1, \lambda_{b_i})$-normal. It is enough to show (3.12) and (3.13). Note that trivially, (3.66) implies (3.12) and (3.67) implies (3.13). Moreover, it was proven in [30] that (3.66) and (3.67) hold. Therefore x_i is V-nice. So, by Theorem 3.2.10, we see that x is Q-distribution normal as claimed. □

CHAPTER 5

CONSTRUCTION OF A NUMBER THAT IS Q-NORMAL AND NOT Q-DISTRIBUTION NORMAL

In this chapter[1], we construct a specific example of a basic sequence Q and a real number x such that x is Q-normal yet not Q-distribution normal. Moreover, the Q-distribution normality of x fails in a particularly strong fashion. Not only does $\{T_{Q,n}(x)\}_{n=1}^{\infty}$ fail to be uniformly distributed mod 1, but $\lim_{n\to\infty} T_{Q,n}(x) = 0$.

The construction of a basic sequence Q and a real number x that is Q-normal but not Q-distribution normal is far more difficult. We will first need to define a sequence of weightings $\nu_1, \nu_2, \ldots$ and blocks $P_{b,w}$. After this, we will prove a number of technical lemmas from which the above stated facts follow.

If we let b be a positive integer, then we define

$$\nu_b^{(1)}(j) = \begin{cases} \frac{1}{2^b} & \text{if } 0 \le j \le b-1 \\ \frac{2^b-b}{2^b} & \text{if } j = b \\ 0 & \text{if } j > b \end{cases} . \tag{5.1}$$

[1]This chapter appears in a joint work with C. Altomare [2]

For a block $B = (b_1, \ldots, b_k)$, we define

$$\nu_b^{(k)}(B) = \prod_{j=1}^{k} \nu_b^{(1)}(b_j). \tag{5.2}$$

Note that ν_b is a $(b, 2^b)$-uniform weighting. Since each $\nu_b^{(k)}$ is determined by $\nu_b^{(1)}$, we refer to $\nu_b^{(k)}$ as ν_b throughout.

Next, we define $P_{b,w}$. Let b and w be positive integers. Denote by $P_1, P_2, \ldots, P_{(b+1)^w}$ the blocks in base $b+1$ of length w written in lexicographic order. Let

$$P_{b,w} = 2^{bw}\nu_b(P_1)P_1 2^{bw}\nu_b(P_2)P_2 \cdots 2^{bw}\nu_b(P_{(b+1)^w})P_{(b+1)^w}. \tag{5.3}$$

In order to get upper and lower bounds for $N(B, P_{b,w})$ for a base $b+1$ block B, we need to calculate the length of $P_{b,w}$. We must first compute $\nu_b(B)$. This calculation is facilitated by the following definition:

Definition 5.0.19. *Given a base $b+1$ block $B = (b_1, \ldots, b_w)$, set*

$$g_b(B) = |\{j : b_j = b\}|. \tag{5.4}$$

Lemma 5.0.20. *If B is a base $b+1$ block of length w, then*

$$2^{bw}\nu_b(B) = \left(2^b - b\right)^{g_b(B)}. \tag{5.5}$$

Proof.

$$2^{bw}\nu_b(B) = 2^{bw} \cdot \left(\frac{2^b - b}{2^b}\right)^{g_b(B)} \cdot \left(\frac{1}{2^b}\right)^{w - g_b(B)} = \left(2^b - b\right)^{g_b(B)}. \tag{5.6}$$

□

Lemma 5.0.21. *If b and w are positive integers, then*

$$|P_{b,w}| = w \cdot 2^{bw}. \tag{5.7}$$

Proof. Fix m such that $0 \leq m \leq w$. Clearly, the number of i such that $g_b(P_i) = m$ is $\binom{w}{m} b^{w-m}$. By Lemma 5.0.20 and the definition of $P_{b,w}$, each block P_i is concatenated $\left(2^b - b\right)^m$ times in forming $P_{b,w}$, with each one of these blocks having length w. It follows that the total number of digits contained in all copies of each block P_i is

$$w \cdot \binom{w}{m} \cdot \left(2^b - b\right)^m \cdot b^{w-m}. \tag{5.8}$$

In order to obtain an expression for the length $|P_{b,w}|$ of $P_{b,w}$, we sum over all possible values of m. Therefore

$$|P_{b,w}| = \sum_{m=0}^{w} w \cdot \binom{w}{m} \cdot \left(2^b - b\right)^m \cdot b^{w-m} = w \cdot (2^b - b + b)^w = w \cdot 2^{bw} \tag{5.9}$$

by the binomial theorem.

□

Lemma 5.0.22. *Let w, k, and b be positive integers such that $k \leq w$. If B is a block of length k in base $b+1$, then*

$$N(B, P_{b,w}) \geq (w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)}. \tag{5.10}$$

Proof. $P_{b,w}$ is defined as the concatenation of copies of the blocks $P_i = (p_{i,1}, \ldots, p_{i,w})$. In order to get this lower bound on $N(B, P_{b,w})$ it is enough to show that the number of occurrences of B inside some copy of some P_i is exactly

$$(w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)}. \tag{5.11}$$

Consider a block P_i containing B. Since B starts at position s in P_i for some s such that $1 \leq s \leq w-k+1$, this leaves exactly $w-k$ digits of P_i undetermined. Let

$$M = \{j : p_{i,j} = b \text{ and } j \notin [s, s+k-1]\} \tag{5.12}$$

and let $m = |M|$. Thus m is the number of times that the block P_i takes on the value b outside of B. Clearly $0 \leq m \leq w-k$.

Note that since m is the number of b's in P_i outside B and $g_b(B)$ is number of b's in P_i inside B, we see that $g_b(B) + m$ is the total number of b's in P_i. By Lemma 5.0.20, exactly

$$\left(2^b - b\right)^{g_b(B)+m} \tag{5.13}$$

copies of P_i are concatenated in forming $P_{b,w}$. Let S be the total number of occurrences of B in blocks P_i that have exactly m occurrences of b outside of B. Since there are $w-k+1$ choices for s, $\binom{w-k}{m}$ choices for M, $w-k-m$ undetermined positions after choosing M, and each undetermined position has b possible values, we see that

$$S = (w-k+1) \cdot \binom{w-k}{m} \cdot \left(2^b - b\right)^{g_b(B)+m} \cdot b^{w-k-m}. \tag{5.14}$$

So, to count the number of times that B occurs in $P_{b,w}$, we sum over m from 0 to $w-k$ and use the binomial theorem to get

$$N(B, P_{b,w}) \geq \sum_{m=0}^{w-k} (w-k+1) \cdot \binom{w-k}{m} \cdot \left(2^b - b\right)^{g_b(B)+m} \cdot b^{w-k-m} \tag{5.15}$$

$$= (w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \sum_{m=0}^{w-k} \binom{w-k}{m} \cdot \left(2^b - b\right)^{m} \cdot b^{w-k-m}$$

$$= (w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot (2^b)^{w-k}.$$

□

We will need the following definition in the proof of Lemma 5.0.24.

Definition 5.0.23. *Let B, C, and D be blocks with $|B| \geq 2$. Suppose that $B = (b_1, \ldots, b_k)$, $C = (c_1, \ldots, c_m)$, and $D = (d_1, \ldots, d_t)$. We say that B straddles C and D if there is an integer s in $[2, k]$, an integer e in $[1, m]$, and an integer f in [1,t] such that $(b_1, \ldots, b_{s-1}) = (c_e, \ldots, c_m)$ and $(b_s, \ldots, b_k) = (d_1, \ldots, d_f)$.*

Intuitively, B straddles C and D if B starts in C and ends in D. It is worth noting that with this definition, if $|B| = 1$ then there are no choices of C and D for which B straddles C and D.

Lemma 5.0.24. *Let w, k, and b be positive integers such that $k \leq w$. If B is a block of length k in base $b+1$, then*

$$N(B, P_{b,w}) \leq w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)} + (k-1)(b+1)^w. \tag{5.16}$$

Proof. Note that $P_{b,w}$ has the form $1C_1 1C_2 \cdots 1C_t$ for some length w blocks $C_1, \ldots, C_t$ and some t. In proving Lemma 5.0.22, we showed that the number of occurrences of B in some P_i is exactly

$$(w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)}. \tag{5.17}$$

When (5.17) is added to an upper bound for the number of occurrences of B in $P_{b,w}$ that straddle C_i and C_{i+1} for some i, we obtain an upper bound for $N(B, P_{b,w})$. Consider a block B that straddles the blocks

$$C_i = (c_{i,1}, \ldots, c_{i,w}) \text{ and } C_{i+1} = (c_{i+1,1}, \ldots, c_{i+1,w}) \tag{5.18}$$

for some i. In this case, B starts at position s in C_i for some s such that $w-k+2 \leq s \leq w$. Define

$$B_1 = (c_{i,s}, \ldots, c_{i,w}), B_2 = (c_{i+1,1}, \ldots, c_{i+1,k-w+s-1}), \tag{5.19}$$

$$B_2' = (c_{i,1}, \ldots, c_{i,k-w+s-1}), \text{ and } B_1' = (c_{i+1,s}, \ldots, c_{i+1,w}). \tag{5.20}$$

Note that since $k \leq w$, these four sets are pairwise disjoint.

If $C_i = C_{i+1}$, then $B_1 = B_1'$ and $B_2 = B_2'$. Since the blocks B_1 and B_2' are both contained in C_i and

$$|B_1| + |B_2'| = |B_1| + |B_2| = |B| = k, \tag{5.21}$$

we see that k positions of C_i are determined. Thus there are $w - k$ undetermined positions in C_i.

Let

$$M = \{j : c_{i,j} = b \text{ and } j \notin [k - w + s, s - 1]\} \tag{5.22}$$

and let $m = |M|$. Therefore m is the number of times the block C_i takes on the value b outside $B_2' \cup B_1$. We again note that $0 \leq m \leq w - k$.

Since m is the number of b's in C_i not determined by B, we know that $g_b(B_1)$ is number of b's in C_i inside B_1 and $g_b(B_2')$ is number of b's in C_i inside B_2'. Thus,

$$g_b(C_i) = g_b(B_1) + g_b(B_2') + m = g_b(B_1) + g_b(B_2) + m = g_b(B) + m \tag{5.23}$$

is the total of number of b's in C_i. By Lemma 5.0.20, it follows that exactly $\left(2^b - b\right)^{g_b(B)+m}$ copies of C_i are concatenated in forming $P_{b,w}$. For a fixed m, define S_m to be the total number of occurrences of B straddling some C_i and C_{i+1} such that $C_i = C_{i+1}$ that have exactly m occurrences of b not determined by B. Since there are $k-1$ choices for s, $\binom{w-k}{m}$ choices for M, $w-k-m$ undetermined positions after choosing M and each undetermined position has b possible values, we see that for a fixed m,

$$S_m \leq (k-1) \cdot \binom{w-k}{m} \cdot \left(2^b - b\right)^{g_b(B)+m} \cdot b^{w-k-m}. \tag{5.24}$$

To obtain an upper bound for the number of times B occurs in $P_{b,w}$ straddling some C_i and C_{i+1} such that $C_i = C_{i+1}$, we need only sum over m from 0 to $w-k$ and use the binomial theorem to get

$$S := \sum_{m=0}^{w-k} S_m \leq (k-1) \sum_{m=0}^{w-k} \binom{w-k}{m} \cdot \left(2^b - b\right)^{g_b(B)+m} \cdot b^{w-k-m} \tag{5.25}$$

$$= \left(2^b - b\right)^{g_b(B)} (k-1) \sum_{m=0}^{w-k} \binom{w-k}{m} \left(2^b - b\right)^m b^{w-k-m}$$

$$= (k-1) \left(2^b - b\right)^{g_b(B)} \left(2^b\right)^{w-k}.$$

Next, we let S' be the number of occurrences of B straddling the blocks C_i and C_{i+1} such that C_i and C_{i+1} are not equal. Let Z denote the set of all i such that $C_i \neq C_{i+1}$. Since the C_i's are written in lexicographic order, it follows that Z has no more elements than the number of base $b+1$ blocks of length w. So Z has at most $(b+1)^w$ elements. For each i in Z, there are at most $k-1$ occurrences of B straddling C_i and C_{i+1}. Therefore,

$$S' \leq (k-1) \cdot (b+1)^w. \tag{5.26}$$

For each occurrence of B in $P_{b,w}$, either B occurs inside C_i for some i, B straddles some C_i and C_{i+1} for which $C_i = C_{i+1}$, or B straddles some C_i and C_{i+1} for which $C_i \neq C_{i+1}$. We determined an upper bound for the number of occurrences of B inside some C_i in Lemma 5.0.22. In the proof of the current lemma, we showed that S is an upper bound for the number of occurrences of B straddling some C_i and C_{i+1} for which $C_i = C_{i+1}$. Also in the proof of the current lemma, we have seen that S' is an upper bound for the number of occurrences of B straddling some C_i and C_{i+1} for which $C_i \neq C_{i+1}$. Putting these three facts together, we see that

$$N(B, P_{b,w}) \leq (w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)} + S + S' \tag{5.27}$$

$$\leq (w-k+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)} + (k-1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)} + (k-1) \cdot (b+1)^w$$

$$= w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-k)} + (k-1) \cdot (b+1)^w.$$

□

We now want to show that $P_{b,w}$ is (ϵ, k, ν_b)-normal. First we need a technical lemma:

Lemma 5.0.25. *If m, b, k and w are positive integers such that $b \geq 6$ and $m \leq k \leq w/2$, then*

$$(m-1)(b+1)^w \leq k \cdot 2^{b(w-m)}. \tag{5.28}$$

Proof. Since $m \le k$ and $k \le w/2$, it follows that

$$1 \ge 2^{b(-w/2+m)} = 2^{-bw/2} \cdot 2^{mb} = (2^{-b/2})^w \cdot 2^{mb} \tag{5.29}$$

$$\ge \left((b+1)2^{-b}\right)^w \cdot 2^{mb} \ge \binom{m-1}{k} \cdot \frac{(b+1)^w 2^{bm}}{2^{bw}}, \tag{5.30}$$

where (5.29) to (5.30) is due to $b+1 \le 2^{b/2}$ for $b \ge 6$. Therefore,

$$1 \ge \binom{m-1}{k} \cdot \frac{(b+1)^w 2^{bm}}{2^{bw}}. \tag{5.31}$$

Multiplying both sides of (5.31) by $k \cdot 2^{b(w-m)}$, the lemma follows.

□

Lemma 5.0.26. *Let b, k and w be positive integers such that $b \ge 6$ and $k \le w/2$. If $\epsilon = \frac{k}{w}$, then $P_{b,w}$ is (ϵ, k, ν_b)-normal.*

Proof. By definition, $P_{b,w}$ is (ϵ, k, ν_b)-normal if for all blocks B in base $b+1$ of length $m \le k$

$$\nu(B)|P_{b,w}|(1-\epsilon) \le N(B, P_{b,w}) \le \nu(B)|P_{b,w}|(1+\epsilon). \tag{5.32}$$

Therefore by Lemma 5.0.22 and Lemma 5.0.24, it is enough to show that

$$\nu(B)|P_{b,w}|(1-\epsilon) \le (w-m+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} \tag{5.33}$$

and

$$w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} + (m-1)(b+1)^w \leq \nu(B)|P_{b,w}|(1+\epsilon). \tag{5.34}$$

To show (5.33), we write

$$(1-\epsilon)|P_{b,w}|\nu_b(B) = \left(1 - \frac{k}{w}\right) w \cdot 2^{bw} \left(2^b - b\right)^{g_b(B)} 2^{-bm} \tag{5.35}$$

$$= (w-k) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} < (w-m+1) \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)}.$$

Next, to show (5.34), we write

$$w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} + (m-1)(b+1)^w \leq w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} + k \cdot 2^{b(w-m)} \tag{5.36}$$

$$\leq w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)} + k \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)}$$

$$= (1+\epsilon) w \cdot \left(2^b - b\right)^{g_b(B)} \cdot 2^{b(w-m)},$$

where the first inequality follows from Lemma 5.0.25.

□

Theorem 5.0.27. *For $i \leq 5$, let $x_i = (0,1)$, $b_i = 2$ and $l_i = 0$. If for $i \geq 6$ we let $x_i = P_{i,i^2}$, $b_i = 2^i$ and $l_i = 2^{4i^2}$, then x is Q-normal.*

Proof. For each $i \geq 1$, we shall define numbers p_i, k_i, ϵ_i, and weightings μ_i in order to define a BFF W such that $\{x_i\}_{i=1}^{\infty}$ is W-good. Thus, we have only to verify (3.65), (3.66) and (3.67) of Theorem 3.3.13 to conclude that x is Q-normal.

For $i \leq 5$, we define $p_i = 2$, $k_i = 1$ and $\mu_i = \lambda_2$. For $i \geq 6$, set $p_i = i$, $k_i = i$ and $\mu_i = \nu_i$. Define $\epsilon_1 = .9$, $\epsilon_1 = .8$, $\epsilon_1 = .7$, $\epsilon_1 = .6$, $\epsilon_1 = .5$ and $\epsilon_i = 1/i$ for $i \geq 6$. Let

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}. \tag{5.37}$$

We note that since μ_i is (p_i, b_i)-uniform, it follows by definition that W is a BFF. Since $\lim_{i\to\infty} k_i = \lim_{i\to\infty} i = \infty$, we see that $R(W)$ is the set of all non-negative integers. So, it is enough to show that conditions (3.65), (3.66) and (3.67) hold for all non-negative integers k. First note that $|x_i| = i^2 \cdot 2^{i^3}$ for $i \geq 6$.

To show (3.65), note that

$$\lim_{i\to\infty} |x_i| \Bigg/ \left(\frac{2^{ik}}{\frac{1}{i-1} - \frac{1}{i}} \right) = \lim_{i\to\infty} \frac{i^2 \cdot 2^{i^3}}{2^{ik} \cdot i(i-1)} = \infty. \tag{5.38}$$

To show (3.66), notice that

$$\lim_{i\to\infty} \frac{l_{i-1}}{l_i} \cdot \frac{|x_{i-1}|}{|x_i|} \cdot i \cdot 2^{ik} \leq \lim_{i\to\infty} 1 \cdot \left(\frac{i-1}{i}\right)^2 \cdot \frac{2^{(i-1)^3+ki}}{2^{i^3}} \cdot i \tag{5.39}$$

$$\leq \lim_{i\to\infty} 1 \cdot 2^{-3i^2+(3+k)i-1} \cdot i = 0.$$

And finally, to show (3.67), we write

$$\lim_{i\to\infty} \frac{1}{l_i} \cdot \frac{|x_{i+1}|}{|x_i|} \cdot 2^{ik} = \lim_{i\to\infty} \left(\frac{i+1}{i}\right)^2 \cdot \frac{2^{(i+1)^3+ki}}{2^{4i^2} \cdot 2^{i^3}} \tag{5.40}$$

$$\leq \lim_{i\to\infty} 2\cdot 2^{-i^2+(3+k)i+1} = 0.$$

This shows that $\{x_i\}_{i=1}^{\infty}$ is W-good. Therefore x is Q-normal by Theorem 3.3.13. ∎

Theorem 5.0.28. *If $\{x_i\}$, $\{b_i\}$, and $\{l_i\}$ are defined as in Theorem 5.0.27, then* $\lim_{n\to\infty} T_{Q,n}(x) = 0$.

Proof. To prove Theorem 5.0.28 we use the trick which is usually used to prove the irrationality of x. For more information see e.g. [19]. Note that

$$T_{Q,n}(x) = q_1\cdots q_n x \pmod 1 = \frac{E_{n+1}}{q_{n+1}} + \frac{E_{n+2}}{q_{n+1}q_{n+2}} + \cdots \tag{5.41}$$

Given n, define $j = j(n)$ as the unique integer satisfying

$$L_{j-1} < n+1 \leq L_j. \tag{5.42}$$

Note that $q_{n+1} = b_j = 2^j$ and $E_{n+1} \leq j$ by construction. Additionally, note that

$$\frac{E_{n+2}}{q_{n+1}q_{n+2}} + \frac{E_{n+3}}{q_{n+1}q_{n+2}q_{n+3}} + \cdots \leq \frac{1}{q_{n+1}}\left[\frac{E_{n+2}}{q_{n+2}} + \frac{E_{n+3}}{q_{n+2}q_{n+3}} + \cdots\right] \leq \frac{1}{q_{n+1}}\cdot 1 = \frac{1}{q_{n+1}}. \tag{5.43}$$

Therefore, since $0 \leq E_{n+1} \leq j$, we see that

$$T_{Q,n}(x) = \frac{E_{n+1}}{q_{n+1}} + \left[\frac{E_{n+2}}{q_{n+1}q_{n+2}} + \frac{E_{n+3}}{q_{n+1}q_{n+2}q_{n+3}} + \cdots\right] \leq \frac{j}{2^j} + \frac{1}{2^j} \to 0. \tag{5.44}$$

∎

Corollary 5.0.29. *If $\{x_i\}$, $\{b_i\}$, and $\{l_i\}$ are defined as in Theorem 5.0.27, then x is not Q-distribution normal.*

CHAPTER 6

CONSTRUCTION OF Q-DISTRIBUTION NORMAL NUMBERS FOR ARBITRARY Q THAT ARE INFINITE IN LIMIT

In this chapter, given an arbitrary basic sequence Q that is infinite in limit, we will provide an explicit construction of an uncountable family of Q-distribution normal sequences. This provides a partial answer to a problem posed by P. Laffer in [24]. We first make the following definition that will be studied more extensively later in this thesis:

Definition 6.0.30. *Given a basic sequence Q, a real number $x \in [0, 1)$ is Q-dense if the sequence*

$$\{T_{Q,n}(x)\}_{n=1}^{\infty} \tag{6.1}$$

is dense in $[0, 1)$.

P. Laffer asked the following:

Problem 6.0.31. *Given an arbitrary basic sequence Q, construct a real number that is Q-distribution normal.*

P. Laffer provided a construction for a Q-dense number for an arbitrary basic sequence Q. Our construction relies heavily on Theorem 2.4.6 and it appears that our construction is unlikely to be able to be modified to work for Q that are not almost infinite in limit. However, it is likely that it may be modified to account for basic sequences Q that are almost infinite in limit by use of Theorem 2.4.5.

We will use the following definition from [23]:

Definition 6.0.32. *For $0 \leq \delta < 1$ and $\epsilon > 0$, a finite sequence $x_1 < x_2 < \cdots < x_N$ in $[0,1)$ is called an almost-arithmetic progression-(δ, ϵ) if there exists an η, $0 < \eta \leq \epsilon$, such that the following conditions are satisfied:*

$$0 \leq x_1 \leq \eta + \delta\eta; \tag{6.2}$$

$$\eta - \delta\eta \leq x_{n+1} - x_n \leq \eta + \delta\eta \text{ for } 1 \leq n \leq N-1; \tag{6.3}$$

$$1 - \eta - \delta\eta \leq x_N < 1. \tag{6.4}$$

Almost arithmetic progressions were introduced by P. O'Neil in [33]. He proved that a sequence $\{x_n\}_n$ of real numbers in $[0,1)$ is uniformly distributed mod 1 if and only if the following holds: for any three positive real numbers δ, ϵ, and ϵ', there exists a positive integer N such that for all $n > N$, the initial segment $x_1, x_2, \ldots, x_n$ can be decomposed into an almost-arithmetic progression-(δ, ϵ) with at most N_0 elements left over, where $N_0 < \epsilon' N$.

In this section, we will construct a sequence of digits $E_1, E_2, E_3, \ldots$ that will be generated by concatenating an infinite list of almost-arithmetic progressions such that if $x = 0.E_1E_2E_3\ldots$ w.r.t. Q, then x will be Q-distribution normal.

For the rest of this chapter, we will fix a basic sequence Q that is infinite in limit.

6.1 Basic Definitions

We will construct the sequence $l_1, l_2, l_3, \ldots$ of positive integers in terms of Q. First,the following definition will be needed:

Definition 6.1.1. *For each positive integer j, we define*

$$n_j = \min\{N : q_m \geq 2j^2 \textit{ for all } m \geq N\}. \tag{6.5}$$

We now recursively define the sequence $l_1, l_2, l_3, \ldots$:

Definition 6.1.2. *We set*

$$l_1 = \max(n_2 - 1, 1). \tag{6.6}$$

Given $l_1, l_2, \ldots, l_{i-1}$, we define l_i to be the smallest positive integer such that

$$l_1 + 2l_2 + 3l_3 + \ldots + il_i \geq n_{i+1} - 1. \tag{6.7}$$

Thus, we have

$$l_i = \max(\min\{k : l_1 + 2l_2 + \ldots + (i-1)l_{i-1} + ik \geq n_{i+1} - 1\}, 1). \tag{6.8}$$

Additionally, for any non-negative integer i, we set

$$L_i = \sum_{j=1}^{i} jl_j = l_1 + 2l_2 + \ldots + il_i. \tag{6.9}$$

Lemma 6.1.3. *Suppose that a is a positive integer, $c \leq a$ is a positive integer and $q \geq 2a^2$ is a positive integer. Then there exists at least two integers F such that*

$$\frac{F}{q} \in \left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right]. \tag{6.10}$$

Proof. We assume, for contradiction, that there are fewer than two solutions to (6.10). Thus, there exists an integer F such that

$$\frac{F}{q} < \frac{c}{a} - \frac{1}{2a^2} \tag{6.11}$$

and

$$\frac{c}{a} + \frac{1}{2a^2} < \frac{F+2}{q}, \tag{6.12}$$

so

$$\left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right] \not\subseteq \left[\frac{F}{q}, \frac{F+2}{q}\right]. \tag{6.13}$$

By (6.13), we conclude that

$$\left(\frac{c}{a} + \frac{1}{2a^2}\right) - \left(\frac{c}{a} - \frac{1}{2a^2}\right) < \frac{F+2}{q} - \frac{F}{q}, \tag{6.14}$$

so

$$\frac{1}{a^2} < \frac{2}{q}. \tag{6.15}$$

Cross multiplying (6.15) gives

$$q < 2a^2, \tag{6.16}$$

which contradicts $q \geq 2a^2$.

