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SERIES EXPANSIONS FOR THE ALL-TIME MAXIMUM OF α-STABLE RANDOM WALKS

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Abstract

We study random walks whose increments are α-stable distributions with shape parameter $1 < \alpha < 2$. Specifically, assuming a mean increment size which is negative, we provide series expansions in terms of the mean increment size for the probability that the all-time maximum of an α-stable random walk is equal to zero and, in the totally skewed to the left case of skewness parameter $\beta = -1$, for the expected value of the all-time maximum of an α-stable random walk. Our series expansions generalize previous results for Gaussian random walks. Key ingredients in our proofs are Spitzer's identity for random walks, the stability property of α-stable random variables and Zolotarev's integral representation for the CDF of an α-stable random variable. We also discuss an application of our results to a problem arising in queueing theory.

Keywords: random walk, queueing, heavy tail

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1. Introduction

The class of α-stable distributions is an extension of the class of Gaussian distributions to random variables with infinite variance and sometimes even infinite mean. It is in fact well known [11, 28, 32] that the class of α-stable distributions corresponds to the set of weak limit points in generalized versions of the central limit theorem for sums of i.i.d. random variables with an infinite second moment. In this paper, we study several problems related to the all-time maximum of a random walk whose increments are i.i.d. α-stable random variables with finite, negative mean. Specifically, let $\{X_k, k \geq 1\}$ be

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an i.i.d. sequence of α-stable random variables with shape parameter $1 < \alpha < 2$, skewness parameter $-1 \leq \beta \leq 1$, shift parameter $-\nu < 0$ and scale parameter 1. See (1) below for the characteristic function of the sequence $\{X_k, k \geq 1\}$. One may consult Samorodnitsky and Taqqu [28] or Zolotarev [35] for detailed discussions regarding α-stable random variables. We also provide the necessary background information on α-stable distributions in Section 2 below. For the moment, we note that since $1 < \alpha < 2$, we have that

$$-\infty \ < \ \mathbb{E}_\nu[X_1] = -\nu < 0 \qquad \text{and} \qquad \mathrm{Var}_\nu(X_1) = +\infty.$$

Now set $S_0 = 0$ and

$$S_n \quad = \quad \sum_{k=1}^{n} X_k, \quad n \geq 1.$$

We then define $S = \{S_n, n \geq 0\}$ to be the random walk associated with the sequence $\{X_k, k \geq 1\}$. Our main quantity of interest in this paper is the all-time maximum of S. More precisely, we wish to study the random variable

$$M \quad = \quad \sup_{n \geq 0} \ S_n.$$

Note that since we have assumed that $-\nu < 0$, it is immediate by the strong law of large numbers [9] that M is well-defined (meaning with probability one). In the present paper we will be concerned with studying $\mathbb{P}_\nu(M = 0)$ and $\mathbb{E}_\nu[M]$ as functions of the drift parameter $-\nu < 0$.

The main results which we obtain in this paper provide full series expansion for both $\mathbb{P}_\nu(M = 0)$ and $\mathbb{E}_\nu[M]$ as functions of the drift parameter $-\nu < 0$. One may consult Theorems 3.1 and 3.2 of Section 3 below for the precise form of our series expansions. Our results contribute to a long line of research [5, 8, 15, 16, 22, 29] aimed at obtaining series expansions for random walks. For a survey of such results one may consult Janssen and van Leeuwaarden [16]. The class of α-stable random variables includes the class of Gaussian random variables as a special case by setting $\alpha = 2$ and $\beta = 0$ and so our results are most closely related to those of Chang and Peres [8] and Janssen and van Leeuwaarden [15, 16], both of whom studied the Gaussian random walk. Specifically, in [8] and [16] a series expansion is provided in terms of $-\nu$ for $\mathbb{P}_\nu(M = 0)$ and in [15] series expansions are given in terms of $-\nu$ for $\mathbb{E}_\nu[M]$ and indeed all of the cumulants of M. Our series expansions are also valid for the Gaussian case of $\alpha = 2$ and $\beta = 0$ and similar to [8] and [15, 16] the coefficients of our expansions may be expressed in terms of the analytic continuation of the Riemann zeta function to points along the negative portion of the real axis. At the same time the expansions we obtain also reveal additional coefficients which were effectively set equal to zero in the series expansions of [8] and [15, 16]. Indeed, the series expansions of [8] and [15, 16] were expressed entirely in terms of odd or even powers of $-\nu < 0$. This was presumably due to the symmetry of the Gaussian distribution. The series expansions which we obtain in the present paper have non-zero coefficients for all integer powers of $-\nu < 0$, with the exception again of the symmetric case of $\beta = 0$.

The proofs of our main results in this paper rely as in Chang and Peres [8] and Janssen and van Leeuwaarden [15, 16] upon Spitzer's identities [31] for random walks in order to express $\mathbb{P}_\nu(M = 0)$ and $\mathbb{E}_\nu[M]$ as infinite sums involving either certain probabilities or expectations, respectively, associated with the partial sums of the random walk S. Due to the stability property of α-stable random variables [28], it is immediate that the partial sums expressed in Spitzer's identities [31] are α-stable random variables themselves. Unfortunately, although α-stable random variables do admit simple closed form expressions for their characteristic functions, it turns out that, with the exception of a few special cases, there are no known closed form expressions for their distribution functions. Therefore, in the present paper we rely upon Zolotarev's [35] integral representation for the distribution function of an α-stable random variable in order to continue the analysis after expressing either $\mathbb{P}_\nu(M = 0)$ or $\mathbb{E}_\nu[M]$ as an infinite sum using Spitzer's identity [31]. In fact, a significant portion of the paper is devoted to analyzing certain functions appearing in the integral representation of Zoltarev [35]. Once these functions have been properly analyzed, we then borrow techniques from complex analysis originally developed by Riemann [27] for analytically continuing the zeta function in order to finish the proof.

We now complete this section by noting that the all-time maximum of a random walk also plays an important role in more applied fields. In queueing theory [33], the limiting delay distribution for the $G/G/1$ queue may be expressed in terms of the all-time maximum of an associated random walk. This problem is discussed in further detail in Section 7. Moreover, applications may also be found in fields as diverse as financial engineering [6, 7], insurance mathematics [2], operations management [12, 18, 34], and sequential analysis [30].