□

Definition 6.1.4. *Let*

$$S_Q = \left\{(a,b,c) \in \mathbb{N}^3 : b \leq l_a, c \leq a\right\} \tag{6.17}$$

and define $h : S_Q \to \mathbb{N}$ *by*

$$h(a,b,c) = L_{a-1} + (b-1)a + c. \tag{6.18}$$

Lemma 6.1.5. *The function* h *is a bijection between* S_Q *and* $\mathbb{N}$.

Proof. Starting at $n = 1$, put l_1 boxes of length 1, followed by l_2 boxes of length 2, l_3 boxes of length 3, and so on. Then the position of component c of the b^{th} box of length a is at

$$1l_1 + 2l_2 + \ldots + (a-1)l_{a-1} + (b-1)a + c = h(a,b,c), \tag{6.19}$$

so h is clearly a bijection between S_Q and $\mathbb{N}$.

□

Definition 6.1.6. *The sequence* $F = \left\{F_{(a,b,c)}\right\}_{(a,b,c)\in S_Q}$ *is a* Q*-special sequence if* $F_{(a,b,1)} = 0$ *for* $(a,b,1) \in S_Q$ *and*

$$\frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right] \tag{6.20}$$

for $(a,b,c) \in S_Q$ *with* $c > 1$.

Given a Q-special sequence F, Lemma 6.1.5 allows us to define $E_F = \{E_{F,n}\}_{n=1}^{\infty}$ as follows:

Definition 6.1.7. *Suppose that* F *is a* Q*-special sequence. For any positive integer* n*, we define*

$$E_{F,n} = F_{h^{-1}(n)} \tag{6.21}$$

and let $E_F = \{E_{F,n}\}_{n=1}^{\infty}$.

Given a Q-special sequence F, we will show that the number

$$\sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n} \tag{6.22}$$

is Q-distribution normal. It should be noted that by Lemma 6.1.3, there are uncountable many Q-special sequences. This will allow us to construct uncountably many Q-distribution normal numbers.

Definition 6.1.8. *If F is a Q-special sequence and $(a, b, 1) \in S_Q$, then we define*

$$y_{F,a,b} = \left\{ \frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \right\}_{c=1}^{a} \tag{6.23}$$

and let

$$D^*_{F,a,b} = D^*(y_{F,a,b}). \tag{6.24}$$

We will also write

$$y_F = 1 y_{F,1,1} 1 y_{F,1,2} \ldots 1 y_{F,1,l_1} 1 y_{F,2,1} 1 y_{F,2,2} \ldots 1 y_{F,2,l_2} 1 y_{F,3,1} 1 y_{F,3,2} \ldots 1 y_{F,3,l_3} 1 y_{F,4,1} \ldots . \tag{6.25}$$

Definition 6.1.9. *If F is a Q-special sequence, we define*

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n}. \tag{6.26}$$

Thus, by construction,

$$y_F = \left\{ \frac{E_{F,n}}{q_n} \right\}_{n=1}^{\infty}. \tag{6.27}$$

By Theorem 2.4.6, x_F is Q-distribution normal if and only if y_F is uniformly distributed mod 1.

6.2 Basic Lemmas

We will use the following theorem from [31]:

Theorem 6.2.1. *Let $x_1 < x_2 < \cdots < x_N$ be an almost arithmetic progression-(δ, ϵ) and let η be the positive real number corresponding to the sequence according to Definition 6.0.32. Then*

$$D_N^* \leq \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}} \text{ for } \delta > 0 \tag{6.28}$$

and

$$D_N^* \leq \min\left(\eta, \frac{1}{N}\right) \text{ for } \delta = 0. \tag{6.29}$$

We will use the following corollary of Theorem 6.2.1 in our proof:

Corollary 6.2.2. *Let $x_1 < x_2 < \cdots < x_N$ be an almost arithmetic progression-(δ, ϵ) and let η be the positive real number corresponding to the sequence according to Definition 6.0.32. Then*

$$D_N^* \le \frac{1}{N} + \delta. \tag{6.30}$$

Lemma 6.2.3. *If F is a Q-special sequence, then the sequence $y_{F,a,b}$ is an almost arithmetic progression-$\left(\frac{1}{a}, \frac{1}{a}\right)$ and*

$$D_{F,a,b}^* \le \frac{2}{a}. \tag{6.31}$$

Proof. The case $a = 1$ is trivial, so suppose that $a > 1$. To show that $y_{F,a,b}$ is an almost arithmetic progression-$\left(\frac{1}{a}, \frac{1}{a}\right)$, we first note that since $F_{(a,b,1)} = 0$,

$$0 \le \frac{F_{(a,b,1)}}{q_{h^{-1}(a,b,1)}} \le \frac{1}{a} + \frac{1}{a^2}, \tag{6.32}$$

so (6.2) holds.

Next, suppose that $2 \le c \le a - 1$. By construction,

$$\frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right] \tag{6.33}$$

and

$$\frac{F_{(a,b,c+1)}}{q_{h^{-1}(a,b,c+1)}} \in \left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right], \tag{6.34}$$

so

$$\frac{F_{(a,b,c+1)}}{q_{h^{-1}(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \leq \left(\frac{c}{a} + \frac{1}{2a^2}\right) - \left(\frac{c-1}{a} - \frac{1}{2a^2}\right) \tag{6.35}$$

and

$$\frac{F_{(a,b,c+1)}}{q_{h^{-1}(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \geq \left(\frac{c}{a} - \frac{1}{2a^2}\right) - \left(\frac{c-1}{a} + \frac{1}{2a^2}\right). \tag{6.36}$$

Combining (6.35) and (6.36), we see that

$$\frac{1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,c+1)}}{q_{h^{-1}(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \leq \frac{1}{a} + \frac{1}{a^2}, \tag{6.37}$$

so (6.3) holds.

Lastly, by construction,

$$\frac{a-1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,a)}}{q_{h^{-1}(a,b,a)}} < \frac{a-1}{a} + \frac{1}{a^2}, \tag{6.38}$$

so

$$1 - \frac{1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,a)}}{q_{h^{-1}(a,b,a)}} \leq 1 - \frac{1}{a} + \frac{1}{a^2} < 1 \tag{6.39}$$

and we have verified (6.4). Therefore, $y_{F,a,b}$ is an almost arithmetic progression-$\left(\frac{1}{a}, \frac{1}{a}\right)$. By Corollary 6.2.2,

$$D^*_{F,a,b} \leq \frac{1}{a} + \frac{1}{a} = \frac{2}{a}. \tag{6.40}$$

$\square$

Throughout the rest of this chapter, for a given n, the letter $i = i(n)$ is the unique integer satisfying

$$L_i < n \leq L_{i+1}. \tag{6.41}$$

Given a positive integer n, let

$$m = n - L_i. \tag{6.42}$$

Note that m can be written uniquely as

$$m = \alpha(i+1) + \beta \tag{6.43}$$

with

$$0 \leq \alpha \leq l_{i+1} \text{ and } 0 \leq \beta < i+1. \tag{6.44}$$

We define α and β as the unique integers satisfying these conditions.

Recall that $D^*(z)$ is bounded above by 1 for all finite sequences z of real numbers in $[0,1)$. By Corollary 3.2.5,

$$D_n^*(y_F) \leq f_i(\alpha,\beta) := \frac{\left(\sum_{j=1}^{i} l_j \cdot j \cdot \frac{2}{j}\right) + \alpha \cdot (i+1) \cdot \frac{2}{i+1} + \beta}{\left(\sum_{j=1}^{i} j l_j\right) + (i+1)\alpha + \beta} \tag{6.45}$$

$$= \frac{\left(\sum_{j=1}^{i} 2 l_j\right) + 2\alpha + \beta}{\left(\sum_{j=1}^{i} j l_j\right) + (i+1)\alpha + \beta}. \tag{6.46}$$

Note that $f_i(\alpha, \beta)$ is a rational function of α and β. We consider the domain of f_i to be $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, where $\mathbb{R}_0^+$ is the set of all non-negative real numbers. Given a Q-special sequence F, we now give an upper bound for $D_n^*(y_F)$. Since $D_n^*(y_F)$ is at most $f_i(\alpha, \beta)$, it is enough to bound $f_i(\alpha, \beta)$ from above on $[0, l_{i+1}] \times [0, i]$.

Lemma 6.2.4. *If* $i > 2$*,*

$$\sum_{j=1}^{i} j l_j > \sum_{j=1}^{i} 2 l_j, \tag{6.47}$$

and

$$(w, z) \in \{0, \ldots, l_{i+1}\} \times \{0, \ldots, i\}, \tag{6.48}$$

then

$$f_i(w, z) < f_i(0, i+1) = \frac{\left(\sum_{j=1}^{i} 2 l_j\right) + i + 1}{\left(\sum_{j=1}^{i} j l_j\right) + i + 1}. \tag{6.49}$$

Proof. To bound $f_i(w, z)$, we first compute its partial derivatives $\frac{\partial f_i}{\partial z}(w, z)$ and $\frac{\partial f_i}{\partial w}(w, z)$. We will show that $\frac{\partial f_i}{\partial w}(w, z)$ is always negative and $\frac{\partial f_i}{\partial z}(w, z)$ is always positive. Note that this is enough to prove Lemma 6.2.4 since $w \geq 0$ and $z < i + 1$.

First, we note that $f_i(w, z)$ is a rational function of w and z of the form

$$f_i(w, z) = \frac{C + Dw + Ez}{F + Gw + Hz}, \tag{6.50}$$

where

$$C = \sum_{j=1}^{i} 2l_j, \; D = 2, \; E = 1, \tag{6.51}$$

$$F = \sum_{j=1}^{i} jl_j, \; G = i + 1, \text{ and } H = 1. \tag{6.52}$$

Therefore,

$$\frac{\partial f_i}{\partial w}(w, z) = \frac{D(F + Gw + Hz) - G(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{6.53}$$

$$= \frac{D(F + Hz) - G(C + Ez)}{(F + Gw + Hz)^2}$$

and

$$\frac{\partial f_i}{\partial z}(w, z) = \frac{E(F + Gw + Hz) - H(C + Dw + Ez)}{(F + Gw + Hz)^2} \tag{6.54}$$

$$= \frac{E(F + Gw) - H(C + Dw)}{(F + Gw + Hz)^2}.$$

Thus, the sign of $\frac{\partial f_i}{\partial w}(w, z)$ does not depend on w and the sign of $\frac{\partial f_i}{\partial z}(w, z)$ does not depend on z. We will show that $f_i(w, z)$ is a decreasing function of w by proving that

$$D(F + Hz) < G(C + Ez). \tag{6.55}$$

Similarly, we show that $f_i(w, z)$ is an increasing function of z by verifying that

$$E(F + Gw) > H(C + Dw). \tag{6.56}$$

Substituting the values in (6.52) into (6.55), we need to prove that

$$2\left(\sum_{j=1}^{i} jl_j + z\right) < (i+1)\left(\sum_{j=1}^{i} 2l_j + z\right). \tag{6.57}$$

Distributing both sides, we see that (6.57) is equivalent to

$$2\sum_{j=1}^{i} jl_j + 2z < 2\left((i+1)\sum_{j=1}^{i} l_j\right) + (i+1)z. \tag{6.58}$$

However, $i > 2$ by hypothesis and

$$\sum_{j=1}^{i} jl_j < (i+1)\sum_{j=1}^{i} l_j, \tag{6.59}$$

so (6.55) holds.

By substituting the values in (6.52) into (6.56), we need to prove that

$$\sum_{j=1}^{i} jl_j + (i+1)w > \sum_{j=1}^{i} 2l_j + 2w. \tag{6.60}$$

By (6.47), we know that

$$\sum_{j=1}^{i} jl_j > \sum_{j=1}^{i} 2l_j \tag{6.61}$$

and $i > 2$, so (6.56) holds.

□

Set

$$\bar{\epsilon}_i = f_i(0, i+1) = \frac{\left(\sum_{j=1}^{i} 2l_j\right) + i + 1}{\left(\sum_{j=1}^{i} jl_j\right) + i + 1}. \tag{6.62}$$

We will now prove a series of lemmas to show that $\bar{\epsilon}_i \to 0$. The following was proven by O. Toeplitz in [50]:

Theorem 6.2.5. *Let $\{\gamma_{n,k} : 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers such that:*

1. $\lim_{n\to\infty} \gamma_{n,k} = 0$ *for each* $k \in \mathbb{N}$*;*
2. $\lim_{n\to\infty} \sum_{k=1}^{n} \gamma_{n,k} = 1$*;*
3. *there exists $C > 0$ such that for all positive integers n: $\sum_{k=1}^{n} |\gamma_{n,k}| \leq C$.*

   Then for any convergent sequence $\{\alpha_n\}$, the transformed sequence $\{\beta_n\}$ given by

$$\beta_n = \sum_{k=1}^{n} \gamma_{n,k}\alpha_k, n \geq 1, \tag{6.63}$$

   is also convergent and

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha_n. \tag{6.64}$$

We will need the following theorem that follows from Theorem 6.2.5:

Theorem 6.2.6. *Let L be a real number and $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive real numbers such that*

$$\sum_{n=1}^{\infty} b_n = \infty \tag{6.65}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L. \tag{6.66}$$

Then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L. \tag{6.67}$$

Proof. Let $\alpha_n = \frac{a_n}{b_n}$ and let

$$\gamma_{n,k} = \frac{b_k}{b_1 + b_2 + \ldots + b_n}. \tag{6.68}$$

We now verify that $\{\gamma_{n,k}\}$ satisfies the hypothesis of Theorem 6.2.5. Clearly,

$$\lim_{n\to\infty} \gamma_{n,k} = 0 \tag{6.69}$$

for all k, as $\sum_{n=1}^{\infty} b_n = \infty$. Next, we note that

$$\sum_{k=1}^{n} \gamma_{n,k} = \sum_{k=1}^{n} \frac{b_k}{b_1 + b_2 + \ldots + b_n} = \frac{b_1 + b_2 + \ldots + b_n}{b_1 + b_2 + \ldots + b_n} = 1, \tag{6.70}$$

so the second condition of Theorem 6.2.5 is satisfied. The third condition is trivially satisfied for $C = 1$ as $\gamma_{n,k} > 0$ for all n and k.

Thus, if

$$\beta_n = \sum_{k=1}^{n} \gamma_{n,k}\alpha_k = \sum_{k=1}^{n} \frac{a_k}{b_1 + b_2 + \ldots + b_n} = \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n}, \tag{6.71}$$

then by Theorem 6.2.5, we see that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = \lim_{n\to\infty} \frac{a_n}{b_n} = L. \tag{6.72}$$

□

We may now show that $\bar{\epsilon}_i \to 0$.

Lemma 6.2.7.

$$\lim_{n\to\infty} \bar{\epsilon}_i = 0. \tag{6.73}$$

Proof. We will first show that $\lim_{i\to\infty} \bar{\epsilon}_i \to 0$. The lemma will then follow as $i = i(n)$ is an increasing function of n.

We apply Theorem 6.2.6 with $a_1 = 2l_1 + 2$, $b_1 = l_1 + 2$ and for $j > 1$,

$$a_j = 2l_j + 1 \text{ and } b_j = jl_j + 1. \tag{6.74}$$

We see that

$$a_1 + a_2 + \ldots + a_i = \left(\sum_{j=1}^{i} 2l_j\right) + i + 1 \tag{6.75}$$

and

$$b_1 + b_2 + \ldots + b_i = \left(\sum_{j=1}^{i} j l_j \right) + i + 1. \tag{6.76}$$

Since

$$\lim_{i \to \infty} \frac{a_i}{b_i} = \lim_{i \to \infty} \frac{2l_i + 1}{il_i + 1} = 0, \tag{6.77}$$

we see that

$$\lim_{i \to \infty} \bar{\epsilon}_i = \lim_{i \to \infty} \frac{\left(\sum_{j=1}^{i} 2l_j \right) + i + 1}{\left(\sum_{j=1}^{i} jl_j \right) + i + 1} = \lim_{i \to \infty} \frac{a_i}{b_i} = 0. \tag{6.78}$$

□

6.3 Main Theorem

We now prove the main theorem of this chapter.

Theorem 6.3.1. *Suppose that F is a Q-special sequence. If*

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n}, \tag{6.79}$$

then x_F is Q-distribution normal.

Proof. Suppose that n is large enough so that $i > 2$ and

$$(i-2)l_i > l_1. \tag{6.80}$$

Then, clearly,

$$il_i + 2l_2 + l_1 > 2l_i + 2l_2 + 2l_1. \tag{6.81}$$

We also note that

$$jl_j > 2l_j \text{ for } j > 2. \tag{6.82}$$

Combining (6.81) and (6.82), we see that

$$\sum_{j=1}^{i} jl_j > \sum_{j=1}^{i} 2l_j. \tag{6.83}$$

By Lemma 6.2.4, we see that $D_n^*(y_F) < \bar{\epsilon}_{i(n)}$. By Lemma 6.2.7, $\bar{\epsilon}_{i(n)} \to 0$, so the sequence y_F is uniformly distributed mod 1. Thus, by Theorem 2.4.6, x_F is Q-distribution normal.

□

Suppose that F_1 and F_2 are Q-special sequences and that $F_1 \neq F_2$. Then we see that $x_{F_1} \neq x_{F_2}$, so we have counstructed uncountably many Q-distribution normal numbers. We will now see that while Theorem 6.3.1 allows us to construct Q-distribution normal numbers, none of these numbers will be simply Q-ratio normal.

Proposition 6.3.2. *Suppose that Q is infinite in limit and that F is a Q-special sequence. Then x_F is not simply Q-ratio normal.*

Proof. We will show that the digit 1 may only occur finitely often in the Q-Cantor series expansion of x_F. Suppose that $(a,b,2) \in F$ and $a \geq 2$. Then, by construction, we have

$$\frac{F_{(a,b,2)}}{q_{h^{-1}(a,b,2)}} \in \left[\frac{1}{a} - \frac{1}{2a^2}, \frac{1}{a} + \frac{1}{2a^2}\right] \tag{6.84}$$

and $q_{h^{-1}(a,b,2)} \geq 2a^2$. Thus, we see that

$$\frac{F_{(a,b,2)}}{q_{h^{-1}(a,b,2)}} \geq \frac{1}{a} - \frac{1}{2a^2}, \tag{6.85}$$

so

$$\frac{F_{(a,b,2)}}{q_{h^{-1}(a,b,2)}} \geq \frac{1}{a} - \frac{1}{2a^2} \tag{6.86}$$

and

$$F_{(a,b,2)} \geq \left(\frac{1}{a} - \frac{1}{2a^2}\right) q_{h^{-1}(a,b,2)} \geq \left(\frac{1}{a} - \frac{1}{2a^2}\right) \cdot 2a^2 = 2a - 1 > 1. \tag{6.87}$$

Thus, by (6.87), $F_{(a,b,2)} > 1$ when $a \geq 2$. Since $F_(a,b,1) = 0$ whenever $(a,b,1) \in S_Q$, there are infinitely many n such that $E_{F,n} = 0$, so x_F is not simply Q-ratio normal.

□

We will see in Theorem 8.6.13 and Theorem 8.6.15 that the set

$$\Theta_Q = \{x_F : F \text{ is a } Q\text{-special sequence}\} \tag{6.88}$$

is an example of a perfect set that is also nowhere dense.

6.4 Examples

Using Theorem 6.3.1, we will now give examples of Q-distribution normal numbers for two different basic sequences. First, we recall the relevent definitions that are needed for our construction. We fix a basic sequence Q that is infinite in limit and define

$$n_j = \min\{N : q_m \geq 2j^2 \text{ for all } m \geq N\}, \tag{6.89}$$

$$l_1 = \max(n_2 - 1, 1), \tag{6.90}$$

and for $i > 1$, l_i is the smallest positive integer such that

$$l_1 + 2l_2 + 3l_3 + \ldots + il_i \geq n_{i+1} - 1. \tag{6.91}$$

We also need

$$S_Q = \left\{(a, b, c) \in \mathbb{N}^3 : b \leq l_a, c \leq a\right\} \tag{6.92}$$

and for $(a, b, c) \in S_Q$

$$h(a, b, c) = L_{a-1} + (b-1)a + c. \tag{6.93}$$

For $(a, b, 1) \in S_Q$, $F_{(a,b,1)} = 0$. For $(a, b, c) \in S_Q$ with $c > 1$, we let $F_{(a,b,c)}$ be any integer that satisfies

$$\frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right]. \tag{6.94}$$

Lastly, if F is a Q-special sequence, we define

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \dots q_n}. \tag{6.95}$$

6.4.1 Example for a Fast Growing q_n

We first consider the basic sequence $Q = \{q_n\}$, where

$$q_n = n^2 + 1. \tag{6.96}$$

It will help to inspect the first several values of q_n:

n	1	2	3	4	5	6	7	8	9	10	11
q_n	2	5	10	17	26	37	50	65	82	101	122

From this table, we see that $n_1 = 1$, $n_2 = 3$, $n_3 = 5$, $n_4 = 6$, and $n_5 = 7$. Thus,

$$l_1 = \max(3-1, 1) = 2. \tag{6.97}$$

So, l_2 is the smallest positive integer that satisfies

$$2 + 2l_2 \geq 5 - 1, \tag{6.98}$$

which implies that $l_2 = 1$. Similarly, l_3 is the smallest positive integer such that

$$2 + 2 \cdot 1 + 3l_3 \geq 6 - 1, \tag{6.99}$$

so $l_3 = 1$. We may conclude that $l_4 = 1$ in the same manner. We will now construct a Q-special sequence F. We first note that

$$F_{(1,1,1)} = F_{(1,2,1)} = F_{(2,1,1)} = 0. \tag{6.100}$$

The first non-trivial value to compute is $F_{(2,1,2)}$, which must satisfy

$$\frac{F_{(2,1,2)}}{q_4} = \frac{F_{(2,1,2)}}{17} \in \left[\frac{3}{8}, \frac{5}{8}\right]. \tag{6.101}$$

Clearly $7, 8, 9$, and 10 are all solutions but we will choose $F_{(2,1,2)} = 7$.

Next, we note that $F_{(3,1,1)} = 0$ and that $F_{(3,1,2)}$ and $F_{(3,1,3)}$ satisfy

$$\frac{F_{(3,1,2)}}{q_6} = \frac{F_{(3,1,2)}}{37} \in \left[\frac{5}{18}, \frac{7}{18}\right] \tag{6.102}$$

and

$$\frac{F_{(3,1,3)}}{q_7} = \frac{F_{(3,1,3)}}{50} \in \left[\frac{11}{18}, \frac{13}{18}\right], \tag{6.103}$$

respectively. We choose the solutions $F_{(3,1,2)} = 11$ and $F_{(3,1,3)} = 31$.

Next, $F_{(4,1,1)} = 0$ and $F_{(4,1,2)}$, $F_{(4,1,13)}$, and $F_{(4,1,4)}$ satisfy

$$\frac{F_{(4,1,2)}}{q_9} = \frac{F_{(4,1,2)}}{82} \in \left[\frac{7}{32}, \frac{9}{32}\right], \tag{6.104}$$

$$\frac{F_{(4,1,3)}}{q_{10}} = \frac{F_{(4,1,3)}}{101} \in \left[\frac{15}{32}, \frac{17}{32}\right], \tag{6.105}$$

and

$$\frac{F_{(4,1,4)}}{q_{11}} = \frac{F_{(4,1,4)}}{122} \in \left[\frac{23}{32}, \frac{25}{32}\right], \tag{6.106}$$

respectively. We choose the solutions $F_{(4,1,2)} = 18$, $F_{(4,1,13)} = 48$, and $F_{(4,1,4)} = 88$. Thus, we see that if

$$E_F = (0, 0, 0, 7, 0, 11, 31, 0, 18, 48, 88, \ldots), \tag{6.107}$$

then by Theorem 6.3.1, the number

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n} \tag{6.108}$$

is Q-distribution normal.

6.4.2 An Example for a Non-Increasing q_n

In this subsection, we will let $Q = \{q_n\}$ be defined by

$$q_n = \lfloor (4 + (-1)^n/2) n^{3/4} \rfloor. \tag{6.109}$$

We once again write the first few values of q_n:

n	1	2	3	4	5	6	7	8	9
q_n	3	7	7	12	11	17	15	21	18

The main differences with the previous example are that q_n is non-increasing so we will need to take more care computing the values of $\{n_j\}$. Additionally, q_n grows slower, so we will have larger values of l_i.

We start as before and see that $n_1 = 1$, $n_2 = 4$, and $n_3 = 9$. We can also show that $n_4 = 21$, but need more values than those displayed in the chart. As before,

$$l_1 = \max(4 - 1, 1) = 3, \tag{6.110}$$

so l_2 is the smallest positive integer such that

$$3 + 2l_2 \geq 9 - 1. \tag{6.111}$$

Thus, $l_2 = 3$. We can also see that l_3 is the smallest positive integer satisfying

$$3 + 2 \cdot 3 + 3l_3 \geq 21 - 1, \tag{6.112}$$

so $l_3 = 4$.

Since $l_1 = 3$, we see that

$$F_{(1,1,1)} = F_{(1,2,1)} = F_{(1,3,1)} = 0. \tag{6.113}$$

We also note that since $l_2 = 3$, we can immediately say that

$$F_{(2,1,1)} = F_{(2,2,1)} = F_{(2,3,1)} = 0 \tag{6.114}$$

and that $F_{(2,1,2)}$, $F_{(2,2,2)}$, and $F_{(2,3,2)}$ satisfy

$$\frac{F_{(2,1,2)}}{q_5} = \frac{F_{(2,1,2)}}{11} \in \left[\frac{3}{8}, \frac{5}{8}\right], \tag{6.115}$$

$$\frac{F_{(2,2,2)}}{q_7} = \frac{F_{(2,2,2)}}{15} \in \left[\frac{3}{8}, \frac{5}{8}\right], \tag{6.116}$$

and

$$\frac{F_{(2,3,2)}}{q_9} = \frac{F_{(2,3,2)}}{18} \in \left[\frac{3}{8}, \frac{5}{8}\right], \tag{6.117}$$

respectively. We choose the solutions $F_{(2,1,2)} = 5$, $F_{(2,2,2)} = 6$, and $F_{(2,3,2)} = 7$. Thus, we see that if

$$E_F = (0, 0, 0, 0, 5, 0, 6, 0, 7, \ldots), \tag{6.118}$$

then by Theorem 6.3.1, the number

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n} \tag{6.119}$$

is Q-distribution normal.

6.5 Conjectures

We make the following conjecture:

Conjecture 6.5.1. *If $p(n)$ is a positive integer valued non-constant polynomial and $q_n = p(n)$, then there exists some $M > 0$ such that*

$$l_i < M \text{ for } i = 1, 2, 3, \ldots. \tag{6.120}$$

Conjecture 6.5.1 should imply the more interesting conjecture:

Conjecture 6.5.2. *Suppose that $p(n)$ is a positive integer valued non-constant polynomial and that $q_n = p(n)$ and F is a Q-special sequence. If*

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \ldots q_n}, \tag{6.121}$$

then

$$D^* \left(\left\{ \frac{E_{F,n}}{q_n} \right\}_{m=0}^{n-1} \right) = O(n^{-1}). \tag{6.122}$$

We remark that if Conjecture 6.5.2 is true, then (6.122) will provide an improved discrepancy estimate over that of Theorem 2.4.10 for those Q-distribution normal numbers that are in Θ_Q.

CHAPTER 7

MEASURE OF SETS OF Q-NORMAL AND Q-DISTRIBUTION NORMAL NUMBERS

A. Rényi showed in [38] that if Q is infinite in limit, then almost every real number is simply Q-normal if and only if Q is 1-divergent. In this chapter, we improve upon this result and show that almost every real number is Q-normal of order k if and only if Q is k-divergent. Additionally, we will provide improved asymptotics on $N_n^Q(B,x)$ for typical real numbers x.

7.1 Strongly Normal Numbers

In this section, we develop notions of normality that are stronger than Q-normality, Q-ratio normality, and Q-distribution normality that coincide with normality in the case of the b-ary expansion. These notions will mainly be used in this chapter to prove the measure theoretic typicality of various types of normal numbers. They will also be crucial when we consider various winning sets associated with normal numbers.

We will first need to make definitions similar to those of $N_n^Q(B,x)$ and $Q_n^{(k)}$:

Definition 7.1.1. *Given a real number $x \in [0,1)$, a basic sequence Q, a block B of length k, a positive integer $p \in [1,k]$, and a positive integer n, we will denote by $N_{n,p}^Q(B,x)$ the number of times the block B occurs in the Q-Cantor series expansion of x with starting position of the form $j \cdot k + p$ for $0 \le j < \frac{n}{k}$.*

Definition 7.1.2. *Given positive integers n and k, we define*

$$\rho(n,k) = \max\left\{ i \in \mathbb{Z} : i < \frac{n}{k} \right\}. \tag{7.1}$$

Definition 7.1.3. *Given a basic sequence Q and positive integers n, p, and k with $p \in [1,k]$, we write*

$$Q_{n,p}^{(k)} = \sum_{j=0}^{\rho(n,k)} \frac{1}{q_{jk+p} q_{jk+p+1} \cdots q_{jk+p+k-1}}. \tag{7.2}$$

We will note the following simple result:

Lemma 7.1.4. *Given a real number $x \in [0,1)$, a basic sequence Q, a block B of length k, a positive integer $p \in [1,k]$, and a positive integer n, we have*

$$N_{n,1}^Q(B,x) + N_{n,2}^Q(B,x) + \ldots + N_{n,k}^Q(B,x) = N_n^Q(B,x) + O(k) \tag{7.3}$$

and

$$Q_{n,1}^{(k)} + Q_{n,2}^{(k)} + \ldots + Q_{n,k}^{(k)} = Q_n^{(k)} + O(k). \tag{7.4}$$

Proof. This follows directly from the definition of $N_n^Q(B,x)$ and $Q_{n,p}^{(k)}$.

□

We wish to make the remark that it is not true in general that $\lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$ if Q is k-divergent. The following basic sequence was suggested by C. Altomare (verbal communication). We consider the basic sequence $Q = \{q_n\}$, defined as follows:

$$q_n = \begin{cases} \max(2, \lfloor n^{1/4} \rfloor) & \text{if } n \equiv 0 \pmod 4 \\ \max(2, \lfloor n^{1/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 1 \pmod 4 \\ \max(2, \lfloor n^{3/4} \rfloor) & \text{if } n \equiv 2 \pmod 4 \\ \max(2, \lfloor n^{3/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 3 \pmod 4 \end{cases}. \tag{7.5}$$

To see that Q is 2-divergent and $\lim_{n\to\infty} Q_{n,1}^{(2)} < \infty$, we reason as follows. Note that for all $m \geq 4$,

$$\frac{1}{q_{4m}q_{4m+1}} = \frac{1}{\lfloor (4m)^{1/4} \rfloor \cdot \lfloor (4m+1)^{1/4} \log^2(4m+1) \rfloor} \geq \frac{1}{\sqrt{4m+1}\log^2(4m+1)}. \tag{7.6}$$

However, we remark that

$$\sum_{n=16}^{\infty} \frac{1}{q_n q_{n+1}} \geq \sum_{m=4}^{\infty} \frac{1}{\sqrt{4m+1}\log^2(4m+1)} = \infty, \tag{7.7}$$

so Q is 2-divergent. We also note that if $n > 16$ and $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$, then

$$\frac{1}{q_n q_{n+1}} \approx \frac{1}{n \log^2 n}. \tag{7.8}$$

So, since

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty, \tag{7.9}$$

we have that $\lim_{n\to\infty} Q_{n,1}^{(2)} < \infty$.

Due to this example, we will need to make the following definition:

Definition 7.1.5. *Let k be a positive integer. Then a basic sequence Q is strongly k-divergent if for all positive integers p with $p \in [1, k]$, we have*

$$\lim_{n\to\infty} Q_{n,p}^{(k)} = \infty. \tag{7.10}$$

A basic sequence Q is strongly fully divergent if it is strongly k-divergent for all k.

Definition 7.1.6. *Suppose that Q is a basic sequence. A real number x in $[0, 1)$ is strongly Q-normal of order k if for all Q-admissable blocks B of length $m \le k$ and all $p \in [1, m]$, we have*

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B, x)}{Q_{n,p}^{(m)}} = 1. \tag{7.11}$$

Definition 7.1.7. *Suppose that Q is a basic sequence. A real number x in $[0, 1)$ is strongly Q-ratio normal of order k if for all Q-admissable blocks B_1 and B_2 of length $m \le k$ and for all $p \in [1, m]$, we have*

$$\lim_{n\to\infty} \frac{N_{n,p}^Q(B_1,x)}{N_{n,p}^Q(B_2,x)} = 1. \tag{7.12}$$

Definition 7.1.8. *Suppose that Q is a basic sequence. A real number x in $[0,1)$ is strongly Q-distribution normal if for all positive integers k and $p \in [1,k]$, the sequence*

$$\{T_{Q,kn+p}(x)\}_{n=1}^{\infty} \tag{7.13}$$

is uniformly distributed mod 1.

We wish to note the following criteria for checking the strong Q-distribution normality of a real number:

Theorem 7.1.9. *If Q is a basic sequence, then x is Q-distribution normal if and only if for all intervals I with rational endpoints, and positive integers k and p with $p \in [1,k]$, we have that*

$$\lim_{n\to\infty} \frac{A_n(I, \{T_{Q,kn+p}(x)\}_{i=0}^{n-1})}{n} = \lambda(I). \tag{7.14}$$

Proof. Given k and p, we set

$$q_0' = q_1 q_2 \cdots q_p. \tag{7.15}$$

For $i \geq 1$, we put

$$q_i' = q_{ik+p+1} q_{ik+p+2} \cdots q_{ik+p+k}. \tag{7.16}$$

We set

$$Q' = \{q'_i\}_{i=0}^{\infty} \tag{7.17}$$

and apply Theorem 2.4.12 to Q'.

□

Definition 7.1.10. *Suppose that Q is a basic sequence. A real number x is strongly simply Q-normal if it is strongly Q-normal of order 1 and strongly simply Q-ratio normal if it is strongly Q-ratio normal of order 1. x is strongly Q-normal if it is strongly Q normal of order k for all k. x is strongly Q-ratio normal if it is strongly Q-ratio normal of order k for all k.*

We will now see that many of these notions are equivalent in the case of the b-ary expansion. We recall the following standard results that may be found in [23]:

Lemma 7.1.11. *Whenever x is normal in base b, so is rx for any rational number r.*

Theorem 7.1.12. *The real number x is normal in base b if and only if x is simply normal to all the bases $b, b^2, b^3, \ldots$.*

Theorem 7.1.13. *Let $k \geq 2$ be an integer. A real number x is normal in base b if and only if it is normal in base b^k.*

We may now prove:

Theorem 7.1.14. *Suppose that $b \geq 2$ is an integer and that $q_n = b$ for all n. Then a real number x is strongly Q-normal if and only if it is Q-normal.*

Proof. First, we assume that x is strongly Q-normal, so x is simply normal in base b^k for all k. Thus, by Theorem 7.1.12, x is normal in base b.

Let k and p be positive integers with $p \in [1, k]$ and suppose that $x = 0.E_1E_2E_3\ldots$ w.r.t. Q. We set

$$y := b^{p-1}x \pmod 1 = 0.E_pE_{p+1}E_{p+2}\ldots \text{ w.r.t. } Q. \tag{7.18}$$

By Lemma 7.1.11, y is normal in base b. So by Theorem 7.1.13, y is normal in base b^k. Since k and p were arbitrary, x is strongly Q-normal.

□

Corollary 7.1.15. *Suppose that $b \geq 2$ is an integer and that $q_n = b$ for all n. Then the following are equivalent*

1. *x is strongly Q-normal;*
2. *x is Q-normal;*
3. *x is strongly Q-distribution normal;*
4. *x is Q-distribution normal;*

5. *x is strongly Q-ratio normal;*

6. *x is Q-ratio normal.*

Proof. These follow by Theorem 7.1.14 and the equivalence of normality, ratio normality, and distribution normality in base b.

□

We will now show that for more general basic sequences Q that strong normality implies normality. We will first need the following lemma:

Lemma 7.1.16. *Let k be a positive integer and suppose that the sequences*

$$X_m = \{x_{m,n}\}_{n=1}^{\infty} \tag{7.19}$$

are uniformly distributed mod 1 for $m \leq k$. Then the sequence

$$Z = (x_{1,1}, x_{2,1}, \ldots, x_{k,1}, x_{1,2}, x_{2,2}, \ldots, x_{k,2}, x_{1,3}, x_{2,3}, \ldots) \tag{7.20}$$

is uniformly distributed mod 1.

Proof. For clarity, we will only prove the case where $k = 2$. Lemma 7.1.16 will follow similarly for larger values of k. Let an arbitrary interval $I \subset [0, 1)$ and a real number $\epsilon > 0$ be given. Suppose that $X = \{x_n\}_{n=1}^{\infty}$ and $Y = \{y_n\}_{n=1}^{\infty}$ are uniformly distributed mod 1. We want to show that the sequence

$$Z = (x_1, y_1, x_2, y_2, x_3, y_3 \ldots) \tag{7.21}$$

is uniformly distributed mod 1. Set

$$P_n = A_n(I, X),\ Q_n = A_n(I, Y),\ \text{and } R_n = A_n(I, Z). \tag{7.22}$$

Note that

$$R_{2n} = P_n + Q_n \tag{7.23}$$

and

$$R_{2n-1} = P_n + Q_{n-1}. \tag{7.24}$$

Since the sequences X and Y are uniformly distributed mod 1, there exists a positive integer M such that for all $n > M$, we have

$$\left| \frac{P_n}{n} - 1 \right| < \epsilon \tag{7.25}$$

and

$$\left| \frac{Q_n}{n} - 1 \right| < \epsilon. \tag{7.26}$$

However, (7.25) and (7.26) are equivalent to

$$n - n\epsilon < P_n < n + n\epsilon \tag{7.27}$$

and

$$n - n\epsilon < Q_n < n + n\epsilon. \tag{7.28}$$

By adding (7.27) and (7.28) and substituting (7.23), we arrive at the inequality

$$2n - 2n\epsilon < R_{2n} < 2n + 2n\epsilon, \tag{7.29}$$

so

$$\left|\frac{R_{2n}}{2n} - 1\right| < \epsilon. \tag{7.30}$$

Using similar reasoning, we add (7.27) and (7.28) and substitute (7.24) to get

$$\left|\frac{R_{2n-1}}{2n-1} - 1\right| < \epsilon. \tag{7.31}$$

Combining (7.30) and (7.31), we see that

$$\lim_{n\to\infty} \frac{R_n}{n} = 1, \tag{7.32}$$

so Z is uniformly distributed mod 1.

∎

Theorem 7.1.17. *If Q is a basic sequence and x is strongly Q-distribution normal, then x is Q-distribution normal.*

Proof. Suppose that x is strongly Q-distribution normal. The sequences

$$\{T_{Q,2n}(x)\}_{n=0}^{\infty} \tag{7.33}$$

and

$$\{T_{Q,2n+1}(x)\}_{n=0}^{\infty} \tag{7.34}$$

are uniformly distributed mod 1. By Lemma 7.1.16, the sequence

$$(T_{Q,0}(x), T_{Q,1}(x), T_{Q,2}(x), T_{Q,3}(x), T_{Q,4}(x), T_{Q,5}(x), \ldots) = \{T_{Q,n}(x)\}_{n=0}^{\infty} \tag{7.35}$$

is uniformly distributed mod 1, so x is Q-distribution normal.

□

We will now need the following lemma:

Lemma 7.1.18. *If $g_1, g_2, \ldots, g_n$ are non-negative functions on the natural numbers, then*

$$o(g_1) + o(g_2) + \ldots + o(g_n) = o(g_1 + g_2 + \ldots + g_n). \tag{7.36}$$

Proof. Suppose that $f_i = o(g_i)$ for $i \in [1, n]$, so

$$\lim_{m \to \infty} \frac{f_i(m)}{g_i(m)} = 0. \tag{7.37}$$

We let

$$g(m) = \max_{1 \le i \le n} g_i(m). \tag{7.38}$$

Thus,

$$\lim_{m \to \infty} \frac{f_i(m)}{g(m)} = 0, \tag{7.39}$$

so

$$\lim_{m \to \infty} \frac{\sum_{i=1}^{n} f_i(m)}{g(m)} = 0. \tag{7.40}$$

However,

$$g(m) \leq \sum_{i=1}^{n} g_i(m), \tag{7.41}$$

so

$$\lim_{m \to \infty} \frac{\sum_{i=1}^{n} f_i(m)}{\sum_{i=1}^{n} g_i(m)} = 0. \tag{7.42}$$

□

Theorem 7.1.19. *If Q is a basic sequence and x is strongly Q-normal of order k, then x is Q-normal of order k.*

Proof. Let $m \leq k$ be a positive integer and let B be a block of length k. Since x is strongly Q-normal of k, we know that for all $p \in [1, m]$,

$$N_{n,p}^{Q}(B, x) = Q_{n,p}^{(k)} + o(Q_{n,p}^{(k)}). \tag{7.43}$$

Thus, by Lemma 7.1.18, we see that

$$N_n^Q(B,x) = \sum_{p=1}^{m} N_{n,p}^Q(B,x) = \sum_{p=1}^{m} \left(Q_{n,p}^{(k)} + o(Q_{n,p}^{(k)})\right) \tag{7.44}$$

$$= \sum_{p=1}^{m} Q_{n,p}^{(k)} + o\left(\sum_{p=1}^{m} Q_{n,p}^{(k)}\right) = Q_n^{(k)} + o\left(Q_n^{(k)}\right), \tag{7.45}$$

so

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1. \tag{7.46}$$

Therefore, x is Q-normal.

□

Theorem 7.1.20. *If Q is a basic sequence and x is strongly Q-ratio normal of order k, then x is Q-ratio normal of order k.*

Proof. Let $m \leq k$ be a positive integer and let B_1 and B_2 be blocks of length m. Since x is strongly Q-ratio normal of k, we know that for all $p \in [1,m]$,

$$\frac{N_{n,p}^Q(B_1,x)}{N_{n,p}^Q(B_2,x)} = 1 + o(1), \tag{7.47}$$

so

$$N_{n,p}^Q(B_1,x) = N_{n,p}^Q(B_2,x) + o\left(N_{n,p}^Q(B_2,x)\right). \tag{7.48}$$

Thus, by Lemma 7.1.18, we see that

$$N_n^Q(B_1,x)=\left(\sum_{p=1}^{m} N_{n,p}^Q(B_1,x)\right)+O(m) \tag{7.49}$$

$$=\sum_{p=1}^{m}\left(N_{n,p}^Q(B_2,x)+o\left(N_{n,p}^Q(B_2,x)\right)\right)+O(m)=\sum_{p=1}^{m} N_{n,p}^Q(B_2,x)+o\left(N_{n,p}^Q(B_2,x)\right)$$

$$=N_{n,p}^Q(B_2,x)+o\left(N_{n,p}^Q(B_2,x)\right),$$

so

$$\lim_{n\to\infty}\frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)}=1. \tag{7.50}$$

Therefore, x is Q-ratio normal.

□

Corollary 7.1.21. *Suppose that Q is a basic sequence. If x is strongly Q-normal, then x is Q-normal. If x is strongly Q-ratio normal, then x is Q-ratio normal.*

Proposition 7.1.22. *If k is a positive integer, Q is a basic sequence, and a real number x is strongly Q-normal of order k, then x is also strongly Q-ratio normal of order k.*

Proof. Let $m \leq k$ be a positive integer and let B_1 and B_2 be blocks of length k. Since x is strongly Q-normal of k, we know that for all $p \in [1,m]$, we have

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_1,x)}{Q_{n,p}^{(m)}} = 1 \tag{7.51}$$

and

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_2,x)}{Q_{n,p}^{(m)}} = 1. \tag{7.52}$$

Dividing (7.51) by (7.52), we arrive at

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_1,x)}{N_{n,p}^{Q}(B_2,x)} = \lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_1,x)/Q_{n,p}^{(m)}}{N_{n,p}^{Q}(B_2,x)/Q_{n,p}^{(m)}} \tag{7.53}$$

$$= \frac{\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_1,x)}{Q_{n,p}^{(m)}}}{\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B_2,x)}{Q_{n,p}^{(m)}}} = \frac{1}{1} = 1,$$

so x is strongly Q-ratio normal of order k.

□

Corollary 7.1.23. *If Q is a basic sequence and a real number x is strongly Q-normal, then x is also strongly Q-ratio normal.*

It is natural to ask the following:

Problem 7.1.24. *Let Q be an arbitrary basic sequence. Must all Q-normal numbers also be strongly Q-normal? Are Q-distribution normal numbers necessarily strongly Q-distribution normal?*

7.2 Random Variables Associated With Normality

For this section, we must recall a few basic notions from probability theory. Given a random variable X, we will denote the expected value of X as $\mathrm{E}[X]$. We will denote the variance of X as $\mathrm{Var}[X]$. Recall that

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2. \tag{7.54}$$

Lastly, $\mathrm{P}(X = j)$ will represent that probability that $X = j$.

We consider x as a random variable which has uniform distribution on the interval $[0, 1)$. If

$$x = \sum_{n=1}^{\infty} \frac{E_n(x)}{q_1 q_2 \cdots q_n}, \tag{7.55}$$

then we consider $E_1(x), E_2(x), E_3(x), \ldots$ to be random variables. So for all n, we have

$$\mathrm{P}(E_n(x) = j) = \begin{cases} \frac{1}{q_n} & \text{if } 0 \leq j \leq q_n - 1 \\ 0 & \text{if } j \geq q_n \end{cases}. \tag{7.56}$$

Lemma 7.2.1. *If Q is a basic sequence, then the random variables*

$$E_1(x), E_2(x), E_3(x), \ldots \tag{7.57}$$

are independent.

Proof. Suppose that n is a positive integer and that $0 \leq F_j < q_j - 1$ for all j. Then

$$\mathrm{P}\left(E_1(x) = F_1, E_2(x) = F_2, \ldots, E_n(x) = F_n\right) \tag{7.58}$$

$$= \lambda\left(\{x \in [0,1) : x = 0.F_1F_2\ldots F_n \text{ w.r.t. } Q\}\right) = \frac{1}{q_1 q_2 \cdots q_n}$$

$$= \frac{1}{q_1} \cdot \frac{1}{q_2} \cdots \frac{1}{q_n} = \mathrm{P}\left(E_1(x) = F_1\right) \cdot \mathrm{P}\left(E_2(x) = F_2\right) \cdots \mathrm{P}\left(E_n(x) = F_n\right).$$

∎

Given a basic sequence Q, we define the following random variables:

Definition 7.2.2. *If b is a natural number, then*

$$r_{b,n}^{Q}(x) = \begin{cases} 1 & \text{if } E_n(x) = b \\ 0 & \text{if } E_n(x) \neq n \end{cases}. \tag{7.59}$$

Definition 7.2.3. *If B is a block of length k, then*

$$r_{B,i,p}^{Q}(x) = \begin{cases} 1 & \text{if } E_{ik+p,k}(x) = B \\ 0 & \text{if } E_{ik+p,k}(x) \neq B \end{cases}. \tag{7.60}$$

Lemma 7.2.4. *For all non-negative integers b, the random variables*

$$r^Q_{b,1}(x), r^Q_{b,2}(x), r^Q_{b,3}(x), \ldots \tag{7.61}$$

are independent.

Proof. This follows directly from Lemma 7.2.1 as the random variables

$$E_1(x), E_2(x), E_3(x), \ldots \tag{7.62}$$

are independent.

□

Lemma 7.2.5. *If $B = (b_1, b_2, \ldots, b_k)$ is a block of length k, then*

$$r^Q_{B,i,p}(x) = r^Q_{b_1,ik+p}(x) \cdot r^Q_{b_2,ik+p+1}(x) \cdots r^Q_{b_k,ik+p+k-1}(x). \tag{7.63}$$

Proof. By definition,

$$r^Q_{B,i,p}(x) = \begin{cases} 1 & \text{if } E_{ik+p,k} = B \\ 0 & \text{if } E_{ik+p,k} \neq B \end{cases}, \tag{7.64}$$

or in other words, $r^Q_{B,i,p}(x) = 1$ if

$$r^Q_{b_1,ik+p}(x) = r^Q_{b_2,ik+p+1}(x) = \ldots = r^Q_{b_k,ik+p+k-1}(x) = 1 \tag{7.65}$$

and $r^Q_{B,i,p}(x) = 0$ otherwise.

□

Corollary 7.2.6. *For all blocks B of length k and positive integers $p \in [1, k]$, the random variables $r^Q_{B,0,p}(x), r^Q_{B,1,p}(x), r^Q_{B,2,p}(x), \ldots$ are independent.*

Proof. Using Lemma 7.2.4 and Lemma 7.2.5, we see that for all $i_1, i_2 \geq 0$ and $p_1, p_2 \in [1, k]$, we have

$$\mathrm{E}\left[r^Q_{B,i_1,p_1}(x) \cdot r^Q_{B,i_2,p_2}(x)\right] = \mathrm{E}\left[\left(\Pi_{j=0}^{k-1} r^Q_{b_j,i_1k+p_1+j}(x)\right) \cdot \left(\Pi_{j=0}^{k-1} r^Q_{b_j,i_2k+p_2+j}(x)\right)\right] \tag{7.66}$$

$$= \left(\Pi_{j=0}^{k-1} \mathrm{E}\left[r^Q_{b_j,i_1k+p_1+j}(x)\right]\right) \cdot \left(\Pi_{j=0}^{k-1} \mathrm{E}\left[r^Q_{b_j,i_2k+p_2+j}(x)\right]\right)$$

$$= \mathrm{E}\left[\Pi_{j=0}^{k-1} r^Q_{b_j,i_1k+p_1+j}(x)\right] \cdot \mathrm{E}\left[\Pi_{j=0}^{k-1} r^Q_{b_j,i_2k+p_2+j}(x)\right]$$

$$= \mathrm{E}\left[r^Q_{B,i_1,p_1}(x)\right] \cdot \mathrm{E}\left[r^Q_{B,i_2,p_2}(x)\right].$$

□

Lemma 7.2.7. *If $B = (b_1, b_2, \ldots, b_k)$ is a block of length k, then*

$$E\left[r^Q_{B,i,p}(B, x)\right] = \frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}} \tag{7.67}$$

and

$$Var\left[r^Q_{B,i,p}(B, x)\right] = \frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p} q_{ik+p+1} \cdots q_{ik+p+k-1}}\right)^2. \tag{7.68}$$

Proof. We first compute the expected value of $r^Q_{B,i,p}(x)$. By Lemma 7.2.4 and Lemma 7.2.5, we see that

$$\mathrm{E}\left[r^Q_{B,i,p}(x)\right] = \mathrm{E}\left[r^Q_{b_1,ik+p}(x) \cdot r^Q_{b_2,ik+p+1}(x) \cdots r^Q_{b_k,ik+p+k-1}(x)\right] \tag{7.69}$$

$$= \mathrm{E}\left[r^Q_{b_1,ik+p}(x)\right] \cdot \mathrm{E}\left[r^Q_{b_2,ik+p+1}(x)\right] \cdots \mathrm{E}\left[r^Q_{b_k,ik+p+k-1}(x)\right]$$

$$= \frac{1}{q_{ik+p}} \cdot \frac{1}{q_{ik+p+1}} \cdots \frac{1}{q_{ik+p+k-1}} = \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}}.$$

Next, computing the variance, we recall that

$$\mathrm{Var}\left[r^Q_{B,i,p}(x)\right] = \mathrm{E}\left[r^Q_{B,i,p}(x)^2\right] - \mathrm{E}\left[r^Q_{B,i,p}(x)\right]^2. \tag{7.70}$$

However, since $r^Q_{B,i,p}(x)$ may only be 0 or 1, we see that

$$\left(r^Q_{B,i,p}(x)\right)^2 = r^Q_{B,i,p}(x), \tag{7.71}$$

so

$$\mathrm{Var}\left[r^Q_{B,i,p}(x)\right] = \frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1} \cdots q_{ik+p+k-1}}\right)^2. \tag{7.72}$$

∎

7.3 Typicality of Normal Numbers

We will need the following law of the iterated logarithm[1]:

Theorem 7.3.1. *Let $X_1, X_2, \ldots, X_n$ be independent random variables. Assume there exists a constant $c > 0$ such that $|X_j| < c$ for all j. Let $F_j = E[X_j], V_j = Var[X_j]$ and*

$$t_n = \sum_{j=1}^{n} V_j. \tag{7.73}$$

If $t_n \to \infty$, then, with probability one,

$$\limsup_{n\to\infty} \frac{X_1 + X_2 + \ldots + X_n - F_1 - F_2 - \ldots F_n}{\sqrt{2t_n \log\log t_n}} = 1. \tag{7.74}$$

Corollary 7.3.2. *Under the same assumptions of Theorem 7.3.1, with probability one,*

$$X_1 + X_2 + \ldots + X_n = F_1 + F_2 + \ldots + F_n + O\left(t_n^{1/2}(\log\log t_n)^{1/2}\right). \tag{7.75}$$

We will also need the Borel-Cantelli Lemma:[2]

[1]See [16].