2. Background on α-stable Random Variables

Several parameterizations exist for the characteristic function of an α-stable distribution and for the remainder of this paper we use the parameterization (B) given by Theorem C.3 of Zolotarev [35]. Specifically, let $S_\alpha(\beta,\gamma,1)$ denote a generic α-stable random variable which under the probability measure $\mathbb{P}$ has shape parameter $0<\alpha<2, \alpha\neq 1$, skewness parameter $-1\leq\beta\leq 1$, shift parameter $\gamma\in\mathbb{R}$ and scale parameter 1. Then, the characteristic function of $S_\alpha(\beta,\gamma,1)$ is given by

$$\ln\mathbb{E}[\exp(itS_\alpha(\beta,\gamma,1))] \quad = \quad it\gamma-|t|^\alpha(\exp(-i(\pi/2)\beta K(\alpha)\mathrm{sgn}\ t)), \quad t\in\mathbb{R}, \tag{1}$$

where $K(\alpha)=\alpha-1+\mathrm{sgn}(1-\alpha)$. The case of $\alpha=1$ may be treated separately. We note that (1) may be extended to the case of $\alpha=2$ and $\beta=0$ in which case we recover the characteristic function of a normal random variable with a mean of γ and a variance of two. One may consult Samorodnitsky and Taqqu [28] or Zolotarev [35] for in-depth discussions of α-stable random variables.

The shape parameter $0<\alpha<2$ of an α-stable random variable controls the rate of decay of the tails of the distribution and the skewness parameter $-1\leq\beta\leq 1$ specifies the degree of asymmetry in the distribution. If $\beta>0$, then the distribution is skewed to the right, while if $\beta<0$, then the distribution is skewed to the left, and the case of $\beta=0$ corresponds to the distribution being symmetric about the origin. The role of the the shift parameter γ is intuitively clear from (1). For the remainder of this paper, we will be concerned for the most part with the cases of $1<\alpha<2$ and $-1\leq\beta\leq 1$. In these cases, the distribution of $S_\alpha(\beta,\gamma,1)$ admits a density whose support is the entire real line. Moreover, for $1<\alpha<2$, we have the asymptotics as x approaches ∞ given by

$$\mathbb{P}(S_\alpha(\beta,\gamma,1)>x) \quad \sim \quad x^{-\alpha}C_\alpha\frac{1+\tilde{\beta}}{2}\cos((\pi/2)\beta K(\alpha)), \tag{2}$$

where $\tilde{\beta}=\tan((\pi/2)\beta K(\alpha))\cot((\pi/2)\alpha)$ and $C_\alpha=(1-\alpha)(\Gamma(2-\alpha)\cos((\pi/2)\alpha))^{-1}$. The asymptotic (2) also holds for $\mathbb{P}(S_\alpha(\beta,\gamma,1)<-x)$ as $x\to\infty$ with $\tilde{\beta}$ replaced by

$-\tilde{\beta}$. Note from (2) that if $1 < \alpha < 2$, then $S_\alpha(\beta, \gamma, 1)$ will only have finite moments up to order $p < \alpha$.

It is clear from (2) and the ensuing discussion that if $\beta = 1$, then the right tail of the distribution of $S_\alpha(\beta, \gamma, 1)$ is heavier than the left tail, while the situation is reversed if $\beta = -1$. We therefore refer to $S_\alpha(\beta, \gamma, 1)$ in the case of $\beta = 1$ as being "totally skewed to the right", while if $\beta = -1$ we say that $S_\alpha(\beta, \gamma, 1)$ is "totally skewed to the left". Specifically, if $\beta = 1$, then as $x \to \infty$,

$$\mathbb{P}(S_\alpha(1, \gamma, 1) < -x) \quad \sim \quad B_\alpha \left(\frac{x}{\alpha}\right)^{-\alpha/(2(\alpha-1))} \exp(-(\alpha-1)(x/(\alpha))^{\alpha/(\alpha-1)}), \tag{3}$$

where $B_\alpha = (2\pi\alpha(\alpha-1))^{-1/2}$. If $\beta = -1$, then (3) holds with $> x$ instead of $< -x$.

Now, for each $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, let $G_\alpha(x; \beta, \gamma, 1)$ denote the CDF of an $S_\alpha(\beta, \gamma, 1)$ random variable. For the remainder of the paper we suppress the dependence of $G_\alpha(x; \beta, \gamma, 1)$ on the parameters of the distribution and simply write $G(x)$. Our proofs in the present paper rely heavily upon Zolotarev's [35] integral representations for G. We therefore now provide the necessary results. For each $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, let

$$\theta \;=\; \theta(\beta) \;=\; \beta(\alpha - 2)/\alpha, \tag{4}$$

and define the function $U_{\alpha,\theta}$ by setting

$$U_{\alpha,\theta}(\varphi) \;=\; \left(\frac{\sin((\pi/2)\alpha(\varphi+\theta))}{\cos((\pi/2)\varphi)}\right)^{\alpha/(1-\alpha)} \frac{\cos((\pi/2)((\alpha-1)\varphi + \alpha\theta))}{\cos((\pi/2)\varphi)}, \tag{5}$$

for $-\theta < \varphi \leq 1$. It then follows by (2.2.27) of Zolotarev [35] that for $1 < \alpha < 2$, $-1 \leq \beta \leq 1$, $\gamma = 0$ and scale parameter 1, we may represent the CDF G of an $S_\alpha(\beta, 0, 1)$ random variable on the positive half-line via the integral expression

$$G(x) \;=\; 1 - \frac{1}{2}\int_{-\theta}^{1} \exp(-x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) d\varphi, \;\; x > 0. \tag{6}$$

Moreover, differentiating (6) with respect to x and passing the derivative underneath the integral sign, one obtains as in (2.2.18) of Zolotarev [35] the representation on the positive half-line for the pdf $g_\alpha(x; \beta, 0, 1)$ of an $S_\alpha(\beta, 0, 1)$ random variable given by

$$g(x) \;=\; \frac{1}{2}\frac{\alpha}{\alpha-1} x^{1/(\alpha-1)} \int_{-\theta}^{1} U_{\alpha,\theta}(\varphi) \exp(-x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) d\varphi, \;\; x > 0. \tag{7}$$

Note that in (7) we suppress the dependence of $g_\alpha(x; \beta, 0, 1)$ on the parameters of the distribution, which we continue to do for the remainder of the paper.

3. Main Results

As stated in Section 1, our main results in this paper provide series expansions for both $\mathbb{P}_\nu(M = 0)$ and $\mathbb{E}_\nu[M]$ as a function of the drift parameter $-\nu < 0$. We now state these results in order.

3.1. Series Expansion for $\mathbb{P}_\nu(M = 0)$

In this section, we provide a series expansion for the quantity $\mathbb{P}_\nu(M = 0)$ as a function of the drift parameter ν. In general, it turns out that $\mathbb{P}_\nu(M = 0) \to 0$ as $\nu \to 0$ and so it is convenient to first renormalize $\mathbb{P}_\nu(M = 0)$ in order to obtain a non-degenerate quantity in the limit as ν tends to zero. Specifically, for each $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, let

$$\vartheta \quad = \quad \frac{1}{2}\frac{\alpha}{\alpha - 1}(1 + \theta), \tag{8}$$

where we recall the definition of θ from (4) above. We then define the normalized version of $\mathbb{P}_\nu(M = 0)$ by setting

$$\pi(\nu) \quad = \quad \nu^{-\vartheta}\mathbb{P}_\nu(M = 0), \quad \nu > 0. \tag{9}$$

A straightforward calculation using the facts that $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, shows that $\vartheta \geq 1$ for all parameter combinations with strict equality holding only in the totally skewed to the right case of $\beta = 1$ for all stability parameters $1 < \alpha < 2$.