[2]See [4].

Theorem 7.3.3. *(The Borel Cantelli Lemma) If* $\sum_{n=1}^{\infty} P(A_n) < \infty$*, then* $P(A_n \text{ i.o.}) = 0$.

Definition 7.3.4. *Let* Q *be a basic sequence and* B *be a block of length* k*. If* $m = ik + p$*, we let*

$$F_m^{(k)} = E\left[r_{B,i,p}^Q(x)\right] \tag{7.76}$$

and

$$V_m^{(k)} = Var\left[r_{B,i,p}^Q(x)\right]. \tag{7.77}$$

Given the expected values $F_1^{(k)}, F_2^{(k)}, \ldots, F_n^{(k)}$*, we define*

$$F_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} \tag{7.78}$$

and

$$t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} V_{ik+p}^{(k)}. \tag{7.79}$$

We remark that $F_{n,p}^{(k)} = Q_{n,p}^{(k)}$ *and will use this fact frequently and without mention.*

Lemma 7.3.5. *If* Q *is a basic sequence and* $n, k,$ *and* p *are positive integers with* $p \in [1, k]$*, then*

$$\frac{1}{2} F_{n,p}^{(k)} \leq t_{n,p}^{(k)} < F_{n,p}^{(k)}. \tag{7.80}$$

Proof. We see that

$$t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} V_{jk+p}^{(k)} \tag{7.81}$$

$$= \sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right)^2 \right)$$

$$< \sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right) = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} = F_{n,p}^{(k)}.$$

To show the other direction of the inequality, we recall that since Q is a basic sequence, $q_m \geq 2$ for all m so for all i

$$\sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right)^2 \right) \tag{7.82}$$

$$\geq \sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \frac{1}{2^k} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} \right) \right)$$

$$\geq \sum_{i=0}^{\rho(n,k)} \frac{1}{2} \cdot \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} = \frac{1}{2} F_{n,p}.$$

∎

Lemma 7.3.6. *If Q is a basic sequence and B is a block of length k, then for almost every real number x is $[0,1)$, we have*

$$N_{n,p}^{Q}(B,x) = F_{n,p}^{(k)} + O\left(\sqrt{F_{n,p}^{(k)}} \left(\log\log F_{n,p}^{(k)} \right)^{1/2} \right). \tag{7.83}$$

Proof. We consider two cases. The first case is when

$$\lim_{n\to\infty} F_{n,p}^{(k)} < \infty. \tag{7.84}$$

We see that

$$\lim_{n\to\infty} F_{n,p}^{(k)} = \lim_{n\to\infty} \sum_{i=0}^{\rho(n,k)} \mathrm{P}\left(r_{B,i,p}^{Q} = 1\right) < \infty, \tag{7.85}$$

so by Theorem 7.3.3, we have

$$\mathrm{P}\left(r_{B,i,p}^{Q} = 1 \text{ i.o. }\right) = 0. \tag{7.86}$$

Thus, for almost every $x \in [0, 1)$,

$$\lim_{n\to\infty} N_{n,p}^{Q}(B, x) < \infty \tag{7.87}$$

and (7.83) holds.

Second, we consider the case where

$$\lim_{n\to\infty} F_{n,p}^{(k)} = \infty. \tag{7.88}$$

By Lemma 7.3.5, we have

$$\lim_{n\to\infty} t_{n,p}^{(k)} \geq \lim_{n\to\infty} F_{n,p}^{(k)} = \infty. \tag{7.89}$$

Note that

$$N_{n,p}^Q(B,x) = \sum_{i=0}^{\rho(n,k)} r_{B,i,p}(x), \tag{7.90}$$

so by Corollary 7.3.2,

$$N_{n,p}^Q(B,x) = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} + O\left(\sqrt{t_{n,p}^{(k)}}\left(\log\log t_{n,p}^{(k)}\right)^{1/2}\right) \tag{7.91}$$

for almost every $x \in [0,1)$. By Lemma 7.3.5, $t_{n,p}^{(k)} < F_{n,p}^{(k)}$, so for almost every $x \in [0,1)$, we have

$$N_{n,p}^Q(B,x) = F_{n,p}^{(k)} + O\left(\sqrt{F_{n,p}^{(k)}}\left(\log\log F_{n,p}^{(k)}\right)^{1/2}\right). \tag{7.92}$$

□

Lemma 7.3.6 allows us to prove the following results on strongly normal numbers:

Theorem 7.3.7. *Suppose that Q is strongly k-divergent and infinite in limit. Then almost every $x \in [0,1)$ is strongly Q-normal of order k.*

Proof. Let B be a block of length $m \leq k$ and $p \in [1,m]$. Then by Lemma 7.3.6 for almost every $x \in [0,1)$, we have that

$$N_{n,p}^Q(B,x) = F_{n,p}^{(m)} + O\left(\sqrt{F_{n,p}^{(m)}}\left(\log\log F_{n,p}^{(m)}\right)^{1/2}\right), \tag{7.93}$$

so

$$\frac{N_{n,p}^{Q}(B,x)}{Q_{n,p}^{(m)}} = \frac{F_{n,p}^{(m)}}{Q_{n,p}^{(m)}} + O\left(\frac{\sqrt{F_{n,p}^{(m)}}\left(\log\log F_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right). \tag{7.94}$$

However, Q is strongly k-divergent, so $Q_{n,p}^{(m)} \to \infty$ and

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B,x)}{Q_{n,p}^{(m)}} = \lim_{n\to\infty} \frac{F_{n,p}^{(m)}}{Q_{n,p}^{(m)}} + O\left(\frac{\sqrt{F_{n,p}^{(m)}}\left(\log\log F_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right) = 1. \tag{7.95}$$

Since there are finitely many choices of m and p and only countably many choices of B, the result follows.

□

We also note the simple corollary:

Corollary 7.3.8. *If Q is strongly fully divergent and infinite in limit, then almost every real $x \in [0,1)$ is strongly Q-normal.*

We now work torwards proving a result much stronger than Corollary 7.3.8 on the typicality of Q-normal numbers. We will need the following lemma in addition to Lemma 7.3.6:

Lemma 7.3.9. *If Q is a basic sequence and k and p are positive integers with $p \in [1,k]$, then*

$$\sum_{p=1}^{k}\left(F_{n,p}^{(k)} + O\left(\sqrt{F_{n,p}^{(k)}}\left(\log\log F_{n,p}^{(k)}\right)^{1/2}\right)\right) = Q_{n}^{(k)} + O\left(\sqrt{Q_{n}^{(k)}}\left(\log\log Q_{n}^{(k)}\right)^{1/2}\right). \tag{7.96}$$

Proof. We first note that

$$\sum_{p=1}^{k} F_{n,p}^{(k)} \leq Q_n^{(k)} + \left(Q_n^{(k)} - Q_{n-k}^{(k)}\right). \tag{7.97}$$

Since $Q_n^{(k)} - Q_{n-k}^{(k)} \to 0$, we see that

$$\sum_{p=1}^{k} F_{n,p}^{(k)} = Q_n^{(k)} + o(1). \tag{7.98}$$

Next, we note that

$$\sum_{p=1}^{k} \sqrt{F_{n,p}^{(k)}} \left(\log\log F_{n,p}^{(k)}\right)^{1/2} \leq k \sqrt{\sum_{p=1}^{k} F_{n,p}^{(k)} \left(\log\log \sum_{p=1}^{k} F_{n,p}^{(k)}\right)^{1/2}}. \tag{7.99}$$

By (7.98) and (7.99), we see that

$$\sum_{p=1}^{k} O\left(\sqrt{F_{n,p}^{(k)}} \left(\log\log F_{n,p}^{(k)}\right)^{1/2}\right) = O\left(\sqrt{Q_n^{(k)}} \left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{7.100}$$

Combining (7.98) and (7.100), we have

$$\sum_{p=1}^{k} \left(F_{n,p}^{(k)} + O\left(\sqrt{F_{n,p}^{(k)}} (\log\log F_{n,p}^{(k)})^{1/2}\right)\right) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}} \left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{7.101}$$

□

Theorem 7.3.10. *If Q is a basic sequence and B is a block of length k, then for almost every real number x is $[0,1)$, we have*

$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{7.102}$$

Proof. We first note that

$$\sum_{p=1}^{k} N_{n,p}(B,x) \leq N_n^Q(B,x) + \left(N_n^Q(B,x) - N_{n-k}^Q(B,x)\right) \tag{7.103}$$

$$= N_n^Q(B,x) + O(1), \tag{7.104}$$

so

$$N_n^Q(B,x) = \sum_{p=1}^{k} N_{n,p}(B,x) + O(1). \tag{7.105}$$

Thus, by (7.105) and Lemma 7.3.6, for almost every $x \in [0,1)$, we have

$$N_n^Q(B,x) = \sum_{p=1}^{k}\left(F_{n,p}^{(k)} + O\left(\sqrt{F_{n,p}^{(k)}}\left(\log\log F_{n,p}^{(k)}\right)^{1/2}\right)\right) + O(1). \tag{7.106}$$

Applying Lemma 7.3.9 to (7.106), we see that for almost every $x \in [0,1)$,

$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{7.107}$$

□

We recall the following standard result on infinite products:

Lemma 7.3.11. *If* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that* $0 \le a_n < 1$ *for all* n*, then the infinite product*

$$\prod_{n=1}^{\infty}(1-a_n) \tag{7.108}$$

converges if and only if the sum

$$\sum_{n=1}^{\infty} a_n \tag{7.109}$$

is convergent.

Theorem 7.3.12. *Suppose that* Q *is a basic sequence that is infinite in limit. Then almost every real number in* $[0,1)$ *is* Q*-normal of order* k *if and only if* Q *is* k*-divergent.*

Proof. First, we suppose that Q is k-divergent. Then by Theorem 7.3.10, for almost every $x \in [0,1)$, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = \lim_{n\to\infty} \frac{Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right)}{Q_n^{(k)}} \tag{7.110}$$

$$= 1 + \lim_{n\to\infty} \frac{O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right)}{Q_n^{(k)}} = 1.$$

We now suppose that Q is k-convergent.[3] Set

[3] We will use similar reasoning to that found in [38].

$$B = (0, 0, \dots, 0) \ (k \text{ zeros}). \tag{7.111}$$

We will show that the set of real numbers in $[0, 1)$ whose Q-Cantor series expansion does not contain the block B has positive measure. Call this set V. We see that

$$\lambda(V) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{q_n q_{n+1} \cdots q_{n+k-1}}\right). \tag{7.112}$$

Set

$$a_n = q_n q_{n+1} \cdots q_{n+k-1}. \tag{7.113}$$

Since Q is k-convergent, we have $\sum a_n < \infty$. Thus, $\lambda(V) > 0$ by Lemma 7.3.11.

□

Corollary 7.3.13. *Suppose that Q is a basic sequence that is infinite in limit. Then almost every real number in $[0, 1)$ is Q-normal if and only if Q is fully divergent.*

Now that we have investigated the typicality of Q-normal numbers, we may turn our attention to Q-distribution normality. We will need the following lemma from [23]:

Lemma 7.3.14. *Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence of distinct integers. Then the sequence $\{a_n x\}_{n=1}^{\infty}$ is uniformly distributed mod 1 for almost all real numbers x.*

This allows us to give an easy proof that for any basic sequence Q, almost every real number is strongly Q-distribution normal[4].

Theorem 7.3.15. *If Q be a basic sequence, then almost every real x in $[0,1)$ is strongly Q-distribution normal.*

Proof. Let k be a positive integer and let $p \in [1,k]$. Set

$$a_n = q_1 q_2 \cdots q_{kn+p}. \tag{7.114}$$

Since $q_n \geq 2$ for all n, the sequence $\{a_n\}$ consists of distinct elements. Thus, by Lemma 7.3.14, the sequence

$$\{a_n x \pmod 1\}_{n=1}^{\infty} = \{T_{Q,kn+p}(x)\}_{n=1}^{\infty} \tag{7.115}$$

is uniformly distributed mod 1 for almost every real x. Since there are countably many choices for k and p, almost every real number is strongly Q-distribution normal.

□

Corollary 7.3.16. *Let Q be a basic sequence. Then almost every real x in $[0,1)$ is Q-distribution normal.*

---

[4]Theorem 7.3.15 is a slightly stronger version of Corollary 7.3.16, which was known to many authors (see, for example, [24]).

Corollary 7.3.17. *Suppose that Q is k-divergent and infinite in limit. Then almost every real x in $[0,1)$ is Q-normal of order k and strongly Q-distribution normal. If Q is fully divergent, then almost every real x in $[0,1)$ is Q-normal and strongly Q-distribution normal.*

Proof. The set of strongly Q-distribution normal numbers has full measure by Theorem 7.3.15. If Q is k-divergent, then the set of Q-normal numbers of order k has full measure by Theorem 7.3.7. If Q is fully divergent, then the set of Q-normal numbers has full measure by Corollary 7.3.13. The corollary follows as the intersection of two sets of full measure has full measure. □

7.4 Another Approach

We will now see that Theorem 7.1.19 allows us to prove that almost every real number is Q-normal for fully divergent Q in a different way. We will now proceed in a manner similar to that of [38]. We will need the following lemma:[5]

Lemma 7.4.1. *If the random variables $\eta_1, \eta_2, \ldots, \eta_n, \ldots$ are independent, $E[\eta_n] = 0$, and*

$$\sum_{n=1}^{\infty} Var[\eta_n] / b_n^2 < \infty, \tag{7.116}$$

[5]Lemma 7.4.1 is discussed in [38] and follows from Kolmogorov's three series test and Kronecker's lemma.

then $\frac{1}{b_n}\sum_{k=1}^{n}\eta_k$ *tends to* 0 *with probability* 1*, provided*

$$0 < b_n \leq b_{n+1} \; (n = 1, 2, \ldots) \tag{7.117}$$

and

$$\lim_{n\to\infty} b_n = \infty. \tag{7.118}$$

We now need to define new random variables to proceed:

Definition 7.4.2. *Suppose that* Q *is a basic sequence,* B *is a block of length* k*, and* p *is a positive integer with* $p \in [1, k]$*. Then for all non-negative* i*, define*

$$\eta_{B,i,p}^{Q}(x) = r_{B,i,p}^{Q}(x) - E\left[r_{B,i,p}^{Q}(x)\right] = r_{B,i,p}^{Q}(x) - F_{ik+p}^{(k)}. \tag{7.119}$$

Lemma 7.4.3. *Suppose that* Q *is a basic sequence,* B *is a block of length* k*, and* p *is a positive integer with* $p \in [1, k]$*. Then*

$$E\left[\eta_{B,i,p}^{Q}(x)\right] = 0 \tag{7.120}$$

and

$$Var\left[\eta_{B,i,p}^{Q}(x)\right] = Var\left[r_{B,i,p}^{Q}(x)\right] = F_{ik+p}^{(k)}\left(1 - F_{ik+p}^{(k)}\right). \tag{7.121}$$

The random variables $\eta_{B,i,p}^{Q}(x), \eta_{B,1,p}^{Q}(x), \eta_{B,2,p}^{Q}(x), \ldots$ *are also independent.*

Proof. First, we write

$$\mathrm{E}\left[\eta_{B,i,p}^{Q}(x)\right] = \mathrm{E}\left[r_{B,i,p}^{Q}(x) - F_{ik+p}^{(k)}\right] \tag{7.122}$$

$$= \mathrm{E}\left[r_{B,i,p}^{Q}(x)\right] - F_{ik+p}^{(k)} = F_{ik+p}^{(k)} - F_{ik+p}^{(k)} = 0.$$

Next, by basic properties of variance,

$$\mathrm{Var}\left[\eta_{B,i,p}^{Q}(x)\right] = \mathrm{Var}\left[r_{B,i,p}^{Q}(x) - F_{ik+p}^{(k)}\right] = \mathrm{Var}\left[r_{B,i,p}^{Q}(x)\right]. \tag{7.123}$$

Independence follows directly from Corollary 7.2.6.

□

Theorem 7.4.4. *Suppose that Q is strongly k-divergent and infinite in limit. Then almost every $x \in [0,1)$ is strongly Q-normal of order k.*

Proof. We will apply Lemma 7.4.1. Suppose that B is a block of length $m \leq k$ and that $p \in [1,m]$. Set

$$b_i = \sum_{j=0}^{i-1} F_{jm+p}^{(m)} = F_{im,p}^{(m)}. \tag{7.124}$$

Since Q is strongly k-divergent, we see that

$$\lim_{i\to\infty} b_i = \infty, \tag{7.125}$$

so (7.118) holds. Clearly, $b_i \leq b_{i+1}$, so (7.117) holds. The random variables

$$\eta^Q_{B,0,p}(x), \eta^Q_{B,1,p}(x), \eta^Q_{B,2,p}(x), \ldots \tag{7.126}$$

are independent by Lemma 7.4.3, so we only need verify (7.116).

Since $\lim_{n\to\infty} q_n = \infty$, there exists a positive integer n_B such that $n_B \equiv p \pmod{m}$ and for all i satisfying

$$i \geq 1 + \frac{n_B - p}{m}, \tag{7.127}$$

we have that

$$B < Q_{(i-1)m+p,m}. \tag{7.128}$$

Thus,

$$\sum_{i=1+(n_B-p)/m}^{\infty} \frac{\mathrm{Var}\left[\eta^Q_{B,i-1,p}(x)\right]}{b_i^2} = \sum_{i=1+(n_B-p)/m}^{\infty} \frac{F^{(k)}_{(i-1)m+p}\left(1 - F^{(k)}_{(i-1)m+p}\right)}{\left(\sum_{j=0}^{i-1} F^{(m)}_{jm+p}\right)^2} \tag{7.129}$$

$$\leq \sum_{i=1+(n_B-p)/m}^{\infty} \frac{F^{(k)}_{(i-1)m+p}}{\left(\sum_{j=0}^{i-1} F^{(m)}_{jm+p}\right)\left(\sum_{j=0}^{i-2} F^{(m)}_{jm+p}\right)} = \sum_{i=1+(n_B-p)/m}^{\infty} \frac{F^{(m)}_{im,p} - F^{(m)}_{(i-1)m,p}}{F^{(m)}_{im,p} F^{(m)}_{(i-1)m,p}}$$

$$= \sum_{i=1+(n_B-p)/m}^{\infty} \left(\frac{1}{F^{(m)}_{(i-1)m,p}} - \frac{1}{F^{(m)}_{im,p}}\right) = \lim_{i\to\infty} \left(\frac{1}{F^{(m)}_{n_B-p,p}} - \frac{1}{F^{(m)}_{im,p}}\right)$$

$$= \frac{1}{F^{(m)}_{n_B-p,p}} - 0 < \infty,$$

as Q is strongly k-divergent.

By Lemma 7.4.1, we obtain

$$P\left(\lim_{i\to\infty}\frac{\sum_{j=0}^{i-1}\eta_{B,j,p}^{Q}(x)}{F_{i,p}^{(m)}}=0\right)=1. \tag{7.130}$$

However, $\eta_{B,j,p}^{Q}(x)=r_{B,i,p}^{Q}(x)-F_{ik+p}^{(k)}$, so

$$P\left(\lim_{i\to\infty}\frac{\sum_{j=0}^{i-1}\left(r_{B,j,p}^{Q}(x)-F_{jk+p}^{(k)}\right)}{F_{im,p}^{(m)}}=0\right)=1. \tag{7.131}$$

We remark that if

$$\frac{n-m}{m}\leq i<\frac{n}{m}, \tag{7.132}$$

then

$$\sum_{j=0}^{i-1}r_{B,j,p}^{Q}(x)=N_{n,p}^{Q}(B,x). \tag{7.133}$$

So,

$$P\left(\lim_{n\to\infty}\frac{N_{n,p}^{Q}(B,x)}{Q_{n,p}^{(m)}}=1\right)=1. \tag{7.134}$$

Since $m\leq k$ and $p\in[1,m]$ were arbitrary with only finitely many choices of each, almost every $x\in[0,1)$ is strongly Q-normal of order k.

□

We will now see that Lemma 7.4.1 may also be used to prove a weaker version of Theorem 7.3.15. We make the following definitions:

Definition 7.4.5. *Given a basic sequence Q, a non-negative integer n, and an interval $I \subset [0,1)$, we define*

$$s_{I,n}^{Q} = \begin{cases} 1 & \text{if } T_{Q,n}(x) \in I \\ 0 & \text{if } T_{Q,n}(x) \notin I \end{cases}, \tag{7.135}$$

$$G_{I,n}^{Q} = E\left[s_{I,n}^{Q}\right], \tag{7.136}$$

and

$$\nu_{I,n} = s_{I,n}^{Q} - G_{I,n}^{Q}. \tag{7.137}$$

Lemma 7.4.6. *Given a basic sequence Q, a non-negative integer n, and an open interval $I \subset [0,1)$, we have*

$$\frac{\lfloor q_n \lambda(I) \rfloor}{q_n} - \frac{2}{q_n} \leq G_{I,n}^{Q} \leq \frac{\lfloor q_n \lambda(I) \rfloor}{q_n} + \frac{2}{q_n}. \tag{7.138}$$

Proof. We set

$$t(n) = \#\left\{m : \left[\frac{m}{q_{n+1}}, \frac{m+1}{q_{n+1}}\right) \cap I \neq \emptyset\right\}. \tag{7.139}$$

Then, clearly, we have

$$\lfloor q_n \lambda(I) \rfloor \leq t(n) \leq 2 + \lfloor q_n \lambda(I) \rfloor. \tag{7.140}$$

However,

$$\frac{t(n) - 2}{q_n} \leq \mathrm{P}\left(s_{I,n}^{Q}(x) = 1\right) \leq \frac{t(n)}{q_n}, \tag{7.141}$$

so

$$\frac{\lfloor q_n \lambda(I) \rfloor}{q_n} - \frac{2}{q_n} \leq \mathrm{E}\left[s_{I,n}^{Q}(x)\right] \leq \frac{\lfloor q_n \lambda(I) \rfloor}{q_n} + \frac{2}{q_n}. \tag{7.142}$$

□

Lemma 7.4.7. *Given a basic sequence Q, a non-negative integer n, and an open interval $I \subset [0,1)$, we have*

$$E\left[\nu_{I,n}^{Q}(x)\right] = 0 \tag{7.143}$$

and

$$Var\left[\nu_{I,n}^{Q}(x)\right] \leq G_{I,n}^{Q}. \tag{7.144}$$

Proof. (7.143) follows by definition. To show (7.144), we note that

$$\mathrm{Var}\left[\nu_{I,n}^{Q}(x)\right] = \mathrm{Var}\left[s_{I,n}^{Q}(x) - G_{I,n}^{Q}\right] = \mathrm{Var}\left[s_{I,n}^{Q}(x)\right] \tag{7.145}$$

$$= \mathrm{E}\left[\left(s_{I,n}^{Q}(x)\right)^2\right] - \left(G_{I,n}^{Q}\right)^2 = \mathrm{E}\left[s_{I,n}^{Q}(x)\right] - \left(G_{I,n}^{Q}\right)^2 \leq G_{I,n}^{Q}.$$

□

Lemma 7.4.8. *Given a basic sequence Q that is infinite in limit and an open interval $I \subset [0,1)$, we have*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} G_{I,k}^{Q}}{n\lambda(I)} = 1. \tag{7.146}$$

Proof. For $n \geq 1$, we set $b_n = \lambda(I)$. We also set

$$a_1 = G_{I,0}^{Q} + G_{I,1}^{Q} \text{ and } a_n = G_{I,n}^{Q}, \text{ for } n \geq 2. \tag{7.147}$$

By Lemma 7.4.6, for $n \geq 2$, we have

$$\frac{\frac{\lfloor q_n\lambda(I)\rfloor}{q_n} - \frac{2}{q_n}}{\lambda(I)} \leq \frac{\frac{a_n}{b_n} \leq \frac{\lfloor q_n\lambda(I)\rfloor}{q_n} + \frac{2}{q_n}}{\lambda(I)}. \tag{7.148}$$

Since Q is infinite in limit and by (7.148), we have $\lim_{n\to\infty} \frac{a_b}{b_n} = 1$. Thus, by Theorem 6.2.6, we conclude that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} G_{I,k}^{Q}}{n\lambda(I)} = 1. \tag{7.149}$$

□

We may now prove:

Theorem 7.4.9. *If Q is infinite in limit, then almost every real $x \in [0,1)$ is Q-distribution normal.*

Proof. We will apply Lemma 7.4.1. Suppose that $I \subset [0,1)$ is an open interval with rational endpoints. Set

$$b_n = \sum_{k=0}^{n} G_{I,k}^{Q}. \tag{7.150}$$

Since Q is infinite in limit, there exists a positive integer N such that for $n > N$, we have

$$\frac{\lfloor q_k \lambda(I) \rfloor}{q_k} \geq \frac{1}{2}\lambda(I). \tag{7.151}$$

Thus, by Lemma 7.4.6, we see that

$$\lim_{n\to\infty} b_n \geq \sum_{k=0}^{\infty} \left(\frac{\lfloor q_k \lambda(I) \rfloor}{q_k} - \frac{2}{q_k} \right) \geq \sum_{k=N}^{\infty} \left(\frac{1}{2} \cdot \lambda(I) - \frac{2}{q_k} \right) = \infty, \tag{7.152}$$

so (7.118) holds. Clearly, $b_n \leq b_{n+1}$, so (7.117) holds. The random variables

$$\nu_{I,1}^{Q}(x), \nu_{I,2}^{Q}(x), \nu_{I,3}^{Q}(x), \ldots \tag{7.153}$$

are independent so we only need verify (7.116).