Proposition 3.1. *For each parameter combination $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, we have that $\pi(\nu) \to 1$ as $\nu \to 0$.*

Note that using (9) one has that Proposition 3.1 implies the rough asymptotic regarding $\mathbb{P}_\nu(M = 0)$ given by

$$\mathbb{P}_\nu(M = 0) \quad = \quad \nu^{\vartheta} + o(\nu^{\vartheta}), \quad \text{as} \quad \nu \downarrow 0. \tag{10}$$

In Theorem 3.1 below we provide the full series expansion leading to the rough asymptotic above.

Before stating Theorem 3.1 recall first from Spitzer [31] the fundamental identity

$$\ln \mathbb{P}_\nu(M = 0) \quad = \quad -\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}_\nu(S_n > 0). \tag{11}$$

Moreover, note that using the stability property of α-stable random variables (see Corollary 1.2.9 of [28]) it follows that for each $n \geq 1$, we have that under $\mathbb{P}_0$,

$$X_1 + ... + X_n \quad \stackrel{\mathrm{d}}{=} \quad n^{1/\alpha} X_1, \tag{12}$$

and so we may write

$$\mathbb{P}_\nu(S_n > 0) \quad = \quad \mathbb{P}_0(X_1 > n^{(\alpha-1)/\alpha}\nu). \tag{13}$$

Thus, using the definition of π in (9), we obtain by differentiation that

$$\frac{d}{d\nu} \ln \pi(\nu) \quad = \quad -\vartheta \frac{1}{\nu} + \sum_{n=1}^{\infty} \frac{1}{n^{1/\alpha}} g(n^{(\alpha-1)/\alpha}\nu), \tag{14}$$

where g denotes the density of an $\alpha-$stable random variable. Next, note that using Zolotarev's integral representation (7) above for the density of an α-stable distribution on the positive half-line, we obtain by the monotone convergence theorem [24] along with the fact that $U_{\alpha,\theta}(\varphi) \geq 0$ for $-\theta < \varphi \leq 1$ (see Proposition 5.1 below) that we may write

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^{1/\alpha}} g(n^{(\alpha-1)/\alpha}\nu) \qquad (15) \\ = \quad & \frac{1}{2} \frac{\alpha}{\alpha-1} \nu^{1/(\alpha-1)} \int_{-\theta}^{1} U_{\alpha,\theta}(\varphi) \frac{1}{\exp(\nu^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) - 1} d\varphi. \end{aligned}$$

In addition, by the results of Section 5 below, it follows that we may make the change-of-variables $x = U_{\alpha,\theta}(\varphi)$ in the integral above together with (14) and (15) to arrive at

$$\frac{d}{d\nu} \ln \pi(\nu) \quad = \quad -\mathcal{H}_1(\nu), \quad \nu > 0, \tag{16}$$

where the function $\mathcal{H}_1(\nu)$ is defined by

$$\mathcal{H}_1(\nu) \quad = \quad \vartheta \frac{1}{\nu} + \frac{1}{2} \frac{\alpha}{\alpha-1} \nu^{1/(\alpha-1)} \int_{U_{\alpha,\theta}(1)}^{\infty} \frac{x}{\exp(\nu^{\alpha/(\alpha-1)} x) - 1} (U_{\alpha,\theta}^{-1})'(x) dx, \tag{17}$$

for $\nu > 0$.

Proposition 3.2. *For each parameter combination $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, we have the series expansion*

$$\mathcal{H}_1(\nu) \quad = \quad \frac{1}{\pi} \frac{1}{\alpha} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k/\alpha)}{\Gamma(k)} \sin((\pi/2)k(1+\theta)) \zeta(k/\alpha - k + 1) \nu^{k-1}, \tag{18}$$

valid for all $0 < \nu < \alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$.

Proposition 3.2 together with Proposition 3.1 and (16) now imply Theorem 3.1.

Theorem 3.1. *For each parameter combination* $1 < \alpha < 2$ *and* $-1 \leq \beta \leq 1$*, we have the series expansion*

$$\begin{aligned} & \pi(\nu) \\ = \quad & \exp\left(\frac{1}{\pi}\frac{1}{\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k/\alpha)}{\Gamma(k+1)}\sin((\pi/2)k(1+\theta))\zeta(k/\alpha-k+1)\nu^{k}\right), \end{aligned} \tag{19}$$

valid for all $0 < \nu < \alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$.

In the coefficients of the series expansions above, the function ζ is the analytic continuation of the Riemann zeta function to the complex plane. Indeed, Propositions 3.1 and 3.2 and Theorem 3.1 also hold in the Gaussian case of $\alpha = 2$ and $\beta = 0$. Specifically, it may be shown in this case using the duplication formula for the Gamma function that the series expansion given by (19) above coincides with the series expansions obtained by Chang and Peres [8] and Janssen and van Leeuwaarden [17]. It is also interesting to note that the radius of convergence $\alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha} \to 1$ as $\alpha \to 1$. Hence, the series expansion (19) is valid for all $0 < \nu < 1$ regardless of the value of $1 < \alpha < 2$. We also note that using (9) above, it is immediate that Theorem 3.1 implies a full series expansion for $\mathbb{P}_\nu(M = 0)$ in a neighborhood of zero.

3.2. Series Expansion for $\mathbb{E}_\nu[M]$

In this section, we study the quantity $\mathbb{E}_\nu[M]$. By Theorems 5 and 6 of Kiefer and Wolfowitz [21] we have that $\mathbb{E}_\nu[M] < +\infty$ if and only if $E_\nu[(\max(X_1, 0))^2] < +\infty$. Hence, $\mathbb{E}_\nu[M] = +\infty$ for $\nu > 0$, the exception being the totally skewed to the left case of $\beta = -1$ for all stability parameters $1 < \alpha < 2$. We therefore focus our attention on this special case for the remainder of the section. For each $\nu > 0$ define the quantity

$$\Xi(\nu) \quad = \quad \mathbb{E}_\nu[M] - \frac{1}{\nu^{1/(\alpha-1)}}. \tag{20}$$

Our main result in this section, Theorem 3.1, provides a series expansion for Ξ as a function of ν. We begin with the following result which identifies the limit of Ξ as ν tends to zero.