By Lemma 7.4.7, we see that

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}\left[\nu_{I,n}^{Q}(x)\right]}{b_n^2} \leq \sum_{n=1}^{\infty} \frac{G_{I,n}^{Q}}{\left(\sum_{k=0}^{n} G_{I,k}^{Q}\right)^2} \tag{7.154}$$

$$\leq \sum_{n=1}^{\infty} \frac{G_{I,n}^{Q}}{\left(\sum_{k=0}^{n} G_{I,k}^{Q}\right)\left(\sum_{k=0}^{n-1} G_{I,k}^{Q}\right)} = \sum_{n=1}^{\infty} \frac{b_n - b_{n-1}}{b_n \cdot b_{n-1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{b_{n-1}} - \frac{1}{b_n}\right) = \lim_{n\to\infty} \left(\frac{1}{b_0} - \frac{1}{b_n}\right) = \frac{1}{b_0} - 0 < \infty.$$

By Lemma 7.4.1, we obtain

$$\mathrm{P}\left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \nu_{I,k}^{Q}(x)}{b_n} = 0\right) = 1. \tag{7.155}$$

However, $\nu_{I,k}^{Q}(x) = s_{I,k}^{Q}(x) - G_{I,k}^{Q}$, so

$$\mathrm{P}\left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \left(s_{I,k}^{Q}(x) - G_{I,k}^{Q}\right)}{\sum_{k=0}^{n} G_{I,k}^{Q}} = 0\right) = 1. \tag{7.156}$$

We remark that

$$\sum_{k=1}^{n} s_{I,k}^{Q}(x) = A_n\left(I, \{T_{Q,n}(x)\}_{n=1}^{\infty}\right), \tag{7.157}$$

so

$$\mathrm{P}\left(\lim_{n\to\infty} \frac{A_n\left(I, \{T_{Q,n}(x)\}_{n=1}^{\infty}\right)}{\sum_{k=0}^{n} G_{I,k}^{Q}} = 1\right) = 1. \tag{7.158}$$

Thus, we may apply Lemma 7.4.8 to (7.158) to obtain

$$\mathrm{P}\left(\lim_{n\to\infty} \frac{A_n\left(I, \{T_{Q,n}(x)\}_{n=1}^{\infty}\right)}{n\lambda(I)} = 1\right) = 1, \tag{7.159}$$

so

$$\mathrm{P}\left(\lim_{n\to\infty} \frac{A_n\left(I, \{T_{Q,n}(x)\}_{n=1}^{\infty}\right)}{n} = \lambda(I)\right) = 1. \tag{7.160}$$

However, we recall that by Theorem 2.4.12, we only need to check that (7.160) holds for intervals I with rational endpoints. Since there are only countable many such intervals, almost every real $x \in [0,1)$ is Q-distribution normal.

□

Corollary 7.4.10. *If Q is infinite in limit, then almost every real $x \in [0,1)$ is strongly Q-distribution normal.*

Proof. Suppose that $k > 1$ and $p \in [1,k]$. Then we let

$$Q' = (q_p q_{p+1} \cdots q_{p+k-1}, q_{p+k} q_{p+k+1} \cdots q_{p+2k-1}, \ldots) \tag{7.161}$$

and apply Theorem 7.4.9 to Q'. Since there were countably many choices of k and p, the result follows.

□

7.5 Applications to Ratio Normal Numbers

We are now in a position to compare the prevelance of Q-normal numbers to Q-ratio normal numbers, depending on properties of the basic sequence Q. In particular, we will show that if Q is infinite in limit, then while we cannot guarantee the existance of Q-normal numbers, the set of Q-ratio normal numbers will be dense in $[0,1)$.

We first need the following notation:

Definition 7.5.1. *Suppose that Q is a k-convergent basic sequence. Define*

$$Q_{\infty}^{(k)} = \lim_{n\to\infty} Q_n^{(k)} < \infty. \tag{7.162}$$

Lemma 7.5.2. *If Q is k-convergent, then a real number $x \in [0,1)$ is Q-normal of order k if and only if for all blocks B of length $m \leq k$, we have*

$$\lim_{n\to\infty} N_n^Q(B,x) = Q_{\infty}^{(m)}. \tag{7.163}$$

Proof. This follows directly from the definition of Q-normality of order k.

□

Theorem 7.5.3. *If Q is a basic sequence that is k-convergent for some k, then the set of Q-normal numbers is empty.*

Proof. We make the observation that since $q_n \geq 2$ for all n,

$$Q_{\infty}^{(k)} \leq \frac{1}{2} Q_{\infty}^{(k-1)} \tag{7.164}$$

for all k. Thus, there will exists a $K > 0$ such that for all $k > K$, we have

$$Q_\infty^{(k)} < 1. \tag{7.165}$$

Clearly, by Lemma 7.5.2, the set of numbers that are Q-normal of order K must be empty.

□

We now prove two lemmas that will allow us to use Corollary 7.3.13 to show that the set of real numbers that are Q-ratio normal is dense in $[0, 1)$ for all Q that are infinite in limit.

Lemma 7.5.4. *Suppose that the basic sequence Q is given by $Q = (q_1, q_2, \ldots)$ and that n is a positive integer. Suppose that $x = 0.E_1E_2E_3\ldots$ w.r.t. Q and that x is Q-ratio normal. If we let*

$$Q' = (s_1, s_2, \ldots, s_n, q_{n+1}, q_{n+2}, q_{n+3}, \ldots) \tag{7.166}$$

and

$$y = 0.F_1F_2\ldots F_nE_1E_2E_3\ldots \text{ w.r.t. } Q \tag{7.167}$$

for integers $s_i \geq 2$ and $0 \leq F_i \leq s_i - 1$, then y is Q'-ratio normal.

Proof. Let B_1 and B_2 both be blocks of equal length. Clearly,

$$\lim_{n\to\infty} \frac{N_n^{Q'}(B_1, y)}{N_n^{Q'}(B_2, y)} = \lim_{n\to\infty} \frac{N_n^{Q}(B_1, x)}{N_n^{Q}(B_2, x)} = 1, \tag{7.168}$$

so y is Q'-ratio normal.

□

Proposition 7.5.5. *If Q is infinite in limit, then there exists a real number that is Q-ratio normal.*

Proof. Let Q' be any fully divergent basic sequence. Then we know that there exists a Q'-ratio normal number by Corollary 7.3.13. Let $x = 0.E_1E_2E_3\ldots$ w.r.t. Q' be Q'-ratio normal and suppose that

$$E' = (E'_1, E'_2, \ldots). \tag{7.169}$$

Let

$$M_k = \min\{m : q_n > k \ \forall n \geq m\}. \tag{7.170}$$

Set

$$E_n = \min(E'_n, q_n - 1) \tag{7.171}$$

and put $E = (E_1, E_2, \ldots)$. Suppose that B and B' are two blocks of length k and let

$$l = \max(\max(B), \max(B')) + 2. \tag{7.172}$$

Thus, if $n > M_l$, then $E'_{n,k} = B$ is equivalent to $E_{n,k} = B$ and $E'_{n,k} = B'$ is equivalent to $E_{n,k} = B'$. Since $E_n \leq q_n - 1$ for all n, E is a Q-admissable sequence, so

$$\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{7.173}$$

is Q-ratio normal.

□

Theorem 7.5.6. *If Q is infinite in limit, then the set of numbers that are Q-ratio normal is dense in $[0, 1)$.*

Proof. Let $x = 0.E_1E_2E_3\ldots$ w.r.t Q and let $\epsilon > 0$. Let n be large enough so that

$$\frac{1}{q_1 q_2 \cdots q_n} < \epsilon. \tag{7.174}$$

Let $Q' = (q_{n+1}, q_{n+2}, q_{n+3}, \ldots)$. As Q' is infinite in limit, by Proposition 7.5.5, there exists a real number y that is Q'-ratio normal. Suppose that $y = .F_1F_2\ldots$ w.r.t Q'. We set

$$z = 0.E_1E_2\ldots E_nF_1F_2F_3\ldots \text{ w.r.t. } Q. \tag{7.175}$$

Clearly,

$$|x - z| < \frac{1}{q_1 q_2 \cdots q_n} < \epsilon. \tag{7.176}$$

We also see by Lemma 7.5.4 that z is Q-ratio normal so the set of Q-ratio normal numbers is dense in $[0, 1)$.

□

Lemma 7.5.7. *Suppose that Q' and Q are infinite in limit and that $Q' \leq Q$. If x is Q'-ratio normal, then $\Phi_{Q'}^{Q}(x)$ is Q-ratio normal.*

Proof. This follows directly from the definition of ratio normality.

□

Theorem 7.5.8. *Suppose that Q is infinite in limit. Then there exists a real number that is Q-ratio normal of order k and not Q-normal of order k.*

Proof. We will need to examine two separate cases. First, we will consider the case where Q is k-divergent. Let

$$Q' = Q/2. \tag{7.177}$$

By Theorem 7.3.12, we know that there exists a number that is Q-normal of order k. Suppose that x is Q-normal of order k and let

$$y = \Phi_{Q'}^{Q}(x). \tag{7.178}$$

Since $Q' \leq Q$, we see that y is Q-ratio normal of order k by Lemma 7.5.7. To see that y is not Q-normal of order k, we will compute $I_Q^{(1)}(y)$ and apply Lemma 2.5.8. Since Q is infinite in limit and $Q' = Q/2$, we see that for all $\epsilon > 0$ there exists an integer N such that for all $n > N$, we have

$$(1/2 - \epsilon)q_n \leq q_n' \leq q_n/2. \tag{7.179}$$

Thus,

$$\lim_{n\to\infty} \frac{q_n}{q'_n} = 2, \tag{7.180}$$

so

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{q_k}}{\sum_{k=1}^{n} \frac{1}{q'_k}} = \frac{1}{2}. \tag{7.181}$$

Since x is Q-normal of order k, we know that $I_{Q'}^{1}(x) = 1$. But since $N_n(0,x) = N_n(0,y)$ for all n, we see that

$$I_Q^{(1)}(y) = \lim_{n\to\infty} \frac{N_n(0,y)}{\sum_{k=1}^{n} \frac{1}{q_k}} = \lim_{n\to\infty} \frac{N_n(0,x)}{(1/2)\sum_{k=1}^{n} \frac{1}{q'_k}} = 2I_{Q'}^{(1)}(x) = 2. \tag{7.182}$$

Since $I_Q^{(1)}(y) \neq 1$, y is not Q-normal of order k.

Next, we consider the case where Q is k-convergent. Since Q is infinte in limit, we have shown that there will exist a real number x that is Q-ratio normal. Since x is Q-ratio normal, all blocks B of length $k+1$ will appear at least once. Therefore, all blocks of length k will appear infinitely many times. But Q is k-convergent, so

$$\lim_{n\to\infty} Q_n^{(k)} < \infty \tag{7.183}$$

and each block can occur only finitely many times in the Q-Cantor series expansion of x. Thus, $\Phi_{Q'}^{Q}(x)$ is Q-ratio normal of order k, but not Q-normal of order k.

□

Corollary 7.5.9. *If Q is infinite in limit, then there exists a real number that is Q-ratio normal and not Q-normal.*

CHAPTER 8

TOPOLOGICAL RESULTS

The set of numbers normal in base b is an example of a set which is large in the sense that it has full measure in $[0, 1)$, yet is small in the topological sense that it is a set of the first category. We will show in this chapter that analogous results still hold for the Cantor series expansion when we consider any of the notions of normality that we have discussed.

Additionally, for a basic sequence Q that is infinite in limit, we will examine the set of numbers whose Q-Cantor series expansion contains every possible block of non-negative integers. We will also investigate the set of real numbers x such that $\{T_{Q,n}(x)\}_{n=1}^{\infty}$ is dense in $[0, 1)$.

We will end this chapter by proving that for Q that are infinite in limit, the set $\{x_F : F \text{ is a } Q\text{-special sequence}\}$, defined in chapter 6, is perfect and nowhere dense.

8.1 Basic Definitions

In order to begin, we will need to recall several basic definitions and results:

Definition 8.1.1. *A set $S \subset [0,1)$ is nowhere dense if each open interval I contains an open interval J such that $S \cap J = \emptyset$.*

Definition 8.1.2. *A set $S \subset [0,1)$ is of the first category if it is the countable union of nowhere dense sets.*

Definition 8.1.3. *A set $S \subset [0,1)$ is of the second category if it is the complement of a set of the first category.*

Definition 8.1.4. *A set S is a G_δ set if it can be expressed as a countable intersection of open sets.*

Theorem 8.1.5. *A subset S of a complete metric space is of the second category if and only if S contains a set S' which is an everywhere dense G_δ subset of X.*

Lemma 8.1.6. *A subset S of a complete metric space X is of the second category if $S = X \backslash C$ where C is a countable set.*

Proof. Since $X\backslash\{c\}$ is open for all $c \in C$, we see that

$$S = X\backslash C = \bigcap_{c\in C} (X\backslash\{c\}) \tag{8.1}$$

is a set of the second category.

□

8.2 Q-Normal Numbers

Theorem 8.2.1. *For any basic sequence Q, the set of simply Q-ratio normal numbers is of the first category.*

Proof. We define

$$A_m = \bigcap_{n=m}^{\infty} \{x : N_n^Q((0), x) < 2N_n^Q((1), x)\}. \tag{8.2}$$

Let $A = \bigcup_{m=1}^{\infty} A_m$ and note that A contains the set of simply Q-ratio normal numbers as a number x being simply Q-ratio normal requires that

$$\lim_{n\to\infty} \frac{N_n^Q((0), x)}{N_n^Q((1), x)} = 1. \tag{8.3}$$

We will show that for all m, A_m is nowhere dense. Let a positive integer m and an interval $I \subset [0, 1)$ be given. Let J be a Q-adic interval of order n such that for all $x \in J$, we have

$$N_n^Q((0), x) > 2N_n^Q((1), x). \tag{8.4}$$

Such an interval exists as no matter what the first m digits of x are, one can continue to choose the digit 0 until there are more than twice as many 0s as 1s among the first n digits of the Q-Cantor series expansion of x. Clearly $J \cap A_m = \emptyset$ so A_m is nowhere dense.

□

From Theorem 8.2.1, one of our main results follows:

Corollary 8.2.2. *For any basic sequence Q and for all k, the set of Q-normal numbers of order k and the set of Q-ratio normal numbers of order k are of the first category. Additionally, the set of Q-normal numbers and the set of Q-ratio normal numbers is of the first category.*

Proof. All of these sets are subsets of the set of simply Q-ratio normal numbers.
□

Corollary 8.2.3. *For any basic sequence Q and for all k, the set of strongly Q-normal numbers of order k and the set of strongly Q-ratio normal numbers of order k are of the first category. Additionally, the set of strongly Q-normal numbers and the set of strongly Q-ratio normal numbers is of the first category.*

8.3 Q-Distribution Normal Numbers

We now consider a result for distribution normality that is analogous to Theorem 8.2.1:

Theorem 8.3.1. *For any basic sequence Q, the set of Q-distribution normal numbers is of the first category.*

Proof. We define

$$A_m = \bigcap_{n=m}^{\infty} \left\{ x : \frac{N_n^Q([0,1/2),x)}{n} < 2/3 \right\}. \tag{8.5}$$

Let $A = \bigcup_{m=1}^{\infty} A_m$ and note that A contains the set of Q-distribution normal numbers, as a number x being Q-distribution normal requires that

$$\lim_{n\to\infty} \frac{N_n^Q([0,1/2),x)}{n} = 1/2. \tag{8.6}$$

We will show that for all m, A_m is nowhere dense. Let a positive integer m and an interval $I \subset [0,1)$ be given. Let J be a diadic interval of order n such that for all $x \in J$,

$$\frac{N_n^Q([0,1/2),x)}{n} > 2/3. \tag{8.7}$$

Such an interval exists as no matter how many times the sequence $\{T_{Q,k}(x)\}_{k=0}^{m-1}$ visits the interval $[1/2,1)$, there exists an n such that one can continue to choose the digit 0 until $\{T_{Q,k}(x)\}_{k=0}^{n-1}$ has visited the interval $[0,1/2)$ more than two thirds of the time . Clearly, $J \cap A_m = \emptyset$, so A_m is nowhere dense.

□

Corollary 8.3.2. *For any basic sequence Q, the set of strongly Q-distribution normal numbers is of the first category.*

8.4 Q-Disjunctive Numbers

Definition 8.4.1. *If Q is a basic sequence that is infinite in limit, then a real number x in $[0,1)$ is Q-disjunctive if all blocks appear in the Q-Cantor series expansion of x.*

The goal of this section will be to show that the set of Q-disjunctive numbers is of the second category.

Lemma 8.4.2. *For a basic sequence Q that is infinite in limit, a real number x is Q-disjunctive if and only if every block appears infinitely often in the Q-Cantor series expansion of x.*

Proof. If every block occurs infinitely many times in the Q-Cantor series expansion of x then, clearly, it occurs at least once so x is Q-disjunctive.

Suppose that x is Q-disjunctive and that $B = (b_1, b_2, \ldots, b_k)$ is a block of length k. Then the blocks

$$B_t = (b_1, b_2, \ldots, b_k, t), t \in \mathbb{N} \tag{8.8}$$

will also occur in the Q-Cantor series expansion of x. But there are infinitely many blocks B_t, so the block B appears infinitely often.

□

Definition 8.4.3. *Given a basic sequence Q, a Q-admissable block B, and a real number $x \in [0,1)$, we let V_Q be the set of all real numbers in $[0,1)$ whose Q-Cantor series expansion does not contain an infinite tail of 0s.*

Definition 8.4.4. *We let $S(B,n)$ be the set of real numbers z in $[0,1)\backslash V_Q$ where the block of digits B occurs at least once at position n or later.*

Lemma 8.4.5. *If Q is a basic sequence that is infinite in limit and $B=(b_1,b_2,\ldots,b_k)$ is a block of length k, then $S(B,n)$ is open in $[0,1)\backslash V_Q$.*

Proof. Let $x \in S(B,n)$ and suppose that $x = 0.E_1E_2\ldots$ w.r.t. Q. Thus, there exists $i \geq n$ such that $E_{i,k} = B$. Let $j > i+k-1$ such that $E_j \neq 0$ and $E_j \neq q_n - 1$. Such a j exists as $x \notin V_Q$. We let

$$I = \left(x - \frac{1}{q_1q_2\cdots q_j}, x + \frac{1}{q_1q_2\cdots q_j}\right)\backslash V_Q. \tag{8.9}$$

Thus, if $y \in I$ and $y = .F_1F_2\ldots$ w.r.t. Q, then $F_{i,k} = B$, so $y \in S(B,n)$. Therefore, $S(B,n)$ is open in $[0,1)\backslash V_Q$.

□

Lemma 8.4.6. *If Q is a basic sequence that is infinite in limit and $B=(b_1,b_2,\ldots,b_k)$ is a block of length k, then $S(B,n)$ is dense in $[0,1)$.*

Proof. Let $x \in [0,1)$ and $\epsilon > 0$. Let $m > n$ be large enough so that $(q_1q_2\cdots q_m)^{-1} < \epsilon$ and $B < Q_{m+1,k}$. Suppose that $x = E_1E_2\ldots E_mE_{m+1}\ldots$ w.r.t. Q. Then let $y = .E_1E_2\ldots E_mb_1b_2\ldots b_k010101\ldots$ w.r.t. Q. Clearly, $y \in S(B,n)$ as y does not terminate in a string of 0s. We also see that

$$|x-y| < \frac{1}{q_1q_2\cdots q_m} < \epsilon \tag{8.10}$$

as the digits of y are identical to those of x until after the m^{th} digit.

□

Theorem 8.4.7. *Suppose that Q is a basic sequence that is infinite in limit. Then the set of Q-disjunctive numbers is of the second category.*

Proof. Letting B range over all possible blocks and n over all natural numbers, we get a countably infinite family of sets of the form S(B,n). Let

$$S = \bigcap_{B,n} S(B, n). \tag{8.11}$$

Thus, by Lemma 8.4.2, every member of S is Q-disjunctive. Since V_Q is countable, we know that $[0, 1)\backslash V_Q$ is of the second category in $[0, 1)$ by Lemma 8.1.6.

By Lemma 8.4.5 and Lemma 8.4.6, S is a dense G_δ set in $[0, 1)\backslash V_Q$. So, by Theorem 8.1.5, S is of the second category in $[0, 1)$.

□

Corollary 8.4.8. *If Q is fully divergent, then the set of Q-disjunctive numbers is of the second category and is of full measure.*

Proof. Since Q is fully divergent, it is also infinite in limit. By Theorem 8.4.7, the set of Q-disjunctive numbers is of the second category. However, this set also contains the set of Q-normal numbers and thus has full measure by Corollary 7.3.13.

□

8.5 Q-Dense Numbers

We will now further examine the set of Q-dense numbers, defined in Definition 6.0.30.

Theorem 8.5.1. *Suppose that L and R are countable dense subsets of $[0,1)$ and that $x \in [0,1)$. If $\mathbb{I}$ is a set of intervals that consists of an interval with left endpoint l and right endpoint r for all $l \in L$ and $r \in R$ with $l < r$, then the following are equivalent.*

1. *x is Q-dense.*
2. *For all intervals $I \subset [0,1)$, there exists an n such that $T_{Q,n}(x) \in I$.*
3. *For all intervals $I \subset [0,1)$, there exists infinitely many n such that $T_{Q,n}(x) \in I$.*
4. *For all intervals $I \in \mathbb{I}$, there exists an n such that $T_{Q,n}(x) \in I$.*
5. *For all intervals $I \in \mathbb{I}$, there exists infinitely many n such that $T_{Q,n}(x) \in I$.*

Proof. We will show that 1. $\implies$ 3. $\implies$ 5. $\implies$ 4. $\implies$ 1. and 1. $\implies$ 3. $\implies$ 2. $\implies$ 4. $\implies$ 1.

To show 1. $\implies$ 3., we suppose that x is Q dense. Let $I \subset Q$ and suppose that $y \in I$ is an arbitrary point in the interior of I. Suppose that $\epsilon > 0$ is small enough so that $(y-\epsilon, y+\epsilon) \subset I$. Since x is Q-dense, for all $\epsilon > 0$ there exists infinitely many n such that $|T_{Q,n}(x) - y| < \epsilon$. But then

$$T_{Q,n}(x) \in (y-\epsilon, y+\epsilon) \subset I \tag{8.12}$$

for infinitely many n.

3. $\Longrightarrow$ 5. and 5. $\Longrightarrow$ 4. follow trivially. In showing that 4. $\Longrightarrow$ 1., we consider some $y \in [0,1)$ and let $\epsilon > 0$. Let $l \in L$ and $r \in R$ such that $y < l < r$, $l - y < \epsilon/2$ and $r - l < \epsilon/2$. Let $I = [l, r] \in \mathbb{I}$. We know there much exist some n such that $T_{Q,n}(x) \in I$. Then

$$|T_{Q,n}(x) - y| < \epsilon/2 + \epsilon/2 = \epsilon, \tag{8.13}$$

so x is Q-dense.

3. $\Longrightarrow$ 5. and 2. $\Longrightarrow$ 4. also both follow trivially.

□

Lemma 8.5.2. *If Q is a basic sequence and x is Q-distribution normal, then x is Q-dense.*

Proof. Let $y \in [0,1)$ and $\epsilon > 0$. Set $I = (y - \epsilon/2, y + \epsilon/2)$. Since x is Q-distribution normal, there exists a natural number n such that $T_{Q,n}(x) \in I$. But then $|T_{Q,n}(x) - y| < \epsilon$ so x is Q-dense.

□

We remark that by the different characterizations of Theorem 8.5.1, the concept of Q-dense is analogous to that of Q-disjunctive. Similarly to the differences between Q-normal numbers and Q-distribution normal numbers, there is no inclusion between Q-dense numbers and Q-normal numbers.

Proposition 8.5.3. *Suppose that Q is infinite in limit. Then there exists a real number that is Q-dense, but not Q-disjunctive.*

Proof. Suppose that F is a Q-special sequence. In the proof of Proposition 6.3.2, we saw that the Q-Cantor series expansion of x_F contains only finitely many copies of the digit 1, so x_F is not Q-disjunctive. Since x_F is Q-distribution normal, it is Q-dense.

□

Proposition 8.5.4. *Suppose that Q is infinite in limit. Then there exists a real number that is Q-disjunctive, but not Q-dense.*

Proof. We use similar reasoning to the proof of Theorem 7.5.8, where a number that is Q-ratio normal, but not Q-normal is shown to exist. Set $Q' = Q/2$. By Theorem 7.5.6, there exists a real number x that is Q'-normal. We set

$$y = \Phi_{Q'}^{Q}(x), \tag{8.14}$$

so

$$I_Q^{(k)}(y) = \frac{1}{2}. \tag{8.15}$$

Clearly, y is Q-disjunctive, but there exists an interval $I \subset [1/2, 1)$ where there is no n such that $T_{Q,n}(y) \in I$.

□

We will see in the following lemmas that the set of Q-dense real numbers will also be of the second category. The following definition and lemma will be similar to the ones we already provided for Q-disjunctive numbers:

Definition 8.5.5. *Given a basic sequence Q and an interval I, we let $U(I,n)$ be the set of real numbers z in $[0,1)$ such that there exists a natural number $m \geq n$ with $T_{Q,n}(z) \in I$.*

Theorem 8.5.6. *If Q is a basic sequence and a and b are real numbers such that $0 \leq a < b \leq 1$, then*

$$T_{Q,n}^{-1}((a,b)) = \bigcup_{k=0}^{q_1q_2\cdots q_n} \left(\frac{k+a}{q_1q_2\cdots q_n}, \frac{k+b}{q_1q_2\cdots q_n} \right) \tag{8.16}$$

and

$$T_{Q,n}^{-1}([a,b)) = \bigcup_{k=0}^{q_1q_2\cdots q_n} \left[\frac{k+a}{q_1q_2\cdots q_n}, \frac{k+b}{q_1q_2\cdots q_n} \right). \tag{8.17}$$

If $b < 1$, then

$$T_{Q,n}^{-1}((a,b]) = \bigcup_{k=0}^{q_1q_2\cdots q_n} \left(\frac{k+a}{q_1q_2\cdots q_n}, \frac{k+b}{q_1q_2\cdots q_n} \right] \tag{8.18}$$

and

$$T_{Q,n}^{-1}([a,b]) = \bigcup_{k=0}^{q_1q_2\cdots q_n} \left[\frac{k+a}{q_1q_2\cdots q_n}, \frac{k+b}{q_1q_2\cdots q_n} \right]. \tag{8.19}$$

Proof. We prove only (8.16) as the proofs of (8.17), (8.18) and (8.19) are almost identical.