Proposition 3.3. *For each parameter combination* $1 < \alpha < 2$ *and* $\beta = -1$*, we have* *that*

$$\lim_{\nu\to 0}\Xi(\nu) = -\frac{1}{\pi}\frac{1}{\alpha}\Gamma(-1/\alpha)\sin((\pi/2)(1+\theta))\zeta(1-1/\alpha).$$

Similar to our proof of Theorem 3.1, our proof of Theorem 3.2 also has as its starting point Spitzer's identity [31] for random walks. Specifically, recall that for each $\nu > 0$ we may write

$$\mathbb{E}_\nu[M] = \sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}_\nu[S_n^+]. \tag{21}$$

Next, note that using the stability property of α-stable random variables, it follows after an integration-by-parts that for each $n \geq 1$,

$$\mathbb{E}_\nu[S_n^+] = n^{1/\alpha}\int_{n^{(\alpha-1)/\alpha}\nu}^{\infty}(1-G(x))dx. \tag{22}$$

Similar to the developments in Section 3.1, we now obtain, upon substituting (22) into (21) together with the definition of Ξ in (20) and the Zolotarev [35] integral representation (6) of the CDF of an α-stable random variable, that we may write

$$\frac{d}{d\nu}\Xi(\nu) = \frac{1}{\alpha-1}\frac{1}{\nu^{\alpha/(\alpha-1)}} - \frac{1}{2}\int_{-\theta}^{1}\frac{1}{\exp(\nu^{\alpha/(\alpha-1)}U_{\alpha,\theta}(\varphi))-1}d\varphi.$$

Now, by the results of Section 5 below, it follows that we may make the change-of-variables $x = U_{\alpha,\theta}(\varphi)$ in the integral above in order to obtain

$$\frac{d}{d\nu}\mathbb{E}_\nu[M] = \mathcal{H}_2(\nu), \quad \nu > 0, \tag{23}$$

where the function $\mathcal{H}_2(\nu)$ is defined by

$$\mathcal{H}_2(\nu) = \frac{1}{\alpha-1}\frac{1}{\nu^{\alpha/(\alpha-1)}} + \frac{1}{2}\int_{U_{\alpha,\theta}(1)}^{\infty}\frac{1}{\exp(\nu^{\alpha/(\alpha-1)}x)-1}\left(U_{\alpha,\theta}^{-1}\right)'(x)dx, \tag{24}$$

for $\nu > 0$.

Proposition 3.4. *For each parameter combination* $1 < \alpha < 2$ *and* $\beta = -1$*, we have the series expansion*

$$\mathcal{H}_2(\nu) = \frac{(1+\theta)}{4} + \frac{1}{\pi}\frac{1}{\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k/\alpha)}{\Gamma(k+1)}\sin((\pi/2)k(1+\theta))\zeta(k/\alpha-k)\nu^k, \tag{25}$$

valid for all $0 < \nu < \alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$.

Proposition 3.4 together with Proposition 3.3 and (23) now imply Theorem 3.2.

Theorem 3.2. *For each parameter combination* $1 < \alpha < 2$ *and* $\beta = -1$, *we have the series expansion*

$$\begin{aligned} \Xi(\nu) &= -\frac{1}{\pi}\frac{1}{\alpha}\Gamma(-1/\alpha)\sin((\pi/2)(1+\theta))\zeta(1-1/\alpha) + \frac{(1+\theta)}{4}\nu \\ &\quad + \frac{1}{\pi}\frac{1}{\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k/\alpha)}{\Gamma(k+2)}\sin((\pi/2)k(1+\theta))\zeta(k/\alpha-k)\nu^{k+1}, \end{aligned}$$

valid for all $0 < \nu < \alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$.

Propositions 3.3 and 3.4 and Theorem 3.2 also hold in the Gaussian case of $\alpha = 2$ and $\beta = 0$. Specifically, in this case it may be shown that the series expansion given by Theorem 3.2 agrees with the results of Janssen and van Leeuwaarden [15, 16].

4. Rough Asymptotics

In this section, we provide the proofs of the rough asymptotics which are needed in order to prove our main results of the paper.

4.1. Limiting Behavior of $\mathbb{P}_\nu(M = 0)$

Proof of Proposition 3.1. First note that from the definition of π in (9) and Spitzer's identity (11), we have that for $\nu > 0$,

$$\ln \pi(\nu) = -\vartheta \ln(\nu) - \sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}_\nu(S_n > 0). \tag{26}$$

Next, by stability property (12) and (13) of α-stable random variables, the Zolotarev integral representation (6) for the CDF of an α-stable random variable with $\nu = 0$, the positivity of $U_{\alpha,\theta}(\varphi)$ for $-\theta < \varphi < 1$ guaranteed by Proposition 5.1, the Monotone Convergence Theorem [24] and the series definition of the natural logarithm it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}_\nu(S_n > 0) = -\frac{1}{2}\int_{-\theta}^{1}\ln(1-\exp(-\nu^{\alpha/(\alpha-1)}U_{\alpha,\theta}(\varphi)))d\varphi. \tag{27}$$

Next, note that since $1 < \alpha < 2$, we have that $\alpha/(\alpha-1) > 0$, and so, again using the positivity of $U_{\alpha,\theta}(\varphi)$ for $-\theta < \varphi < 1$ guaranteed by Proposition 5.1 of Section 5.1, we have that for each $-\theta < \varphi < 1$,

$$\ln(1 - \exp(-\nu^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi))) - \ln(\nu^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi))) \quad \to \quad 0 \quad \text{as} \quad \nu \to 0. \qquad (28)$$

Moreover, it is straightforward to show that the quantity on the lefthand side above is, for each $-\theta < \varphi < 1$, monotonically decreasing in $\nu > 0$. Hence, since by (30) below we have that for each $\nu > 0$,

$$\int_{-\theta}^{1} |\ln(\nu^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)))| d\varphi \quad < \quad \infty,$$

it follows by (27), (28) and the Dominated Convergence Theorem [24] that we may write

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}_\nu(S_n > 0) \quad = \quad -\frac{1}{2} \frac{\alpha}{\alpha - 1} (1 + \theta) \ln(\nu) - \frac{1}{2} \int_{-\theta}^{1} \ln(U_{\alpha,\theta}(\varphi)) d\varphi + o(1).$$

Thus, using the definition of ϑ given in (8) together with the identity (26), the above equality now implies that

$$\ln \pi(\nu) \quad = \quad \frac{1}{2} \int_{-\theta}^{1} \ln(U_{\alpha,\theta}(\varphi))) d\varphi + o(1), \quad \nu > 0.$$

Hence, in order to complete the proof it suffices to show that

$$\int_{-\theta}^{1} \ln(U_{\alpha,\theta}(\varphi))) d\varphi \quad = \quad 0. \qquad (29)$$

However note that using the definition of $U_{\alpha,\theta}(\varphi)$ in (5) it follows that

$$\begin{aligned} \ln(U_{\alpha,\theta}(\varphi)) \quad = \quad & \frac{\alpha}{1-\alpha} \ln(\sin((\pi/2)\alpha(\varphi + \theta))) \\ & + \ln(\cos((\pi/2)((\alpha - 1)\varphi + \alpha\theta))) \\ & + \frac{1}{\alpha - 1} \ln(\cos((\pi/2)\varphi)). \end{aligned} \qquad (30)$$

By an appropriate change-of-variables, it is then straightforward to prove the desired result (29).

4.2. Limiting Behavior of $\mathbb{E}_\nu[M]$

Proof of Proposition 3.3. First recall that by (21) and (22) above we have that in the case of $1 < \alpha < 2$ and $\beta = -1$ we may write

$$\mathbb{E}_\nu[M] \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^{(\alpha-1)/\alpha}} \int_{n^{(\alpha-1)/\alpha}\nu}^{\infty} (1 - G(x)) dx, \quad \nu > 0,$$

where G denotes the CDF of an α-stable random variable with $1 < \alpha < 2$ and $\beta = -1$. Next, note that using the Zolotarev integral representation for the CDF of an α-stable random variable given by (6), it follows after a change-of-variables that from the definition of θ from (4) of Section 2 above, the positivity of $U_{\alpha,\theta}(\varphi)$ for $-\theta < \varphi < 1$ guaranteed by Proposition 5.1 of Section 5 below, and the Monotone Convergence Theorem [24], we have that for $\nu > 0$ we may write

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^{(\alpha-1)/\alpha}} \int_{n^{(\alpha-1)/\alpha}\nu}^{\infty} (1 - G(x))dx \\ = \; & \frac{1}{2} \int_{-\theta}^{1} \int_{\nu}^{\infty} \frac{1}{\exp(x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) - 1} dx d\varphi. \end{aligned} \tag{31}$$

We now proceed to analyze the inner integral on the righthand side of (31) above. First note that making the change-of-variables $v = x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)$ in the inner integral in (31), we obtain that for $\nu > 0$,

$$\begin{aligned} & \int_{\nu}^{\infty} \frac{1}{\exp(x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) - 1} dx \\ = \; & \frac{\alpha - 1}{\alpha} \frac{1}{(U_{\alpha,\theta}(\varphi))^{(\alpha-1)/\alpha}} \int_{\nu^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)}^{\infty} \frac{1}{\exp(v) - 1} \frac{1}{v^{1/\alpha}} dv. \end{aligned} \tag{32}$$

Next, note that making use of the generating function of the Bernoulli numbers [1], together with their relationship to the Riemann zeta function [1], it follows that for each $0 < x < 2\pi$ we may write

$$\frac{1}{\exp(x) - 1} \frac{1}{x^{1/\alpha}} = \frac{1}{x^{1/\alpha+1}} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\zeta(-k)}{k!} x^{k-1/\alpha}.$$

Now integrating the above and using the contour integral representation [27] of the Riemann zeta function to evaluate the resulting constant of integration, it follows that for $0 < x < 2\pi$,

$$\begin{aligned} & \int_{x}^{\infty} \frac{1}{\exp(v) - 1} \frac{1}{v^{1/\alpha}} dv - \alpha \frac{1}{x^{1/\alpha}} \\ = \; & \Gamma\left(1 - \frac{1}{\alpha}\right) \zeta\left(1 - \frac{1}{\alpha}\right) + \sum_{k=0}^{\infty} (-1)^{k} \frac{\zeta(-k)}{k!} \frac{1}{k + 1 - \frac{1}{\alpha}} x^{k+1-1/\alpha}. \end{aligned} \tag{33}$$

Thus, substituting (33) into (32) we obtain that for $0 < \nu < (2\pi/U_{\alpha,\theta}(\varphi))$ we may write

$$\begin{aligned} & \int_{\nu}^{\infty} \frac{1}{\exp(x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi)) - 1} dx - (\alpha - 1) \frac{1}{U_{\alpha,\theta}(\varphi)} \frac{1}{\nu^{1/(\alpha-1)}} \\ = \; & \frac{\alpha - 1}{\alpha} \Gamma\left(1 - \frac{1}{\alpha}\right) \zeta\left(1 - \frac{1}{\alpha}\right) \frac{1}{(U_{\alpha,\theta}(\varphi))^{(\alpha-1)/\alpha}} + \mathcal{O}(\nu). \end{aligned} \tag{34}$$

Now substituting (34) into (31) and using the fact from Proposition 5.1 of Section 5.1 below that in the case of $\beta = -1$ we have that $U_{\alpha,\theta}(\varphi) > (\alpha-1)\alpha^{\alpha/(1-\alpha)}$ for $-\theta < \varphi < 1$, it follows by the Dominated Convergence Theorem [24] that

$$\begin{aligned} &\lim_{\nu\to 0} \mathbb{E}_\nu[M] - \frac{(\alpha-1)}{2}\int_{-\theta}^{1}\frac{1}{U_{\alpha,\theta}(\varphi)}d\varphi\frac{1}{\nu^{1/(\alpha-1)}} \quad (35)\\ &= \frac{1}{2}\frac{(\alpha-1)}{\alpha}\Gamma\left(1-\frac{1}{\alpha}\right)\zeta\left(1-\frac{1}{\alpha}\right)\int_{-\theta}^{1}\frac{1}{(U_{\alpha,\theta}(\varphi))^{(\alpha-1)/\alpha}}d\varphi. \end{aligned}$$

In order to complete the proof, it now remains to evaluate each of the integrals appearing in the expression (35) above. We begin with the integral appearing on the lefthand side of (35). First note that using the Zolotarev integral representation of the CDF of an α-stable random variable with $\nu = 0$ given by (6) of Section 2 with x replaced by $x^{(\alpha-1)/\alpha}$, and then integrating the resulting expression from 0 to ∞, we have by the Monotone Convergence Theorem [24] that may write

$$\int_{-\theta}^{1}\frac{1}{U_{\alpha,\theta}(\varphi)}d\varphi \;=\; 2\int_0^\infty(1-G(x^{(\alpha-1)/\alpha}))dx \;=\; \int_0^\infty x^{\alpha/(\alpha-1)}g(x)dx, \quad (36)$$

where the equality on the righthand side above follows after performing an integration-by-parts and an appropriate change-of-variables. We now evaluate the integral on the righthand side of (36). For $a \in \mathbb{C}$ with Re $a > 0$ recall the series definition [13] of the Mittag-Leffler function $E_a : \mathbb{C} \mapsto \mathbb{C}$ given by

$$E_a(s) \;=\; \sum_{k=0}^{\infty}\frac{s^k}{\Gamma(ak+1)}, \quad s\in\mathbb{C}.$$

Then, by Theorem 2.10.3 of Zolotarev [35] together with the duality relationship (2.3.3) of Zolotarev [35], we obtain that for $p > 0$,

$$E_{p/\alpha}(s) \;=\; \int_0^\infty E_p(sx^p)x^{-1/\alpha-1}g(x^{-1/\alpha})dx \;=\; \alpha\int_0^\infty E_p(sx^p)g(x)dx \quad (37)$$

for $s \in \mathbb{C}$, where the equality on the righthand side above follows by an appropriate change-of-variables. Taking the derivative with respect to s on both sides of (37) and setting $s = 0$, we now obtain that

$$\int_0^\infty x^p g(x)dx \;=\; \frac{\Gamma(p)}{\Gamma(p/\alpha)}, \quad p>0, \quad (38)$$

where here we have used the fact that $E_p^{'}(0) = (p\Gamma(p))^{-1}$. Now setting $p = \alpha/(\alpha-1)$ in (38) and using (36) and the fact that $\Gamma(1+z) = z\Gamma(z)$ $z \in \mathbb{C}$, it follows that

$$\frac{(\alpha-1)}{2}\int_{-\theta}^{1}\frac{1}{U_{\alpha,\theta}(\varphi)}d\varphi \;=\; 1. \quad (39)$$