If x is a real number such that

$$x \in \bigcup_{k=0}^{q_1 q_2 \cdots q_n} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n} \right), \tag{8.20}$$

then there exists a natural number k such that

$$x \in \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n} \right). \tag{8.21}$$

We note that

$$q_1 q_2 \cdots q_n x < q_1 q_2 \cdots q_n \frac{k+b}{q_1 q_2 \cdots q_n} = k+b \tag{8.22}$$

and

$$q_1 q_2 \cdots q_n x > q_1 q_2 \cdots q_n \frac{k+a}{q_1 q_2 \cdots q_n} = k+a. \tag{8.23}$$

Thus, since $k+a < q_1 q_2 \cdots q_n x < k+b$, we conclude that $q_1 q_2 \cdots q_n x \in (a,b)$, so

$$T_{Q,n}^{-1}\left((a,b)\right) \supset \bigcup_{k=0}^{q_1 q_2 \cdots q_n} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n} \right). \tag{8.24}$$

To show the other direction of the inclusion, consider a real number x such that there exists an integer k with

$$x \in \left[\frac{k}{q_1 q_2 \cdots q_n}, \frac{k+1}{q_1 q_2 \cdots q_n} \right) \setminus \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n} \right) \tag{8.25}$$

$$= \left[\frac{k}{q_1 q_2 \cdots q_n}, \frac{k+a}{q_1 q_2 \cdots q_n}\right] \bigcup \left[\frac{k+b}{q_1 q_2 \cdots q_n}, \frac{k+1}{q_1 q_2 \cdots q_n}\right]. \tag{8.26}$$

If

$$x \in \left[\frac{k}{q_1 q_2 \cdots q_n}, \frac{k+a}{q_1 q_2 \cdots q_n}\right], \tag{8.27}$$

then

$$q_1 q_2 \cdots q_n x \leq q_1 q_2 \cdots q_n \frac{k+a}{q_1 q_2 \cdots q_n} = k + a \tag{8.28}$$

and

$$q_1 q_2 \cdots q_n x \geq q_1 q_2 \cdots q_n \frac{k}{q_1 q_2 \cdots q_n} = k. \tag{8.29}$$

If

$$x \in \left[\frac{k+b}{q_1 q_2 \cdots q_n}, \frac{k+1}{q_1 q_2 \cdots q_n}\right], \tag{8.30}$$

then

$$q_1 q_2 \cdots q_n x \leq q_1 q_2 \cdots q_n \frac{k+1}{q_1 q_2 \cdots q_n} = k + 1 \tag{8.31}$$

and

$$q_1 q_2 \cdots q_n x \geq q_1 q_2 \cdots q_n \frac{k+b}{q_1 q_2 \cdots q_n} = k + b. \tag{8.32}$$

Thus, if

$$x \in \left[\frac{k}{q_1 q_2 \cdots q_n}, \frac{k+1}{q_1 q_2 \cdots q_n}\right) \setminus \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right), \tag{8.33}$$

then $x \notin (a,b)$, so

$$T_{Q,n}^{-1}\left((a,b)\right) \subset \bigcup_{k=0}^{q_1 q_2 \cdots q_n} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right). \tag{8.34}$$

Therefore, (8.16) holds.

□

Lemma 8.5.7. *If Q is a basic sequence and $I = (a,b)$, then*

$$U(I,n) = \bigcup_{m=n}^{\infty} \bigcup_{k=0}^{q_1 q_2 \cdots q_m} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right). \tag{8.35}$$

Proof. We first note that

$$U(I,n) = \bigcup_{m=n}^{\infty} T_{Q,m}^{-1}(I). \tag{8.36}$$

By Theorem 8.5.6,

$$T_{Q,n}^{-1}(I) = \bigcup_{k=0}^{q_1 q_2 \cdots q_n} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right), \tag{8.37}$$

so

$$U(I,n) = \bigcup_{m=n}^{\infty} \bigcup_{k=0}^{q_1 q_2 \cdots q_m} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right). \tag{8.38}$$

□

Lemma 8.5.8. *If Q is a basic sequence and $I = (a,b)$ is an open interval, then $U(I,n)$ is an open set.*

Proof. By Lemma 8.5.7, $U(I,n)$ is a countable union of open sets.

□

Lemma 8.5.9. *If Q is a basic sequence and $I = (a,b)$ is an open interval, then $U(I,n)$ is dense in $[0,1)$.*

Proof. Let $y \in [0,1)$ and $\epsilon > 0$. Suppose that $m \geq$ is large enough so that

$$\frac{1}{q_1 q_2 \cdots q_m} < \epsilon. \tag{8.39}$$

We note that

$$[0,1) = \bigcup_{k=0}^{q_1 q_2 \cdots q_m - 1} \left[\frac{k}{q_1 q_2 \cdots q_m}, \frac{k+1}{q_1 q_2 \cdots q_m}\right), \tag{8.40}$$

so there exists an integer t such that

$$y \in \left[\frac{t}{q_1 q_2 \cdots q_m}, \frac{t+1}{q_1 q_2 \cdots q_m}\right). \tag{8.41}$$

By Theorem 8.5.6,

$$T_{Q,m}^{-1}(I) = \bigcup_{k=0}^{q_1 q_2 \cdots q_n} \left(\frac{k+a}{q_1 q_2 \cdots q_n}, \frac{k+b}{q_1 q_2 \cdots q_n}\right), \tag{8.42}$$

so let

$$x = \frac{t + \frac{a+b}{2}}{q_1 q_2 \cdots q_m} \in \left(\frac{t+a}{q_1 q_2 \cdots q_n}, \frac{t+b}{q_1 q_2 \cdots q_n} \right). \tag{8.43}$$

Then, clearly, $|x - y| < \epsilon$, so $U(I, n)$ is dense in $[0, 1)$.

□

Theorem 8.5.10. *For any basic sequence Q, the set of Q-dense numbers is of the second category.*

Proof. We let $\mathbb{I}$ be the set of open intervals contained in $[0, 1)$ with rational endpoints. Let

$$U = \bigcap_{I \in \mathbb{I}, n > 0} U(I, n). \tag{8.44}$$

Then by Theorem 8.5.1, every member of U is Q-dense. By Lemma 8.5.8 and Lemma 8.5.9, U is a dense G_δ set in $[0, 1)$. So by Theorem 8.1.5, U is a set of the second category in $[0, 1)$.

□

Corollary 8.5.11. *For any basic sequence Q, the set of Q-dense numbers is of the second category and has full measure.*

Proof. The set of Q-dense numbers is of the second category by Theorem 8.5.10. Since every Q-distribution normal number is Q-dense, the set of Q-dense numbers is of full measure by Corollary 7.3.16.

□

8.6 A Set of Q-Distribution Normal Numbers that is Perfect and Nowhere Dense

In this section, we will use the same conventions established in Chapter 6. That is, we fix a basic sequence Q that is infinite in limit and define

$$n_j = \min\{N : q_m \geq 2j^2 \text{ for all } m \geq N\}, \tag{8.45}$$

$$l_1 = \max(n_2 - 1, 1), \tag{8.46}$$

and for $i > 1$, l_i is the smallest positive integer such that

$$l_1 + 2l_2 + 3l_3 + \ldots + il_i \geq n_{i+1} - 1. \tag{8.47}$$

We also need

$$S_Q = \left\{(a,b,c) \in \mathbb{N}^3 : b \leq l_a, c \leq a\right\} \tag{8.48}$$

and for $(a,b,c) \in S_Q$,

$$h(a,b,c) = L_{a-1} + (b-1)a + c. \tag{8.49}$$

For $(a,b,1) \in S_Q$, set $F_{(a,b,1)} = 0$. For $(a,b,c) \in S_Q$ with $c > 1$, we let $F_{(a,b,c)}$ be any integer that satisfies

$$\frac{F_{(a,b,c)}}{q_{h^{-1}(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right]. \tag{8.50}$$

Lastly, if F is a Q-special sequence, we define

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 q_2 \dots q_n}. \tag{8.51}$$

Definition 8.6.1. *We define Γ_Q to be the set of all Q-special sequences. We also define*

$$\Theta_Q = \{x_F : F \in \Gamma_Q\}. \tag{8.52}$$

We will now need to define the Hausdorff dimension of a subset of $\mathbb{R}$.[1]

Definition 8.6.2. *Let $X \subset \mathbb{R}$. Let $\{I_j\}_{j=1}^{\infty}$ be a sequence of intervals such that*

$$X \subset \bigcup_{j=1}^{\infty} I_j. \tag{8.53}$$

Suppose that $c > 0$. We say that $\{I_j\}_{j=1}^{\infty}$ is a c-covering of X if, for all j, we have

$$\lambda(I_j) < c. \tag{8.54}$$

We let $\mathcal{I}_{X,c}$ denote the set of c-coverings of X. We will also need to define

$$\mu_{c,a}(E) = \inf_{a \in (0,1), \{I_j\} \in \mathcal{I}_{X,c}} \sum_{j=1}^{\infty} \lambda(I_j). \tag{8.55}$$

[1]The definition of Hausdorff dimension we give may be extended to much more general contexts (see [15]). We will only be concerned with the Hausdorff dimension of subsets of $\mathbb{R}$ in this thesis.

Clearly, if $0 < c_2 < c_1$, we have

$$\mu_{c_1,a}(X) \leq \mu_{c_2,a}(X). \tag{8.56}$$

Thus, the limit

$$\mu_a(X) := \lim_{c \to 0^+} \mu_{c,a}(X) \tag{8.57}$$

exists and may be ∞. We note the following:

Proposition 8.6.3. *If $\mu_a(X) < \infty$, then for any $b > a$, we have $\mu_b(X) = 0$.*

Proposition 8.6.4. *There is a unique value $H \in [0,1]$ such that for all $a \in (0,1)$, if $a > H$, we have $\mu_a(X) = 0$ and if $a < H$, we have $\mu_a(X) = \infty$.*

Definition 8.6.5. *The unique value $H = H(X)$ from Proposition 8.6.4 is called the Hausdorff dimension of X.*

Proposition 8.6.6. *Suppose that $X \subset [0,1)$ and $\lambda(X) > 0$. Then the Hausdorff dimension of X is 1.*

By Proposition 8.6.6, only sets of measure 0 may have Hausdorff dimension less than 1, so we may consider Hausdorff dimension to be an indicator of the size of a set of measure 0.

In Chapter 6, we showed that every member of Θ_Q is Q-distribution normal. The goal of this section will be to show that Θ_Q is a perfect, nowhere dense subset of $[0,1)$. We will also investigate the Hausdorff dimension of Θ_Q. First, we first remark that the *existance* of a set of normal numbers that is perfect and nowhere dense should not be surprising. However, constructing a specific example of such a set may not lend itself to an obvious solution. We now show that any subset W of $\mathbb{R}$ with positive Lebesgue measure contains a subset that is perfect and nowhere dense. The following lemma will be needed:

Lemma 8.6.7. *Suppose that $\lambda(E) > 0$. Then for all $\alpha < 1$, there is an open interval I such that*

$$\lambda(E \cap I) > \alpha\lambda(I). \tag{8.58}$$

Proposition 8.6.8. *Suppose that $\lambda(W) > 0$. Then there exists a subset of W that is perfect and nowhere dense.*

Proof. By Lemma 8.6.7, without loss of generality, we can assume that $X = (0,1) \cap W$ has positive Lebesgue measure. We let define the intervals $I_{1,k} = (a_{1,k}, b_{1,k})$, $k = 1,2,3$, so that $a_{1,1} = 0$, $b_{1,1} = a_{1,2}$, $b_{1,2} = a_{1,3}$, $a_{1,3} = 1$, and

$$\lambda(I_{1,k} \cap X) = \frac{1}{3}\lambda(X). \tag{8.59}$$

Set $X^*_{1,k} = I_{1,k} \cap X$. Then, by Lemma 8.6.7, there exist intervals $J_{1,k} \subset I_{1,k}$ such that

$$\lambda(J_{1,k} \cap X^*_{1,k}) = \frac{1}{2}\lambda(X^*_{1,k}). \tag{8.60}$$

We set

$$X_{1,k} = (J_{1,1} \cap X^*_{1,1}) \cup (J_{1,3} \cap X^*_{1,3}). \tag{8.61}$$

We construct X_2 similarly by taking out the analogous portions of the remaining components of X_1, and so on. The resulting set $\bar{X} = \cup X_i$ clearly has measure 0 as $\lambda(X_{n+1}) = \frac{1}{3}\lambda(X_n)$.

To see that the set $\bar{X}$ is nowhere dense, we let $I \subset (0,1)$ be an interval. Then for some j, k, I will contain one of the intervals $I_{j,k}$ that is not contained in X_j.

Finally, if $x \in \bar{X}$, then x is contained in $J_{j_n,k_n} \cap X^*_{j_n,k_n}$ for sequences of integers $\{j_n\}_n, \{k_n\}_n$ where j_n is increasing. We see that $\bar{X}$ is perfect as each of the sets $J_{j_n,k_n} \cap X^*_{j_n,k_n}$ has more than one element and

$$\lim_{n\to\infty} \lambda(J_{j_n,k_n} \cap X^*_{j_n,k_n}) = 0. \tag{8.62}$$

□

It should be noted that the proof of Proposition 8.6.8 can be modified to construct perfect, nowhere dense subsets of sets of positive measure that have positive measure or different Hausdorff dimensions.

We will now work towards showing that Θ_Q is perfect and nowhere dense. In order to proceed, we define a metric, d, on Γ_Q:

Definition 8.6.9. *Suppose that F_1 and F_2 are Q-special sequences. If $F_1 \neq F_2$, we define*

$$n_{F_1,F_2} = \min\{n : E_{F_1,n} \neq E_{F_2,n}\}. \tag{8.63}$$

Thus, we may define $d : \Gamma_Q \times \Gamma_Q \to \mathbb{R}$ by

$$d(F_1, F_2) = \begin{cases} \frac{1}{q_1 q_2 \ldots q_{n_{F_1,F_2}-1}} & \text{if } F_1 \neq F_2 \\ 0 & \text{if } F_1 = F_2 \end{cases}. \tag{8.64}$$

Proposition 8.6.10. *(Γ_Q, d) is a metric space.*

Proof. Let $F_1, F_2, F_3 \in \Gamma_Q$. Trivially, we have $d(F_1, F_2) \geq 0$, $d(F_1, F_2) = d(F_2, F_1)$, and $d(F_1, F_2) = 0$ if and only if $F_1 = F_2$. We now wish to show that

$$d(F_1, F_3) \leq d(F_1, F_2) + d(F_2, F_3). \tag{8.65}$$

We assume that F_1, F_2, and F_3 are distrinct, otherwise (8.65) follows trivially. Set

$$a = n_{F_1,F_2}, b = n_{F_2,F_3}, \text{ and } c = n_{F_1,F_3}. \tag{8.66}$$

We note that at least two of the values a, b, and c are equal with the third value being at least as big as the other two. Without loss of generality, we assume that $a = b$ and $c \geq a$. Thus, since $q_n \geq 2$ for all n, we see that

$$\frac{1}{q_1 q_2 \cdots q_{a-1}} \leq \frac{1}{q_1 q_2 \cdots q_{b-1}} + \frac{1}{q_1 q_2 \cdots q_{c-1}}, \tag{8.67}$$

$$\frac{1}{q_1 q_2 \cdots q_{b-1}} \leq \frac{1}{q_1 q_2 \cdots q_{a-1}} + \frac{1}{q_1 q_2 \cdots q_{c-1}}, \tag{8.68}$$

and

$$\frac{1}{q_1 q_2 \cdots q_{c-1}} \leq \frac{1}{q_1 q_2 \cdots q_{a-1}} + \frac{1}{q_1 q_2 \cdots q_{b-1}}, \tag{8.69}$$

so (8.65) holds.

□

We may now study the topological properties of Θ_Q.

Lemma 8.6.11. *If $F_1, F_2 \in \Gamma_Q$, then*

$$|x_{F_1} - x_{F_2}| \leq d(F_1, F_2). \tag{8.70}$$

Proof. Let $n = n_{F_1,F_2}$. We write the Q-Cantor series expansions of x_{F_1} and x_{F_2} as follows:

$$x_{F_1} = \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_{n-1}}{q_1 q_2 \cdots q_{n-1}} + \frac{E_{F_1,n}}{q_1 q_2 \cdots q_n} + \frac{E_{F_1,n+1}}{q_1 q_2 \cdots q_{n+1}} + \ldots \tag{8.71}$$

and

$$x_{F_2} = \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_{n-1}}{q_1 q_2 \cdots q_{n-1}} + \frac{E_{F_2,n}}{q_1 q_2 \cdots q_n} + \frac{E_{F_2,n+1}}{q_1 q_2 \cdots q_{n+1}} + \ldots, \tag{8.72}$$

so

$$|x_{F_1} - x_{F_2}| = \left| \left(\frac{E_{F_1,n}}{q_1 q_2 \cdots q_{n-1}} - \frac{E_{F_2,n}}{q_1 q_2 \cdots q_{n-1}} \right) + \left(\frac{E_{F_1,n+1}}{q_1 q_2 \cdots q_{n+1}} - \frac{E_{F_2,n+1}}{q_1 q_2 \cdots q_{n+1}} \right) + \dots \right|$$

$$\leq \frac{|E_{F_1,n} - E_{F_2,n}|}{q_1 q_2 \cdots q_n} + \frac{|E_{F_1,n+1} - E_{F_2,n+1}|}{q_1 q_2 \cdots q_{n+1}} + \dots \leq \frac{1}{q_1 q_2 \cdots q_{n-1}} = d(F_1, F_2). \quad (8.73)$$

□

Lemma 8.6.12. *If $F \in \Gamma_Q$, then there exists a sequence of Q-special sequences $F_1, F_2, F_3, \dots$ such that $F \neq F_n$ for all n and*

$$\lim_{n \to \infty} d(F, F_n) = 0. \quad (8.74)$$

Proof. By Lemma 6.1.3, we may define a sequence of Q-special sequences as follows. Let n be any positive integer and put

$$(\alpha, \beta, \gamma) = h^{-1}(n). \quad (8.75)$$

Suppose that

$$F_n = \{F_{n,(a,b,c)}\}_{(a,b,c) \in S_Q}. \quad (8.76)$$

We must now consider three cases. First, if $\gamma \neq 1$, then for $m \neq n$, we set $E_{F_n,m} = E_{F,m}$ and we let $E_{F_n,n} \neq E_{F,n}$ be any value that satisfies

$$\frac{E_{F_n,n}}{q_n} \in \left[\frac{\gamma - 1}{\alpha} - \frac{1}{2\alpha^2}, \frac{\gamma - 1}{\alpha} + \frac{1}{2\alpha^2}\right]. \tag{8.77}$$

Second, suppose that $\gamma = 1$ and $\alpha > 1$. Put

$$(\alpha', \beta', \gamma') = h^{-1}(n+1). \tag{8.78}$$

Then for $m \neq n+1$, we set $E_{F_n,m} = E_{F,m}$ and we let $E_{F_n,n+1} \neq E_{F,n+1}$ be any value that satisfies

$$\frac{E_{F_n,n+1}}{q_{n+1}} \in \left[\frac{\gamma' - 1}{\alpha'} - \frac{1}{2\alpha'^2}, \frac{\gamma' - 1}{\alpha'} + \frac{1}{2\alpha'^2}\right]. \tag{8.79}$$

Third, we consider the case where $\alpha = \gamma = 1$. We let $t = h^{-1}(2, 1, 2)$ and note that $t > n$. Then for $m \neq t$, we set $E_{F_n,m} = E_{F,m}$ and we let $E_{F_n,t} \neq E_{F,t}$ be any value that satisfies

$$\frac{E_{F_n,t}}{q_t} \in \left[\frac{2-1}{2} - \frac{1}{2 \cdot 2^2}, \frac{2-1}{2} + \frac{1}{2 \cdot 2^2}\right] = \left[\frac{3}{8}, \frac{5}{8}\right]. \tag{8.80}$$

Now that we have determined the sequence $\{E_{F_n,m}\}_{m=1}^{\infty}$, for $(a, b, c) \in S_Q$, we set

$$F_{n,(a,b,c)} = E_{F_n,h(a,b,c)}. \tag{8.81}$$

Thus, $F \neq F_n$ for all n and for large enough m, we have

$$d(F, F_m) \leq \max\left(\frac{1}{q_1 q_2 \cdots q_m}, \frac{1}{q_1 q_2 \cdots q_{m-1}}\right) = \frac{1}{q_1 q_2 \cdots q_{m-1}}, \tag{8.82}$$

so $F_n \to F$.

□

Theorem 8.6.13. *The set* Θ_Q *is perfect.*

Proof. Suppose that $x \in \Theta_Q$ and that $x = x_F$. Then, by Lemma 8.6.12, there exist a sequence of Q-special sequences $F_1, F_2, F_3, \ldots$, none of which are equal to F, with $F_n \to F$. Thus, $x \neq x_{F_n}$ for all n. Let $\epsilon > 0$ and suppose that N is large enough so that for all $n > N$, we have

$$d(F, F_n) < \epsilon. \tag{8.83}$$

Clearly,

$$|x - x_{F_n}| \leq d(F, F_n) < \epsilon, \tag{8.84}$$

so $x_{F_n} \to x_F$ and Θ_Q is perfect.

□

Lemma 8.6.14. *If* $a \geq 1$*, then*

$$\frac{a-1}{a} + \frac{1.5}{2a^2} < 1. \tag{8.85}$$

Proof. We rewrite (8.85) as

$$\frac{2a^2 - 2a + 1.5}{2a^2} < 1. \tag{8.86}$$

Thus, to verify (8.86), we need show that

$$2a^2 - 2a + 1.5 < 2a^2. \tag{8.87}$$

However, as $a \geq 1$, we see that

$$-2a + 1.5 < 0, \tag{8.88}$$

so (8.87) follows.

□

Theorem 8.6.15. *The set Θ_Q is nowhere dense.*

Proof. Let $I \subset [0, 1)$ be any interval. We will show that there exists an interval $K \subset I$ such that $\Theta_Q \cap K = \emptyset$. If $I \cap \Theta_Q = \emptyset$, then we are done, so assume that $I \cap \Theta_Q \neq \emptyset$. Thus, there exists a positive integer n and a Q-adic interval J of order n with $J \subset I$. We write

$$J = \left[\frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n}, \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n + 1}{q_1 q_2 \cdots q_n}\right), \tag{8.89}$$

where $E_j \in [0, q_j - 1)$ for $j = 1, 2, \ldots, n$. Put

$$(a, b, c) = h^{-1}(n + 1). \tag{8.90}$$

By Lemma 8.6.14, we may set

$$K = \left[\frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} + \left(\frac{a-1}{a} + \frac{1.5}{2a^2} \right) \frac{1}{q_1 q_2 \cdots q_n}, \right. \tag{8.91}$$

$$\left. , \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n + 1}{q_1 q_2 \cdots q_n} \right).$$

If $J \cap \Theta_Q = \emptyset$, we are done, so assume that $J \cap \Theta_Q \neq \emptyset$. Suppose that $F \in \Gamma_Q$ is such that $x_F \in J$ and

$$x = 0.E_1 E_2 \ldots E_n E_{n+1} E_{n+2} \ldots \text{ w.r.t. } Q. \tag{8.92}$$

By construction, if $c \neq 1$, we have

$$\frac{E_{n+1}}{q_{n+1}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2} \right]. \tag{8.93}$$

If $c = 1$, then $E_{n+1} = 0$. Thus, in either case, we see that

$$x_F \leq \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} + \left(\frac{c-1}{a} + \frac{1}{2a^2} \right) \frac{1}{q_1 q_2 \cdots q_n} \tag{8.94}$$

$$< \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} + \left(\frac{a-1}{a} + \frac{1.5}{2a^2} \right) \frac{1}{q_1 q_2 \cdots q_n},$$

so $x_F \notin K$. Hence, $K \cap \Theta_Q = \emptyset$ and Θ_Q is nowhere dense.

□

We conclude this section with a conjecture and open questions about the Hausdorff dimension of Θ_Q.

Conjecture 8.6.16. *If $p(n)$ is an increasing integer valued polynomial in n with $p(2) \geq 2$ for $n \geq 1$ and $q_n = p(n)$, then the Hausdorff dimension of Θ_Q is 1.*

Problem 8.6.17. *Determine the Hausdorff dimension of Θ_Q in terms of the basic sequence Q. A lower bound should be easy to form for many basic sequences, but what is the exact value? How slow must q_n grow for the Hausdorff dimension of Θ_Q to be 0?*

CHAPTER 9

WINNING SETS

9.1 Introduction

We remarked in the introduction that for most notions of normality, the set of non-normal numbers has full Hausdorff dimension. In fact, this set satisfies a much stronger property that we will study in this section.

In [47], W. Schmidt proposed the following game. Let $\alpha \in (0,1)$, $\beta \in (0,1)$, and $S \subset \mathbb{R}$. We consider the game of two player, black and white, where black first picks any closed interval B_1. Then white picks a closed interval $W_1 \subset B_1$ such that $\lambda(W_1) = \alpha\lambda(B_1)$. Black then picks a closed interval $B_2 \subset W_1$ with $\lambda(B_2) = \beta\lambda(W_1)$. Then white picks a closed interval $W_2 \subset B_2$ such that $\lambda(W_2) = \alpha\lambda(B_2)$, and so on. We say that the set S is (α, β)-*winning* if white can play so that

$$\bigcap_{n=1}^{\infty} W_n \subset S. \tag{9.1}$$

The set S is (α, β)-*losing* if it is not (α, β)-winning.