Next, we analyze the integral on the righthand side of (35). In a similar manner to the above, using the Zolotarev integral representation of the CDF of an α-stable random variable with $\nu = 0$ given by (6) of Section 2, along with the integral representation of the gamma function, we may write

$$\int_{-\theta}^{1} \frac{1}{(U_{\alpha,\theta}(\varphi))^{(\alpha-1)/\alpha}} d\varphi = \left(\frac{1}{2}\frac{\alpha-1}{\alpha}\Gamma\left(\frac{\alpha-1}{\alpha}\right)\right)^{-1} \int_{0}^{\infty} (1-G(x))dx.$$

Hence, performing integration-by-parts on the integral on the righthand side above, then setting $p = 1$ in (38), using Euler's reflection formula [23], and noting that in the case that $\beta = -1$ we have that $\alpha(1+\theta) = 2$, it follows that we may write

$$\begin{aligned} &\left(\frac{1}{2}\frac{\alpha-1}{\alpha}\Gamma\left(\frac{\alpha-1}{\alpha}\right)\zeta\left(\frac{\alpha-1}{\alpha}\right)\right)\int_{-\theta}^{1} \frac{1}{(U_{\alpha,\theta}(\varphi))^{(\alpha-1)/\alpha}} d\varphi \\ = &-\frac{1}{\pi}\frac{1}{\alpha}\Gamma(-1/\alpha)\sin((\pi/2)(1+\theta))\zeta(1-1/\alpha), \end{aligned} \tag{40}$$

which completes our treatment of the righthand side of (35). Now placing (35), (39) and (40) together we obtain the desired result.

5. Analysis of $U_{\alpha,\theta}$ and $U_{\alpha,\theta}^{-1}$

In Section 6, an important role is played in the proofs of Propositions 3.2 and 3.4 by $U_{\alpha,\theta}^{-1}$, the inverse of the function $U_{\alpha,\theta}$ appearing in (6). We therefore devote this entire section to analyzing $U_{\alpha,\theta}$ and its inverse $U_{\alpha,\theta}^{-1}$.

5.1. Analysis of $U_{\alpha,\theta}$

In this section, we begin with the following result regarding the behavior of $U_{\alpha,\theta}$ at the endpoints $-\theta$ and 1.

Proposition 5.1. *For each $1 < \alpha < 2$ and $-1 \le \beta \le 1$, we have that $U_{\alpha,\theta}(-\theta) = +\infty$. Moreover,*

1. *If $1 < \alpha < 2$ and $\beta = -1$, then $U_{\alpha,\theta}(1) = (\alpha-1)\alpha^{\alpha/(1-\alpha)}$.*

2. *If $1 < \alpha < 2$ and $-1 < \beta \le 1$, then $U_{\alpha,\theta}(1) = 0$.*

Proof. We first show that in both cases $U_{\alpha,\theta}(\varphi) \to \infty$ as $\varphi \to -\theta$. Note that since $1 < \alpha < 2$ and $-1 \le \beta \le 1$, it is straightforward to show that $-1 < \theta < 1$. Thus, from

(5) we obtain that

$$\lim_{\varphi \to -\theta} U_{\alpha,\theta}(\varphi) \;=\; \cos((\pi/2)\theta)^{\alpha/(\alpha-1)} \lim_{\varphi \to -\theta} \sin((\pi/2)\alpha(\varphi+\theta))^{\alpha/(1-\alpha)} \;=\; +\infty,$$

where the final equality follows since by the assumption $1 < \alpha < 2$, we have that $\alpha/(1-\alpha) < 0$. We next prove Items 1 and 2.

First suppose that $1 < \alpha < 2$ and $\beta = -1$ and note that this implies that $\theta = (2/\alpha - 1)$ and $\alpha(1+\theta) = 2$. Then, using the asymptotic $\sin(x)/x \to 1$ as $x \to 0$, it is straightforward to obtain from (5) that $U_{\alpha,\theta}(\varphi) \to (\alpha-1)\alpha^{\alpha/(1-\alpha)}$ as $\varphi \to 1$, from which Item 1 follows.

Next suppose that $1 < \alpha < 2$ and $-1 < \beta \leq 1$. It is then straightforward to deduce that $0 < \alpha(1+\theta) < 2$ and hence

$$\lim_{\varphi \to 1} U_{\alpha,\theta}(\varphi) = \sin((\pi/2)\alpha(1+\theta))^{\alpha/(1-\alpha)} \lim_{\varphi \to 1} \cos((\pi/2)\varphi)^{1/(\alpha-1)} \;=\; 0,$$

where the final equality follows since by the assumption $1 < \alpha < 2$, we have that $1/(\alpha-1) > 1$. Thus, Item 2 is proven.

Next we prove that $U_{\alpha,\theta}$ is decreasing on $(-\theta, 1)$. This result may be found elsewhere in the literature (see, for instance, Nolan [25]), however, we provide a separate proof for the sake of completeness.

Proposition 5.2. *For each* $1 < \alpha < 2$ *and* $-1 \leq \beta \leq 1$*, the function* $U_{\alpha,\theta}$ *is decreasing on the interval* $(-\theta, 1)$.

Proof. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be the function defined by $f(z) = iz - z^\alpha \exp(i(\pi/2)\beta(\alpha-2))$ for $z \in \mathbb{C}$. Then, for each $z = x+iy$, write $f(z) = u(x,y)+iv(x,y)$, where $u, v : \mathbb{R}^2 \mapsto \mathbb{R}$. Now let Γ_0 be the level curve of v of level 0 in the region of the x-y plane given by $\{(x,y) \in \mathbb{R}^2 : -\theta(\pi/2) < \arg z < \pi/2\}$, where again we write $z = x+iy$ in order to associate points in $\mathbb{R}^2$ with points in $\mathbb{C}$. It now follows that we may parameterize Γ_0 by writing

$$\Gamma_0(\Phi) \;=\; \left(\frac{\sin((\pi/2)\beta(\alpha-2)+\Phi\alpha)}{\sin((\pi/2)+\Phi)}\right)^{1/(1-\alpha)} e^{i\Phi}, \quad -\theta(\pi/2) < \Phi < (\pi/2). \tag{41}$$

Now using the definition of $f : \mathbb{C} \mapsto \mathbb{C}$ along with (41) it straightforward to see after some algebra that $u(\Gamma_0(\Phi)) = -U_{\alpha,\theta}((2/\pi)\Phi)$ for $-\theta(\pi/2) < \Phi < (\pi/2)$. Now recall [23] that each level curve of v in the x-y plane must follow along an integral curve of the vector field formed by the gradient of u. Hence, in order to complete the proof it suffices by Proposition 5.1 to verify that $f^{'}(z) \neq 0$ along the level curve Γ_0. However, using the fact that $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, a straightforward calculation using the definition of $f : \mathbb{C} \mapsto \mathbb{C}$ verifies that $f^{'}(z) \neq 0$ for Re $z > 0$.

5.2. Analysis of the function $U_{\alpha,\theta}^{-1}$

Note by Propositions 5.1 and 5.2 of Section 5.1 above that for each parameter combination $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$ we have that $U_{\alpha,\theta}$ is a continuous, decreasing function on $(-\theta, 1]$, with $U_{\alpha,\theta}(-\theta) = +\infty$. Thus, we may uniquely define the inverse function of $U_{\alpha,\theta}$, which we denote by $U_{\alpha,\theta}^{-1} : [U_{\alpha,\theta}(1), +\infty) \mapsto (-\theta, 1]$, and we define its associated Laplace transform

$$F(s) \;=\; \int_{U_{\alpha,\theta}(1)}^{\infty} e^{-sx} U_{\alpha,\theta}^{-1}(x)dx, \quad \text{Re } s > 0.$$

Next recall by Theorem 2.4.2 of Zolotarev [35] that the CDF G of an α-stable random variable has a unique analytic continuation to the entire complex plane. Let us therefore continue to denote this extension by $G : \mathbb{C} \mapsto \mathbb{C}$. Also by Theorem 2.4.2 of [35], the function G has the series expansion

$$1 - G(s) \;=\; \frac{1}{2}(1+\theta) + \frac{1}{\pi}\frac{1}{\alpha}\sum_{n=1}^{\infty}(-1)^n \frac{\Gamma(\frac{n}{\alpha})}{\Gamma(n+1)} \sin(n(\pi/2)(1+\theta))s^n, \qquad (42)$$

for $s \in \mathbb{C}$. We note as well that by Theorem 2.4.3 of [35], G is of order $\alpha/(\alpha-1)$ and type $(\alpha-1)\alpha^{\alpha/(1-\alpha)}$. These facts will be used in the remainder of this section.

Proposition 5.3. *For each $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$, we have that*

$$F(s) \;=\; \frac{1}{s}\left(e^{-sU_{\alpha,\theta}(1)} - 2(1 - G(s^{(\alpha-1)/\alpha}))\right), \quad \textit{Re } s \;>\; 0. \qquad (43)$$

Proof. Let $1 < \alpha < 2$ and $-1 \leq \beta \leq 1$ and recall from the Zolotarev integral representation (6) of Section 2 that for real-valued $x > 0$ we may write

$$1 - G(x) \;=\; \frac{1}{2}\int_{-\theta}^{1} \exp(-x^{\alpha/(\alpha-1)} U_{\alpha,\theta}(\varphi))d\varphi, \qquad (44)$$

where we recall from (4) of Section 2 that $\theta(\beta) = \beta(\alpha-2)/\alpha$. Now perform the change-of-variables $u = U_{\alpha,\theta}(\varphi)$ for $-\theta < \varphi \leq 1$, and note that by Propositions 5.1 and 5.2 we have that we may write $\varphi = U_{\alpha,\theta}^{-1}(u)$ for $U_{\alpha,\theta}(1) \leq u < +\infty$. It therefore now follows from (44) and Proposition 5.1 that for real-valued $x > 0$ we may write

$$1 - G(x) \quad = \quad -\frac{1}{2}\int_{U_{\alpha,\theta}(1)}^{\infty} \exp\left(-x^{\alpha/(\alpha-1)}u\right)\left(U_{\alpha,\theta}^{-1}\right)'(u)\,du.$$

Now note that the above equality may be rewritten as

$$\int_{U_{\alpha,\theta}(1)}^{\infty} \exp\left(-xu\right)\left(U_{\alpha,\theta}^{-1}\right)'(u)\,du \quad = \quad -2(1 - G(x^{(\alpha-1)/\alpha})). \tag{45}$$

Then, integrating-by-parts the lefthand side above and using some simple algebra, we arrive at

$$F(x) \quad = \quad \frac{1}{x}\left(e^{-xU_{\alpha,\theta}(1)} - 2(1 - G(x^{(\alpha-1)/\alpha}))\right),$$

for real-valued $x > 0$. The result (43) now holds by the uniqueness [23] of analytic continuation.

We now use Proposition 5.3 to provide a characterization of $\left(U_{\alpha,\theta}^{-1}\right)'$ in terms of the CDF G.

Proposition 5.4. *For each parameter combination* $1 < \alpha < 2$ *and* $-1 \leq \beta \leq 1$*, we have that*

$$\left(U_{\alpha,\theta}^{-1}\right)'(x) \quad = \quad -\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{ix\xi}(1 - G((i\xi)^{\alpha/(\alpha-1)}))d\xi, \quad x > U_{\alpha,\theta}(1). \tag{46}$$

Proof. First note that by Propositions 5.1 and 5.2 we have that $\left(U_{\alpha,\theta}^{-1}\right)'$ is an integrable function on $(U_{\alpha,\theta}(1), +\infty)$. Hence, we may define the Fourier transform

$$f(\xi) \quad = \quad \int_{-\infty}^{+\infty} e^{-i\xi u}1\{u > U_{\alpha,\theta}(1)\}\left(U_{\alpha,\theta}^{-1}\right)'(u)du, \quad \xi \in \mathbb{R}.$$

Now note that taking limits on both sides of (45) in the proof of Proposition 5.3, it follows by the Dominated Convergence Theorem [24] and the continuity of the function G that

$$f(\xi) \quad = \quad -2(1 - G((i\xi)^{\alpha/(\alpha-1)})), \quad \xi \in \mathbb{R}.$$

Hence, since by Proposition 5.2 we have that for each $x > U_{\alpha,\theta}(1)$, the function $\left(U_{\alpha,\theta}^{-1}\right)'$ is of bounded variation in an interval containing x and is continuous at x, the result (46) now follows by the Fourier inversion formula.