Definition 9.1.1. *The set S is α-winning if it is (α, β)-winning for all $\beta < \alpha$.*

Our interest in studying winning sets rests in the following theorem of [47]:

Theorem 9.1.2. *An α-winning set in n-dimensional Euclidean space has Hausdorff dimension n.*

Additionally, winning sets satisfy a very strong property that is not always satisfied by sets with full Hausdorff dimension:

Theorem 9.1.3. *The intersection of countably many α-winning sets is α-winning.*

9.2 Windim of Sets of Non-Normal Numbers

In this section, we will show that for any basic sequence Q, the set of all x such that x is not strongly Q-distribution normal is $1/2$-winning. We will also show that if Q is infinite in limit, then the set of all x such that x is not strongly Q-normal is $1/2$-winning. Therefore, both of these sets will have full Hausdorff dimension. Our proofs will follow along similar lines to those found in [47].

9.2.1 Basic Lemmas and Definitions

Lemma 9.2.1. *Let $\alpha' < \alpha < 1$. Then every α-winning set is α'-winning.*

Definition 9.2.2. *Suppose that $S \subset [0,1)$. If S is not α-winning for any $\alpha > 0$, then we define*

$$\textit{windim } S = 0. \tag{9.2}$$

Otherwise, we define

$$\textit{windim } S = \sup\{\alpha : S \textit{ is } \alpha\textit{-winning}\}. \tag{9.3}$$

Definition 9.2.3. *Throughout the rest of this section, we let*

$$D = \{(x,y) : 0 < x < 1, 0 < y < 1, \textit{ and } 1 + xy - 2x > 0\}. \tag{9.4}$$

Additionally, given any $(\alpha, \beta) \in D$, we let

$$\gamma = \gamma(\alpha, \beta) = 1 + \alpha\beta - 2\alpha. \tag{9.5}$$

Definition 9.2.4. *Given $(\alpha, \beta) \in D$, we say that a basic sequence Q is (α, β)-friendly if for all n, we have that*

$$q_n > \frac{4}{\alpha\beta\gamma}. \tag{9.6}$$

Lemma 9.2.5. *If S is an (α, β)-winning set for all $(\alpha, \beta) \in D$, then*

$$\textit{windim } S = 1/2. \tag{9.7}$$

Proof. Suppose that $(1/2, \beta) \in [0,1)^2$. Then $\gamma = \beta/2 > 0$, so $(1/2, \beta) \in D$. Thus, we may conclude that S is a 1/2-winning set, so windim $S = 1/2$.

∎

We will need the following lemma from [47]:

Lemma 9.2.6. *Suppose that $(\alpha, \beta) \in D$ and let the integer t satisfy $(\alpha\beta)^t < \zeta/2$. Assume that a ball B_k with center b_k and radius ρ_k occurs in some (α, β)-play. Then white can play in such a way that*

$$B_{k+t} \subset (b_k + \rho_k \zeta/2, 1). \tag{9.8}$$

Definition 9.2.7. *Given a basic sequence Q and a natural number d, we let $S(Q,d)$ be the set of real numbers in $[0,1)$ whose Q-Cantor series expansion has at most finitely many copies of the digit d.*

Definition 9.2.8. *Given a basic sequence Q and an integer t, we let the basic sequence $\Psi_{Q,t} = \{\psi_{t,j}\}_{j=1}^{\infty}$ be given by*

$$\psi_{t,j} = q_{(j-1)t+1} \cdot q_{(j-1)t+2} \cdots q_{jt}. \tag{9.9}$$

Lemma 9.2.9. *If Q is infinite in limit, x is strongly Q-ratio normal, $k \geq 0$, $p \in [1,k]$, and B is a block of length k, then*

$$\lim_{n \to \infty} N_{n,p}^Q(B,x) = \infty. \tag{9.10}$$

Proof. Suppose that $B = (b_1, b_2, \ldots, b_k)$. For $j \geq 0$, we define the blocks

$$B_j = (b_1, b_2, \ldots, b_k, j). \tag{9.11}$$

Since x is strongly Q-ratio normal, for all $i, j \geq 0$, we have

$$\lim_{n \to \infty} \frac{N_{n,p}^Q(B_i, x)}{N_{n,p}^Q(B_j, x)} = 1. \tag{9.12}$$

So, for all j there is an n such that $N_{n,p}^Q(B_j, x) \geq 1$. Since there are infinitely many choices for j, the lemma follows.

□

Lemma 9.2.10. *If Q is a basic sequence that is infinite in limit and t is a positive integer, then the set $S(\Psi_{Q,t}, 0)$ is contained in the set of numbers that are not strongly Q-ratio normal.*

Proof. Let

$$B = (0, 0, \ldots, 0) \tag{9.13}$$

be the block of length t of t zeros. Since Q is infinite in limit, by Lemma 9.2.9 all $x \in [0,1)$ that are Q-ratio normal satisfy

$$\lim_{n \to \infty} N_{n,1}^{Q}(B, x) = \infty. \tag{9.14}$$

Let $x \in S(\Psi_{Q,t}, 0)$ and suppose, for contradiction, that x is strongly Q-ratio normal. Thus,

$$\lim_{n \to \infty} N_{n}^{\Psi_{Q,t}}(0, x) < \infty. \tag{9.15}$$

Note that by the definition of $\Psi_{Q,t}$, we have that

$$N_{tn,1}^{Q}(B, x) = N_{n}^{\Psi_{Q,t}}(0, x). \tag{9.16}$$

However, we have a contradiction as (9.14) and (9.16) imply that

$$\lim_{n \to \infty} N_{n}^{\Psi_{Q,t}}(0, x) = \infty, \tag{9.17}$$

contradicting (9.15).

□

We make the following conventions that will be used for the rest of the proofs in this section.

Definition 9.2.11. *Given a closed interval I, we let*

$$\rho(I) = \lambda(I)/2 \tag{9.18}$$

be the radius of I. Given $(\alpha, \beta) \in D$, an (α, β)-friendly basic sequence Q, and a closed interval $B_1 \subset [0, 1)$, we let

$$r = \rho(B_1). \tag{9.19}$$

We also choose the integers $k \geq 1$ and $n_0 \geq 1$ such that

$$\frac{1}{4q_1q_2\cdots q_k} > (\alpha\beta)^{n_0-1}r \geq (\alpha\beta) \cdot \frac{1}{4q_1q_2\cdots q_k}. \tag{9.20}$$

Furthermore, we define the integers $n_1, n_2, \ldots$ by

$$\frac{1}{4q_1q_2\cdots q_{k+j}} > (\alpha\beta)^{n_j-1}r \geq (\alpha\beta) \cdot \frac{1}{4q_1q_2\cdots q_{k+j}}. \tag{9.21}$$

Lemma 9.2.12. *If $(\alpha, \beta) \in D$, then $0 < \gamma < 1$.*

Proof. By definition, $\gamma = 1 + \alpha\beta - 2\alpha$. We note that γ is increasing in β. But $\beta < 1$, so

$$\lambda < 1 + \alpha \cdot 1 - 2\alpha = 1 - \alpha. \tag{9.22}$$

However, $\alpha > 0$, so $\gamma < 1$.

□

Lemma 9.2.13. $n_0 < n_1 < n_2 < \ldots$

Proof. Since Q is (α, β)-friendly, we see that

$$\frac{\alpha\beta}{4} > \frac{1}{q_{k+j+1}\gamma} \tag{9.23}$$

and note that by (9.23),

$$(\alpha\beta) \cdot \frac{1}{4q_1q_2\cdots q_{k+j}} = \frac{1}{q_1q_2\cdots q_{k+j}} \cdot \frac{\alpha\beta}{4} > \frac{1}{q_1q_2\cdots q_{k+j+1}}\frac{1}{\gamma}. \tag{9.24}$$

By Lemma 9.2.12, $\lambda < 4$, so

$$\frac{1}{4q_1q_2\cdots q_{k+j}} > (\alpha\beta)^{n_j-1}r > \frac{1}{q_1q_2\cdots q_{k+j+1}}\frac{1}{\gamma} > \frac{1}{4q_1q_2\cdots q_{k+j+1}} \tag{9.25}$$

and we see that $n_0 < n_1 < n_2 < \ldots$.

□

Lemma 9.2.14. $\rho(B_k) = (\alpha\beta)^{k-1}r$.

Proof. This follows directly by the definition of Schmidt games and the fact that $\rho(B_1) = r$.

□

Lemma 9.2.15. *Suppose that $I = [a, b)$ is an interval contained in $[0, 1)$ with $\lambda(I) < 1/2$ and that Q is an (α, β)-friendly basic sequence. If $j \geq 1$, then white can play so that $B_{n_j} \cap T_{Q,k+j}^{-1}(I) = \emptyset$.*

Proof. Suppose that $B_{n_{j-1}}$ is given. By Theorem 8.5.6,

$$T_{Q,k+j}^{-1}(I) = \bigcup_{t=0}^{q_1 q_2 \cdots q_{k+j}} \left[\frac{t+a}{q_1 q_2 \cdots q_{k+j}}, \frac{t+b}{q_1 q_2 \cdots q_{k+j}}\right). \tag{9.26}$$

Thus, $T_{Q,k+j}^{-1}(I)$ is the union of intervals of length $q_1 q_2 \cdots q_{k+j}$ whose distance apart is greater than

$$\frac{1}{q_1 q_2 \cdots q_{k+j-1}}(1 - \lambda(I)) \geq \frac{1}{2 q_1 q_2 \cdots q_{k+j-1}}. \tag{9.27}$$

By definition of n_{j-1},

$$\frac{1}{q_1 q_2 \cdots q_{k+j-1}} > 4(\alpha\beta)^{n_{j-1}-1} r, \tag{9.28}$$

so by Lemma 9.2.14, we see that

$$\frac{1}{2 q_1 q_2 \cdots q_{k+j-1}} > 2(\alpha\beta)^{n_{j-1}-1} r = 2\rho(B_{n_{j-1}}). \tag{9.29}$$

Therefore, the distance between the intervals that make up $T_{Q,k+j}^{-1}(I)$ is greater than $2\rho(B_{n_{j-1}})$, so white only has to worry about some interval C of length no more than $\frac{1}{q_1 q_2 \cdots q_{k+j}}$.

Let b be the center of $B_{n_{j-1}}$ and let c be the center of C. Without loss of generality, we assume that $c \leq b$. Thus, we see that

$$C \subset \left[0, b + \frac{1}{2} \cdot \frac{1}{q_1 q_2 \cdots q_{k+j}}\right). \tag{9.30}$$

We note that

$$\frac{1}{q_1 q_2 \cdots q_{k+j}} = \frac{1}{q_1 q_2 \cdots q_{k+j-1}} \cdot \frac{1}{q_{k+j}}, \tag{9.31}$$

$$\frac{1}{q_{k+j}} < \frac{\alpha\beta\lambda}{4}, \tag{9.32}$$

and

$$\frac{1}{4} \cdot (\alpha\beta) \cdot \frac{1}{q_1 q_2 \cdots q_{k+j-1}} \leq (\alpha\beta)^{n_{j-1}-1} r. \tag{9.33}$$

So,

$$\frac{1}{2} \cdot \frac{1}{q_1 q_2 \cdots q_{k+j}} < (\alpha\beta)^{n_{j-1}-1} r\lambda/2 = \rho(B_{n_{j-1}}\lambda/2). \tag{9.34}$$

Thus,

$$C \subset \left[0, b + \rho(B_{n_{j-1}}\lambda/2)\right). \tag{9.35}$$

Letting $\delta = n_j - n_{j-1}$, we see that by (9.25),

$$(\alpha\beta)^{\delta} = \frac{(\alpha\beta)^{n_j-1} r}{\alpha\beta)^{n_{j-1}-1} r} < \frac{\left(\frac{1}{4q_1 q_2 \cdots q_{k+j}}\right)}{\left(\frac{1}{\lambda q_1 q_2 \cdots q_{k+j}}\right)} = \lambda/4. \tag{9.36}$$

Thus, Lemma 9.2.6 applies, so white can force $B_{n_j} = B_{n_{j-1}+\delta}$ to be in $(b+\rho(B_{n_{j-1}})\lambda/2, 1)$. So, $C \cap B_{n_j} = \emptyset$ and thus,

$$B_{n_j} \cap T^{-1}_{Q,k+j}(I) = \emptyset. \tag{9.37}$$

□

9.2.2 Main Theorems and Conjectures

Definition 9.2.16. *We say that a basic sequence Q is bounded below by M if*

$$\inf_n q_n < M. \tag{9.38}$$

Definition 9.2.17. *Given a basic sequence Q and an interval I contained in $[0,1)$, we let $U(Q,I)$ be the set of all real numbers x in $[0,1)$ such that there are only finitely many solutions to*

$$T_{Q,n}(x) \in I. \tag{9.39}$$

Theorem 9.2.18. *Suppose that $(\alpha,\beta) \in D$ and Q is an (α,β)-friendly basic sequence that is bounded below by 3. Then $S(Q,0)$ is an (α,β)-winning set.*

Proof. We let

$$I = \left[0, \frac{1}{3}\right]. \tag{9.40}$$

Suppose that $x = 0.E_1E_2\ldots$ w.r.t. Q. Then $E_{n+1} = 0$ if $T_{Q,n}(x) \in I$. So $S(Q,0) \supset U(Q,I)$. By Lemma 9.2.15, $U(Q,I)$ is an (α,β)-winning set, so $S(Q,0)$ is also an (α,β)-winning set.

□

Corollary 9.2.19. *Suppose that Q is infinite in limit. Then the set of real numbers that are not strongly Q-ratio normal numbers is a $1/2$-winning set and thus has full Hausdorff dimension.*

Proof. Let $(\alpha, \beta) \in D$ and let j be large enough so that $\Psi_{Q,t}$ is an (α, β)-friendly basic sequence that is bounded below by 3. By Theorem 9.2.18, $S(\Psi_{Q,t}, 0)$ is an (α, β)-winning set. However, by Lemma 9.2.10, $S(\Psi_{Q,t}, 0)$ is contained in the set of numbers that are not strongly Q-ratio normal. Thus, this set is (α, β)-winning for all $(\alpha, \beta) \in D$. So, by Lemma 9.2.5, the set of non Q-ratio normal numbers is $1/2$-winning. Thus, it also has full Hausdorff dimension by Theorem 9.1.2.

□

A reasonable conjecture and strengthening of Corollary 9.2.19 that holds in the case of the b-ary expansion is the following:

Conjecture 9.2.20. *The set of real numbers that are not Q-ratio normal numbers is a $1/2$-winning set and thus has full Hausdorff dimension.*

Lemma 9.2.21. *If Q is a basic sequence, t is a positive integer, and $I \subset [0, 1)$ is a closed interval, then the set $U(\Psi_{Q,t}, I)$ is contained in the set of numbers that are not strongly Q-distribution normal.*

Proof. By definition, if x is strongly Q-distribution normal, then the sequence

$$\{T_{Q,tn}(x)\}_{n=1}^{\infty} \tag{9.41}$$

is uniformly distributed mod 1 so there must exist infinitely many n such that $T_{Q,tn}(x) \in I$. Clearly, such an x cannot be a member of $U(\Psi_{Q,t}, I)$.

□

Theorem 9.2.22. *If $I = (a, b)$ is an interval contained in $[0, 1)$ with $\lambda(I) < 1/2$, $(\alpha, \beta) \in D$, and Q is an (α, β)-friendly basic sequence, then $U(Q, I)$ is an (α, β)-winning set.*

Proof. Let $(\alpha, \beta) \in D$ and suppose that t is large enough so that $\Psi_{Q,t}$ is (α, β)-friendly. Then by Lemma 9.2.15, $U(\Psi_{Q,t}, I)$ is (α, β)-winning.

□

Theorem 9.2.23. *The set of numbers that are not strongly Q-distribution normal is a $1/2$-winning set and thus has full Hausdorff dimension.*

Proof. This follows directly from Lemma 9.2.5, Theorem 9.1.2, Lemma 9.2.21, and Theorem 9.2.22.

□

Another reasonable conjecture and strengthening of Theorem 9.2.23 that holds in the case of the b-ary expansion is the following:

Conjecture 9.2.24. *The set of numbers that are not Q-distribution normal is a $1/2$-winning set and thus has full Hausdorff dimension.*

We remark that, in a sense, our sets of non-normal numbers are particularly large in light of the following theorem of [47]:

Theorem 9.2.25. *The only α-winning set $S \subset [0,1)$ with $\alpha > 1/2$ is $S = [0,1)$.*

9.3 Windim of other Sets Related to Digits of the Cantor Series Expansion

9.3.1 Basic Lemmas and Definitions

In this section, we fix a basic sequence Q and define the set

$$A = \{x = 0.E_1E_2E_3 \ldots : E_n = 0 \text{ infinitely often}\}. \tag{9.42}$$

For $g > 2$, we put

$$B_g = \{x \in [0,1) : T_{Q,n}(x) \in [0, 1/g) \text{ infinitely often}\}. \tag{9.43}$$

In other words, A is the set of real numbers that contain infinitely many zeros in their Q-Cantor series expansion and B_g is the set of real numbers x such that $T_{Q,n}(x)$ is between 0 and $1/g$ infinitely often.

For a positive integer $g \geq 3$, we also define

$$\alpha_g = \frac{1}{(g-1)^2 + 1}. \tag{9.44}$$

Given a basic sequence Q and positive integers $g \geq 3$ and k, we put

$$\eta_k = \lfloor q_k/g \rfloor. \tag{9.45}$$

We will also need the quantities

$$\Omega_k = \sum_{m=k+1}^{\infty} \frac{\eta_m}{q_1 q_2 \cdots q_m} \tag{9.46}$$

and

$$\omega_k = \frac{1}{q_1 q_2 \cdots q_k} - \Omega_k. \tag{9.47}$$

Definition 9.3.1. *Given a positive integer $g \geq 3$, we will say that a basic sequence Q is (g,t)-amicable if for all positive integers k,*

$$g | q_k \tag{9.48}$$

and for $k \geq t$

$$\frac{1}{q_1 q_2 \cdots q_k} \leq (g-1) \sum_{m=k+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_m}. \tag{9.49}$$

We note the following lemma:

Lemma 9.3.2. *Suppose that Q is (g,t)-amicable. Then for all positive integers k, we have*

$$\frac{\eta_k}{q_k} = \frac{1}{g}, \tag{9.50}$$

$$\Omega_k = \frac{1}{g} \sum_{m=k}^{\infty} \frac{1}{q_1 q_2 \cdots q_m}, \tag{9.51}$$

and

$$\Omega_{k+1} = \Omega_k - \frac{1}{g} \cdot \frac{1}{q_1 q_2 \cdots q_k}. \tag{9.52}$$

Proof. To show (9.50), we note that

$$\frac{\eta_k}{q_k} = \frac{\lfloor q_k/g \rfloor}{q_k} = \frac{1}{g}, \tag{9.53}$$

as $g|q_k$.

Next, we substitute (9.50) into (9.46):

$$\Omega_k = \sum_{m=k+1}^{\infty} \frac{\eta_m}{q_1 q_2 \cdots q_m} = \sum_{m=k+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_{m-1}} \cdot \frac{\eta_m}{q_m} \tag{9.54}$$

$$= \sum_{m=k+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_{m-1}} \cdot \frac{1}{g} = \frac{1}{g} \sum_{m=k}^{\infty} \frac{1}{q_1 q_2 \cdots q_m}, \tag{9.55}$$

showing (9.51).

Lastly, to verify (9.52) we see that

$$\Omega_{k+1} = \sum_{m=k+2}^{\infty} \frac{\eta_m}{q_1 q_2 \cdots q_k} = \left(\sum_{m=k+1}^{\infty} \frac{\eta_m}{q_1 q_2 \cdots q_k} \right) - \frac{\eta_{k+1}}{q_1 q_2 \cdots q_{k+1}} \tag{9.56}$$

$$= \Omega_k - \frac{1}{g} \cdot \frac{1}{q_1 q_2 \cdots q_k}, \tag{9.57}$$

by (9.50).

∎

We will freely use the results of Lemma 9.3.2 frequently and without mention.

Lemma 9.3.3. *Suppose that Q is (g,t)-amicable. Then for $k \geq t$,*

$$\frac{1}{g(g-2)} \cdot \omega_k \leq \Omega_{k+1}. \tag{9.58}$$

Proof. Suppose that $k \geq t$. Since Q is (g,t)-amicable, we know that

$$\frac{1}{q_1 q_2 \cdots q_k} \leq (g-1) \sum_{m=k+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_m}. \tag{9.59}$$

Dividing both sides of (9.59), we see that

$$\frac{1}{g} \cdot \frac{1}{q_1 q_2 \cdots q_k} \leq \frac{g-1}{g} \sum_{m=k+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_m}. \tag{9.60}$$

Adding

$$\frac{g-1}{g} \cdot \frac{1}{q_1 q_2 \cdots q_k} \tag{9.61}$$

to both sides of (9.59) gives

$$\frac{1}{q_1 q_2 \cdots q_k} \leq \frac{g-1}{g} \sum_{m=k}^{\infty} \frac{1}{q_1 q_2 \cdots q_m} = (g-1) \sum_{m=k+1}^{\infty} \frac{\eta_m}{q_1 q_2 \cdots q_m}, \tag{9.62}$$

by (9.50).

Thus, we multiply both sides of (9.62) by $g-1$ to get

$$(g-1)\cdot\frac{1}{q_1q_2\cdots q_k}\le(g-1)^2\Omega_k=(g^2-2g+1)\Omega_k, \tag{9.63}$$

so

$$\frac{1}{q_1q_2\cdots q_k}-\Omega_k\le g(g-2)\Omega_k-(g-2)\frac{1}{q_1q_2\cdots q_k}. \tag{9.64}$$

Dividing both side of (9.64) by $g(g-2)$, we see that

$$\frac{1}{g(g-2)}\cdot\omega_k\le\Omega_k-\frac{1}{g}\frac{1}{q_1q_2\cdots q_k}=\Omega_k-\frac{\eta_{k+1}}{q_1q_2\cdots q_{k+1}}=\Omega_{k+1}. \tag{9.65}$$

□

Lemma 9.3.4. *If $g\ge 3$ is an integer and*

$$x\in(0,(\alpha_g(g-1)g)^{-1}], \tag{9.66}$$

then there exists a k such that

$$\alpha_g^{-1}\Omega_{k+1}<x\le\alpha_g^{-1}\Omega_k. \tag{9.67}$$

Proof. We note that by definition,

$$\lim_{n\to\infty}\Omega_n=0. \tag{9.68}$$

We also see that by (9.52), we have

$$\Omega_{n+1} - \Omega_n = \frac{1}{g} \frac{1}{q_1 q_2 \cdots q_n} > 0. \tag{9.69}$$

Since $\alpha_g > 0$,

$$0 < \ldots < \alpha_g^{-1}\Omega_{n+2} < \alpha_g^{-1}\Omega_{n+1} < \alpha_g^{-1}\Omega_n < \alpha_g^{-1}\Omega_{n-1} < \ldots < \alpha_g^{-1}\Omega_0, \tag{9.70}$$

so there must be a positive integer k such that

$$x \in (\alpha_g^{-1}\Omega_{k+1}, \alpha_g^{-1}\Omega_k]. \tag{9.71}$$

□

We will need to make the following conventions for the next lemma and theorem: given a basic sequence Q and integers $k \geq 1$ and n, we let

$$I_k(n) = \left[\frac{n}{q_1 q_2 \cdots q_k}, \frac{n}{q_1 q_2 \cdots q_k} + \Omega_k\right] \tag{9.72}$$

and

$$K_k = \bigcup_{n=-\infty}^{\infty} I_k(n). \tag{9.73}$$

Lemma 9.3.5. *Suppose that Q is (g,t)-amicable and that I is a closed interval such that*

$$\lambda(I) \leq (\alpha_g (g-1) g^t)^{-1}. \tag{9.74}$$

Then there is a positive integer k and an interval $J \subset I \cap K_k$ such that

$$\lambda(J) = \alpha_g \lambda(I). \tag{9.75}$$

Proof. By Lemma 9.3.4, there must exist a positive integer k such that

$$\alpha_g^{-1}\Omega_{k+1} < \lambda(I) \leq \alpha_g^{-1}\Omega_k. \tag{9.76}$$

We fix this value of k throughout the rest of this proof.

Note that K_k is the disjoint union of intervals of length Ω_k and the compliment of K_k is the disjoint union of intervals of length ω_k. The worst case scenario is when the midpoint c of I is equal to the midpoint of one of the intervals that makes up the compliment of K_k. Thus, we note that in this case,

$$[c, c + \frac{1}{2}\lambda(I)] \subset I \tag{9.77}$$

and

$$[c + \omega_k/2, c + \omega_k/2 + \Omega_k] \subset K_k. \tag{9.78}$$

We now set

$$V = \left[c + \omega_k/2, c + \min\left(\frac{1}{2}\lambda(I), \frac{1}{2}\omega_k + \Omega_k\right)\right]. \tag{9.79}$$

Clearly, $V \subset I \cap K_k$, by (9.78) and (9.79). We will now show that $\lambda(V) \geq \alpha_g \lambda(I)$:

$$\lambda(V) - \alpha_g \lambda(I) = \min\left(\frac{1}{2}\lambda(I), \frac{1}{2}\omega_k + \Omega_k\right) - \frac{1}{2}\omega_k - \alpha_g \lambda(I) \tag{9.80}$$

$$= \min\left(\left(\frac{1}{2} - \alpha_g\right)\lambda(I) - \frac{1}{2}\omega_k, \Omega_k - \alpha_g \lambda(I)\right). \tag{9.81}$$

However,

$$\frac{1}{2} - \alpha_g = \frac{1}{2}\frac{1}{(g-1)^2+1} = \frac{1}{2} \cdot g(g-2)\alpha_g \tag{9.82}$$

and

$$\Omega_k - \alpha_g \lambda(I) \geq 0 \tag{9.83}$$

by (9.76), so

$$\lambda(V) - \alpha_g \lambda(I) \geq \min\left(\frac{1}{2} \cdot g(g-2)\alpha_g \lambda(I) - \frac{1}{2}\omega_k, 0\right). \tag{9.84}$$

However, by Lemma 9.3.3,

$$\frac{1}{g(g-2)} \cdot \omega_k \leq \Omega_{k+1}. \tag{9.85}$$

Since $\lambda(I) > \Omega_{k+1}\alpha_g^{-1}$,

$$\frac{1}{2} \cdot g(g-2)\alpha_g \lambda(I) > \frac{1}{2} \cdot g(g-2)\alpha_g \alpha_g^{-1}\Omega_{k+1} \tag{9.86}$$

$$\geq \frac{1}{2}g(g-2)\frac{1}{g(g-2)}\omega_k = \frac{1}{2} \cdot \omega_k. \tag{9.87}$$

Thus, $\lambda(V) - \alpha_g \lambda(I) \geq 0$, so there exists a closed interval $J \subset V$ such that $\lambda(J) = \alpha_g \lambda(I)$.

□

Lemma 9.3.6. *Suppose that Q is a basic sequence and that $k \geq 1$ and n are integers. If x is in the interior of $I_k(n)$, then there exists a positive integer $m \geq k$ such that*

$$T_{Q,m}(x) \in [0, 1/g). \tag{9.88}$$

Proof. We see that for $x = E_0.E_1E_2\ldots$ w.r.t. Q,

$$T_{Q,m}(x) = \frac{E_{m+1}}{q_{m+1}} + \frac{E_{m+2}}{q_{m+1}q_{m+2}} + \ldots. \tag{9.89}$$

Since x is in the interior of $I_k(n)$, there exists an integer $m \geq k$ such that $E_{m+1} < \eta_{m+1}$, so

$$T_{Q,m}(x) < \frac{\eta_{m+1}}{q_{m+1}} + \frac{\eta_{m+2}}{q_{m+1}q_{m+2}} + \ldots \leq \frac{1}{g}. \tag{9.90}$$

Hence, $T_{Q,m}(x) \in [0, 1/g)$.

□

9.3.2 Main Results and Conjectures

Theorem 9.3.7. *Suppose that t and $q \geq 3$ are positive integers and that Q is (g,t)-amicable. Then B_g is α_g-winning.*

Proof. Given an integer n, we let

$$\frac{n}{q_1 q_2 \cdots q_k} = E_0 + \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_k}{q_1 q_2 \cdots q_k} \tag{9.91}$$

be the Q-Cantor series expansion of $\frac{n}{q_1 q_2 \cdots q_k}$. Thus, we see that

$$I_k(n) = \left[E_0 + \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_k}{q_1 q_2 \cdots q_k}, \right. \tag{9.92}$$

$$\left. , E_0 + \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_k}{q_1 q_2 \cdots q_k} + \Omega_k \right],$$

so if x is in the interior of $I_k(n)$, then by Lemma 9.3.6 there exists a positive integer $m \geq k$ such that $T_{Q,m}(x) \in [0, 1/g)$. So, given a closed subset $C \subset I_k(n)$, there is a positive integer m that depends only on C such that for all $x \in C$,

$$T_{Q,m}(x) \in [0, 1/g). \tag{9.93}$$

We let $\beta \in (0,1)$ and will now show that B_g is (α_g, β)-winning. White will play arbitrarily until black chooses a ball B_{j_1} such that

$$\lambda(B_{j_1}) \leq (\alpha_g (g-1) g^t)^{-1}. \tag{9.94}$$

By Lemma 9.3.5, there exists a positive integer $k_1 \geq t$ such that white can pick a ball $W_{j_1} \subset K_{k_1}$. Suppose that n_1 is such that $W_{j_1} \subset I_{k_1}(n_1)$. Thus, for white's next move, he may force W_{j_1+1} to be contained in the interior of $I_{k_1}(n_1)$. Thus, by Lemma 9.3.6, there exists a positive integer $m_1 \geq k_1$ such that for every $x \in W_{j_1+1}$, we have

$$T_{Q,m_1}(x) \in \left[0, \frac{1}{g}\right). \tag{9.95}$$

White may play again arbitrarily until

$$\lambda(B_{j_2}) \leq \alpha_g^{-1}(g-1)^{-1}g^{-k_1-m_1}. \tag{9.96}$$

Since $k_1 + m_1 > t$, by Lemma 9.3.5, there exists a positive integer $k_2 \geq k_1 + m_1$ such that white may force W_{j_2} to be contained in K_{k_2}. Thus, white may choose W_{j_2+1} to be contained in the interior of $I_{k_2}(n_2)$ for some n_2, and so on. Thus, B_g is (α, β)-winning.

□

Corollary 9.3.8. *Suppose that $q \geq 3$ and t are positive integers and that Q is (g,t)-amicable. Then B_g has full Hausdorff dimension.*

While there were many technical difficulties encountered in generalizing this theorem to the case of the Cantor series expansion, it seems as if Theorem 9.3.7 should not be restricted to (g,t)-amicable basic sequences. I make the following conjectures:

Conjecture 9.3.9. *Suppose that Q is a basic sequence. If $g \geq 3$ is a positive integer, then B_g is α_g-winning.*

Conjecture 9.3.10. *Suppose that Q is a basic sequence. If $g \geq 3$ is a positive integer, then B_g is not α-winning for any $\alpha > \alpha_g$.*

Conjecture 9.3.11. *Suppose that Q is bounded. If $g \geq 3$ is a positive integer, then B_g is α_g-winning, but not α-winning for any $\alpha > \alpha_g$. Thus,*

$$\textit{windim } B_g = \alpha_g. \tag{9.97}$$

Corollary 9.3.12. *Suppose that Q is a basic sequence. For all $g > 2$, B_g has full Hausdorff dimension.*

We will need the following lemma for the next theorem:

Lemma 9.3.13. *If Q is infinite in limit, then $A \subset B_g$ for all $g > 2$.*

Proof. Let $x \in A$ with $x = 0.E_1E_2E_3\ldots$ w.r.t Q. Let N be large enough so that $q_n > g$ for all $n > N$. If $E_n = 0$, then

$$T_{Q,n-1}(x) = \frac{0}{q_n} + \frac{E_{n+1}}{q_nq_{n+1}} + \frac{E_{n+2}}{q_nq_{n+1}q_{n+2}} + \ldots < 0 + \frac{1}{q_n} < \frac{1}{g}, \tag{9.98}$$

so $x \in B_g$.

∎

We may prove the following conjecture if we have a proof of Conjecture 9.3.10:

Conjecture 9.3.14. *Suppose that Q is infinite in limit. Then the set of real numbers that contains infinitely many zeros is not α-winning for any $\alpha > 0$.*

Proof. Let $g > 2$. By Conjecture 9.3.10,

$$\text{windim } B_g = \frac{1}{(g-1)^2 + 1}. \tag{9.99}$$

Since Q is infinite in limit and $A \subset B_g$ by Lemma 9.3.13, we see that

$$\text{windim } A \leq \text{windim } B_g. \tag{9.100}$$

However,

$$\lim_{g \to \infty} \text{windim } B_g = \lim_{g \to \infty} \frac{1}{(g-1)^2 + 1} = 0, \tag{9.101}$$

so windim $A = 0$.

∎

CHAPTER 10

IRRATIONALITY OF CERTAIN CANTOR SERIES EXPANSIONS RELATED TO NORMAL NUMBERS

In this chapter, we will examine how well the last property of normal numbers listed in our intro holds up; that is, can we still conclude that a number normal with respect to any of the notions we have studied so far must be irrational?

10.1 The Alternative Q-Cantor Series Expansion

We will need to consider the following expansion, which is almost identical to the Q-Cantor series expansion:

Definition 10.1.1. *Given a basic sequence Q, the alternative Q-Cantor series expansion of a real x in $[0,1)$ is the (unique) expansion of the form*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{10.1}$$

such that E_n is in $\{0,1,\dots,q_n-1\}$ for all n with $E_n \neq 0$ infinitely often.

We remark that the uniquness of the alternative Q-Cantor series expansion is proven identically to Proposition 2.1.3. Clearly, the Q-Cantor series expansion and alternative Q-Cantor series expansion will be identical except on a countable set of real numbers.

Definition 10.1.2. *For a given basic sequence Q, let $\bar{N}_n^Q(B,x)$ denote the number of times a block B occurs starting at a position no greater than n in the alternative Q-Cantor series expansion of x.*

We now define a notion of normality that we will need in this chapter:

Definition 10.1.3. *Given a basic sequence Q that is infinte in limit, a real number $x \in [0,1)$ is weakly alternative Q-ratio normal of order k if for all Q-admissable blocks B and B' of length k that do not contain the digit 0, we have*

$$\lim_{n\to\infty} \frac{\bar{N}_n^Q(B,x)}{\bar{N}_n^Q(B',x)} = 1. \tag{10.2}$$

We will say that x is weakly alternative Q-ratio normal if it is weakly alternative Q-ratio normal of order k for all positive integers k.

10.2 More on Ratio Normality

We will see that the property of being Q-ratio normal is less dependent on the basic sequence Q and more on digits of an expansion. This will become an important idea when we want to construct rational numbers that satisfy certain weak notions of normality.

Similarly to the definition of normal numbers we can give a reasonable definition for normality of a sequence

$$E = (E_1, E_2, \ldots) \tag{10.3}$$

of non-negative integers. For the remainder of this chapter, we will write

$$\mathbb{W} = \mathbb{N} \cup \{0\}. \tag{10.4}$$

Definition 10.2.1. *We will say that $E \in \mathbb{W}^{\mathbb{N}}$ is ratio normal of order k if there exists a basic sequence Q that is infinite in limit and a real number x in $[0, 1)$ such that $E = \pi_Q(x)$ and x is a Q-ratio normal number of order k. Similarly E will be called ratio normal if it is ratio normal of order k for all k.*

Proposition 10.2.2. *A sequence $E \in \mathbb{W}^{\mathbb{N}}$ is ratio normal of order k if and only if for all $m \leq k$ and blocks B_1 and B_2 of length m, we have*

$$\lim_{n\to\infty} \frac{N_n(B_1, E)}{N_n(B_2, E)} = 1. \tag{10.5}$$

Proof. Suppose that E is ratio normal. Then there exists x and Q with $E = \pi_Q(x)$ such that for all k and blocks B_1 and B_2 of length $k \leq m$,

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = 1. \tag{10.6}$$

Clearly, $N_n^Q(B_1,x) = N_n(B_1,E)$ and $N_n^Q(B_2,x) = N_n(B_2,E)$, so

$$\lim_{n\to\infty} \frac{N_n(B_1,E)}{N_n(B_2,E)} = \lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = 1. \tag{10.7}$$

For the converse, suppose that for all k and blocks B_1 and B_2 of length $k \le m$,

$$\lim_{n\to\infty} \frac{N_n(B_1,E)}{N_n(B_2,E)} = 1. \tag{10.8}$$

Define a basic sequence $Q = \{q_n\}$ by letting $q_n = E_n + 17$ for all n. Suppose that $x = 0.E_1E_2E_3\ldots$ w.r.t. Q. Then, similarly to before, $N_n(B_1,E) = N_n^Q(B_1,x)$ and $N_n(B_2,E) = N_n^Q(B_2,x)$, so

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = \lim_{n\to\infty} \frac{N_n(B_1,E)}{N_n(B_2,E)} = 1 \tag{10.9}$$

and x is Q-ratio normal of order k.

□

Corollary 10.2.3. *A sequence $E \in \mathbb{W}^{\mathbb{N}}$ is ratio normal if and only if for all k and blocks B_1 and B_2 of length k*

$$\lim_{n\to\infty} \frac{N_n(B_1,E)}{N_n(B_2,E)} = 1. \tag{10.10}$$

10.3 The E-engel series

We recall the definition of the Engel series expansion. Suppose that $x \in (0,1)$. We define a sequence of positive integers $q_1, q_2, q_3, \ldots$ as follows. Suppose that q_1 satisfies

$$\frac{1}{q_1} \leq x < \frac{1}{q_1 - 1}. \tag{10.11}$$

Given $q_1, q_2, \ldots, q_{n-1}$, then we determine q_n by the inequality

$$\frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n} \leq x < \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \ldots q_{n-1}(q_n - 1)}. \tag{10.12}$$

If

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n}, \tag{10.13}$$

then our expansion is finite and we do not need to determine $q_{n+1}, q_{n+2}, \ldots$. Otherwise, the Engel series expansion of x is

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n} + \ldots. \tag{10.14}$$

We note that if (10.14) is the Engel series expansion of x and $Q = \{q_n\}$, then (10.14) is the Q-Cantor series expansion of x. Thus, we may think of the Engel series expansion as asking: given a real number $x \in (0,1)$, for what basic sequence Q, will the digits of the Q-Cantor series expansion be $(1, 1, 1, \ldots)$? We can take this idea further. Suppose that

$$E = (E_1, E_2, E_3, \ldots), \tag{10.15}$$

where $E_1, E_2, \ldots$ are positive integers. Then we define the *E-Engel series expansion* of $x \in (0,1)$ as follows. We define a sequence of positive integers $q_1, q_2, q_3, \ldots$. Suppose that q_1 satisfies

$$\frac{E_1}{q_1} \leq x < \frac{E_1}{q_1 - 1}. \tag{10.16}$$

Given $q_1, q_2, \ldots, q_{n-1}$, we determine q_n by the inequality

$$\frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} \leq x < \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \ldots q_{n-1}(q_n - 1)}. \tag{10.17}$$

If

$$x = \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n}, \tag{10.18}$$

then our expansion is finite and we do not need to determine $q_{n+1}, q_{n+2}, \ldots$. Otherwise, the E-Engel series expansion of x is

$$x = \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} + \ldots. \tag{10.19}$$

Thus, we see that for $x \in (0,1)$, either (10.18) or (10.19) is the alternative Q-Cantor series expansion of x. We will see that the E-Engel series expansion will help us settle questions involving rationality of normal numbers.

Example 10.3.1. *Let $x = \sqrt{2} - 1$ and $E = (1, 2, 3, 4, 5, \ldots)$. Then the E-Engel series expansion of x is*

$$x = \frac{1}{3} + \frac{2}{3 \cdot 9} + \frac{3}{3 \cdot 9 \cdot 17} + \frac{4}{3 \cdot 9 \cdot 17 \cdot 33} + \frac{5}{3 \cdot 9 \cdot 17 \cdot 33 \cdot 54} + \ldots. \tag{10.20}$$

Example 10.3.2. *Let $x = \frac{25}{29}$ and $E = (2, 4, 8, 16, 32, \ldots)$. Then the E-Engel series expansion of x is*

$$x = \frac{2}{3} + \frac{4}{3 \cdot 7} + \frac{8}{3 \cdot 7 \cdot 78} + \frac{16}{3 \cdot 7 \cdot 78 \cdot 232}. \tag{10.21}$$

Problem 10.3.3. *For what E does a similar result to Theorem 1.5.3 hold for the E-Engel series expansion?*

10.4 Rationality of Cantor Series Expansions

10.4.1 Classical Results

In this subsection, we list well known results on rationality of sums of the form $\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n}$. All of the results in this section can be found in [16]. For recent progress, see [19].

Theorem 10.4.1. *A necessary and sufficient condition that the infinite series*

$$\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{10.22}$$

be irrational is that, for every integer $B \geq 1$, we can find an integer A and a subsequence $i_1, i_2, \dots$ such that

$$\frac{A}{B} < x_{i_n} < \frac{A+1}{B}, n = 1, 2, 3, \dots \tag{10.23}$$

where $x = x_1$ and, for $i > 1$,

$$x_i = T_{Q,i-1}(x). \tag{10.24}$$

Theorem 10.4.2. *A necessary and sufficient condition that the infinite series*

$$\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{10.25}$$

be irrational, provided that $0 \leq E_j \leq q_j - 1$, is that there exist coprime numbers $0 \leq a \leq b$; a condensation and an integer N such that, for all $i \geq n$,

$$E_i = \frac{a}{b} \cdot (Q_i - 1). \tag{10.26}$$

The following theorems are generally far easier to apply:

Theorem 10.4.3. *Let the integers $q_1, q_2, \ldots$ be such that any integer $b > 0$ divides $q_1 q_2 \ldots q_n$ for all sufficiently large n. Then*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n} \tag{10.27}$$

is rational if, and only if, $E_j = q_j - 1$ for all but a finite number of j, or, if $E_j = 0$ ultimately.

Theorem 10.4.4. *If Q is a basic sequence and*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n} \tag{10.28}$$

is irrational if there exists an irrational number t and a subsequence $1 \leq i_1 < i_2 < \ldots$ such that,

$$\lim_{n \to \infty} \frac{E_{i_n}}{q_{i_n}} = t. \tag{10.29}$$

Theorem 10.4.5. *If Q is a basic sequence and*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n} \tag{10.30}$$

is irrational if $E_j \leq q_j - 2$ infinitely often and if there is a subsequence $1 \leq i_1 < i_2 < \ldots$ such that,

$$\lim_{n \to \infty} \frac{E_{i_n}}{q_{i_n}} = 1. \tag{10.31}$$

Theorem 10.4.6. *If Q is a basic sequence and*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n}, \tag{10.32}$$

then x is irrational if $E_j > 0$ infinitely often and if there is a subsequence $1 \le i_1 < i_2 < \dots$ such that, as $n \to \infty$,

$$\lim_{n\to\infty} \frac{E_{i_n}}{q_{i_n}} = 0 \tag{10.33}$$

and

$$\lim_{n\to\infty} q_{i_n} = \infty. \tag{10.34}$$

10.4.2 Rationality of Normal Numbers

It should be noted that the expression

$$\sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \dots q_n} \tag{10.35}$$

in the preceeding theorems need not necessarily be the Q-Cantor series expansion or the alternative Q-Cantor series expansion as we have neither the restriction that $E_n < q_n - 1$ infinitely often or $E_n > 0$ infinitely often. We will see that two of these theorems are immediately applicable to normality.

Theorem 10.4.7. *Suppose that Q is infinite in limit and 1-divergent. If x is simply Q-normal, then x is irrational.*

Proof. We will use Theorem 10.4.6 to show that x is irrational. Since x is simply Q-normal and $q_n \to \infty$, there exists infinitely many j such that $E_j > 0$.

Let the sequence i_n represent the position of the n^{th} zero in the Q-Cantor series expansion of x. Then, clearly, $\frac{E_{i_n}}{q_{i_n}} = 0$, so

$$\lim_{n\to\infty} \frac{E_{i_n}}{q_{i_n}} = 0. \tag{10.36}$$

Since Q is infinite in limit, we have

$$\lim_{n\to\infty} q_{i_n} = \infty, \tag{10.37}$$

so (10.33) and (10.34) are satisfied and x is irrational.

□

Theorem 10.4.8. *If Q is infinite in limit and x is Q-dense, then x is irrational.*

Proof. We will use Theorem 10.4.6 again. Suppose that $x = 0.E_1E_2E_3\ldots$ w.r.t. Q. Since x is Q-dense, E_n will be greater than $\frac{1}{2}$ for infinitely many n and thus $E_n \neq 0$ for infinitely many n. Since x is Q-dense, there is a sequence i_n such that

$$\lim_{n\to\infty} \frac{E_{i_n}}{q_{i_n}} = 0. \tag{10.38}$$

□

Corollary 10.4.9. *If Q is infinite in limit and x is Q-distribution normal, then x is irrational.*

Proof. By Lemma 8.5.2, x is Q-dense. Thus, by Theorem 10.4.8, x is irrational. □

Even more was shown by P. Laffer in [24]. He showed that

Theorem 10.4.10. *A real number x in $[0, 1)$ is irrational if and only if there exists a basic sequence Q such that x is Q-distribution normal.*

We now turn our attention to a case where normality behaves differently from the b-ary expansion:

Theorem 10.4.11. *Suppose that $F = (F_1, F_2, \ldots)$ is a normal sequence. Set $Q = F + 2$ and $E = F + 1$. Then*

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \ldots q_n} \tag{10.39}$$

is the alternative Q-Cantor series expansion of x, x is weakly alternative Q-ratio normal, and x is rational.

Proof. Since F is a normal sequence, for every integer m there will exist a k such that $F_k + 2 = m$. Thus all positive integers m will eventually divide the product $q_1 q_2 \ldots q_n$ for large enough n. Additionally, $E_j = q_j - 1$ for all j, so by Theorem 10.4.3, x is rational. Clearly, x is weakly alternative Q-ratio normal as F is a normal sequence. □

Theorem 10.4.7 should be compared to the case where $q_n = b$ is constant. In the base b expansion, simple normality is not enough to guarantee irrationality. For example, in base 10, the number

$$0.01234567890123456789 0\ldots \tag{10.40}$$

will be simply normal but is also rational in contrast with Theorem 10.4.7.

Theorem 10.4.10 is entirely consistent with what we expect from studying the b-ary expansion. However, Theorem 10.4.11 suggests a significant departure. To see this, we let $Q = \{b, b, \ldots\}$ and $Q' = Q + 1$. Let x be Q-ratio normal, where

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} \tag{10.41}$$

is its alternative Q-cantor series expansion. We remark that since x is normal in base b, its Q-Cantor series expansion and alternative Q-Cantor series expansions are identical. Set

$$y = \sum_{n=1}^{\infty} \frac{E_n + 1}{(b+1)^n}. \tag{10.42}$$

Then, clearly, y is irrational as the digits $E_n + 1$ cannot be eventually periodic as x is irrational. y is also weakly alternative Q'-ratio normal because x is Q-ratio normal, in contrast with Theorem 10.4.11.

We end this section by giving applications of the E-Engel series expansion. First, we need the following lemma:

Lemma 10.4.12. *Suppose that Q is a basic sequence. Then for all positive integers n, we have*

$$\sum_{m=n+1}^{\infty} \frac{q_m - 1}{q_1 q_2 \cdots q_m} = \frac{1}{q_1 q_2 \cdots q_n}. \tag{10.43}$$

Proof. We write

$$\sum_{m=n+1}^{\infty} \frac{q_m - 1}{q_1 q_2 \cdots q_m} = \sum_{m=n+1}^{\infty} \left(\frac{q_m}{q_1 q_2 \cdots q_m} - \frac{1}{q_1 q_2 \cdots q_m} \right) \tag{10.44}$$

$$= \sum_{m=n+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_{m-1}} - \sum_{m=n+1}^{\infty} \frac{1}{q_1 q_2 \cdots q_m} = \frac{1}{q_1 q_2 \cdots q_n}.$$

□

We may now prove the following:

Theorem 10.4.13. *Suppose that $x \in (0,1)$ is rational. Then there exists a basic sequence Q such that x is weakly alternative Q-ratio normal.*

Proof. Suppose that

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_n} \tag{10.45}$$

is the Engel series expansion of x and that $F = (F_1, F_2, \ldots)$ is any normal sequence. Put

$$E = (E_1, E_2, \ldots) = F + 1 \tag{10.46}$$

and

$$Q' = (q'_1, q'_2, \ldots) = F + 2. \tag{10.47}$$

Then by Lemma 10.4.12 and (10.45), we have

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \ldots + \frac{1}{q_1 q_2 \cdots q_{n-1}} + \sum_{m=n+1}^{\infty} \frac{E_m}{q_1 q_2 \cdots q_n q'_{n+1} q'_{n+2} \cdots q'_m}. \tag{10.48}$$

By Theorem 10.4.11 and (10.48), x is weakly alternative Q-ratio normal.

□

Theorem 10.4.14. *Suppose that $x \in (0,1)$ is irrational. Then there exists a basic sequence Q such that x is weakly alternative Q-ratio normal.*

Proof. Suppose that F is a normal sequence and $E = F + 1$. Suppose that

$$x = \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \ldots + \frac{E_n}{q_1 q_2 \cdots q_n} + \ldots \tag{10.49}$$

is the E-Engel series expansion of x. Then (10.49) is the alternative Q-Cantor series expansion of x, so x is clearly alternative Q-ratio normal.

□

By Theorem 10.4.13 and Theorem 10.4.14, for every $x \in (0,1)$, there is a basic sequence Q such that x is weakly alternative Q-ratio normal. Naturally, this begs the following question:

Problem 10.4.15. *Suppose that $x \in (0,1)$. Must there exist a basic sequence Q such that x is alternative Q-ratio normal or Q-ratio normal?*

CHAPTER 11

OPEN PROBLEMS AND FURTHER INVESTIGATIONS

I wish to propose the following problems that were not posed elsewhere in the text.

11.1 More General Constructions of Normal Numbers

The proof techniques used to prove Theorem 3.2.10, Theorem 3.3.13, and Theorem 6.3.1 appear to apply to more general constructions. In particular, it looks as if they may be used to construct number normal with respect to the Lüroth series expansion. This suggests the following open problem:

Problem 11.1.1. *Use the techniques used to prove Theorem 3.2.10, Theorem 3.3.13, and Theorem 6.3.1 to construct a normal number for any "Bernoulli enough" shift.*

It should be noted that the continued fraction map is in fact only strongly mixing so it may be possible to strengthen the scope of Problem 11.1.1 to include shift transformation that are strongly mixing.

11.2 Ergodic Properties of the Cantor Series Expansion

We next note the following problem proposed by A. Rényi:

Problem 11.2.1. *In what way must the ergodic theorem be strengthened to include Theorem 7.3.12 as a special case?*

It is possible that Problem 11.2.1 may not have a satisfactory conclusion. The b-ary expansion is amenable to study through ergodic theory due to Theorem 1.2.4. We know due to Corollary 5.0.29 and examples we have seen throughout the text that there is no simple variation of Theorem 1.2.4 that applies to the Cantor series expansion.

11.3 More Powerful Tools to Study Normality

As was pointed out in the introduction, normality in some less general settings can be studied through the powerful machinery associated with the theory of uniformly distributed sequences. In particular, estimates of discrepancy and trigonometric sums have proven to be very powerful.

Problem 11.3.1. *Find an effective method to transfer estimates of discrepancy to* (ϵ, k, μ)*-normal sequences that can be used to construct* Q*-normal numbers with theorems like Theorem 3.3.13.*

In fact this suggests far more reaching applications if there is a satisfactory solution of Problem 11.1.1.

Problem 11.3.2. *Transfer many of the constructions of numbers normal in some base b to other contexts. For example, what would a suitable version of the Davenport-Erdős constant look like for the continued fraction expansion?*

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