5.3. The function $\mathcal{G}$

Define the pair of functions $\mathcal{G}^+$ and $\mathcal{G}^-$ by setting

$$\mathcal{G}^+(s) \quad = \quad \int_0^{+\infty} e^{is\xi}(1 - G((i\xi)^{(\alpha-1)/\alpha}))d\xi, \quad \text{Im } s \geq 0, \ s \notin [U_{\alpha,\theta}(1), \infty), \tag{47}$$

and

$$\mathcal{G}^-(s) \quad = \quad -\int_{-\infty}^0 e^{is\xi}(1 - G((i\xi)^{(\alpha-1)/\alpha}))d\xi, \quad \text{Im } s \leq 0, \ s \notin [U_{\alpha,\theta}(1), \infty). \tag{48}$$

In Proposition 5.5 below, we show that both of these functions are well defined on their domain of definition, and, moreover, that they agree for $s \in (-\infty, U_{\alpha,\theta}(1))$. In preparation for this result, let

$$\mathcal{G}(s) \quad = \quad \begin{cases} \mathcal{G}^+(s), & \text{Im } s \geq 0, \ s \notin [U_{\alpha,\theta}(1), \infty), \\ \mathcal{G}^-(s), & \text{Im } s \leq 0, \ s \notin [U_{\alpha,\theta}(1), \infty). \end{cases} \tag{49}$$

Proposition 5.5. *For each $s \in \mathbb{C} \setminus [U_{\alpha,\theta}(1), \infty)$, we have that*

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} i \int_{-\theta}^1 \frac{1}{s - U_{\alpha,\theta}(\varphi)} d\varphi. \tag{50}$$

Proof. First note that by the Zolotarev integral representation (6) of Section 2 and the uniqueness of analytic continuation [23], it follows that for each $\xi \in \mathbb{R}$ we may write

$$1 - G((i\xi)^{(\alpha-1)/\alpha}) \quad = \quad \frac{1}{2} \int_{-\theta}^1 \exp(-i\xi U_{\alpha,\theta}(\varphi)) d\varphi.$$

Now let $s \in \mathbb{C}$ be such that Im $s \geq 0$ and $s \notin [U_{\alpha,\theta}(1), \infty)$. It then follows by (47) and the definition of $\mathcal{G}$ above that we may write

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} \lim_{R \to \infty} \int_0^R \int_{-\theta}^1 \exp(-\xi i(U_{\alpha,\theta}(\varphi) - s)) d\varphi d\xi.$$

Now interchanging the order of integration in the above and performing the inner integration, we obtain that

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} i \int_{-\theta}^1 \frac{1}{s - U_{\alpha,\theta}(\varphi)} d\varphi - \frac{1}{2} i \lim_{R \to \infty} \int_{-\theta}^1 \frac{\exp(-Ri(U_{\alpha,\theta}(\varphi) - s))}{s - U_{\alpha,\theta}(\varphi)} d\varphi. \tag{51}$$

Next note that since Im $s \geq 0$ and $s \notin [U_{\alpha,\theta}(1), \infty)$, it follows by Propositions 5.1 and 5.2 that $s - U_{\alpha,\theta}(\varphi) \neq 0$ for $-\theta < \varphi \leq 1$. Hence, using the Riemann-Lebesgue lemma [4], we obtain that

$$\lim_{R \to \infty} \int_{-\theta}^1 \frac{\exp(-Ri(U_{\alpha,\theta}(\varphi) - s))}{s - U_{\alpha,\theta}(\varphi)} d\varphi \quad = \quad 0. \tag{52}$$

It now follows by (51) and (52) that

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} i \int_{-\theta}^{1} \frac{1}{s - U_{\alpha,\theta}(\varphi)} d\varphi, \quad \text{Im } s \geq 0, \ s \notin [U_{\alpha,\theta}(1), \infty).$$

In a similar manner, it may be shown that (50) holds for Im $s \leq 0, \ s \notin [U_{\alpha,\theta}(1), \infty)$.

Now note that by Propositions 5.1 and 5.2, as well as (50) of Proposition 5.5, we have that the function $\mathcal{G}$ is analytic in $\mathbb{C} \setminus [U_{\alpha,\theta}(1), \infty)$. In the following result, we provide a series expansion for $\mathcal{G}$ in a suitable region in the complex plane. We have the following.

Proposition 5.6. *For each $s \in \mathbb{C} \setminus [U_{\alpha,\theta}(1), \infty)$ with $|s| > (\alpha - 1)\alpha^{\alpha/(1-\alpha)}$, we have the series expansion*

$$\begin{aligned} \frac{1}{i}\mathcal{G}(s) \quad &= \quad \frac{1}{2}(1+\theta)\frac{1}{s} \qquad (53) \\ &+ \frac{1}{\pi}\frac{1}{\alpha}\frac{\alpha-1}{\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\frac{n}{\alpha})\Gamma(n(\frac{\alpha-1}{\alpha}))}{\Gamma(n)}\sin(n(\pi/2)(1+\theta))(-s)^{n(1-\alpha)/\alpha-1}. \end{aligned}$$

Proof. First note that letting $s = iz$ for $z > 0$ in (47), we obtain by the definition of $\mathcal{G}$ above that

$$\mathcal{G}(iz) \quad = \quad \int_0^{+\infty} e^{-z\xi}(1 - G((i\xi)^{(\alpha-1)/\alpha}))d\xi, \quad z > 0. \qquad (54)$$

Next, recall by (42) that the function G has the series expansion

$$1 - G(s) \quad = \quad \frac{1}{2}(1+\theta) + \frac{1}{\pi}\frac{1}{\alpha}\sum_{n=1}^{\infty}(-1)^n \frac{\Gamma(\frac{n}{\alpha})}{\Gamma(n+1)}\sin(n(\pi/2)(1+\theta))s^n, \quad s \in \mathbb{C},$$

and by Theorem 2.4.3 of [35] is of order $\alpha/(\alpha-1)$ and type $(\alpha-1)\alpha^{\alpha/(1-\alpha)}$. Hence, for $z > (\alpha-1)\alpha^{\alpha/(1-\alpha)}$, we obtain from (42), (54) and the Dominated Convergence Theorem [24] along with some simple algebra that

$$\begin{aligned} \mathcal{G}(iz) \quad &= \quad \frac{1}{2}(1+\theta)\frac{1}{z} \qquad (55) \\ &+ \frac{1}{\pi}\frac{1}{\alpha}\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(\frac{n}{\alpha})}{\Gamma(n+1)}\sin(n(\pi/2)(1+\theta))i^{n(\alpha-1)/\alpha}\int_0^{+\infty}e^{-z\xi}\xi^{n(\alpha-1)/\alpha}d\xi. \end{aligned}$$

However, note that after a change-of-variables one may write

$$\int_0^{+\infty} e^{-z\xi}\xi^{n(\alpha-1)/\alpha}d\xi \quad = \quad n\frac{\alpha-1}{\alpha}\Gamma(n(\alpha-1)/\alpha)z^{n(1-\alpha)/\alpha-1}, \quad n \geq 1. \qquad (56)$$

Hence, upon substitution of (56) into (55) and some algebra we obtain that for $z > (\alpha-1)\alpha^{\alpha/(1-\alpha)}$,

$$\begin{aligned}\frac{1}{i}\mathcal{G}(iz) &= \frac{1}{2}(1+\theta)\frac{1}{zi}\\ &+\frac{1}{\pi}\frac{1}{\alpha}\frac{\alpha-1}{\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\frac{n}{\alpha})\Gamma(n(\frac{\alpha-1}{\alpha}))}{\Gamma(n)}\sin(n(\pi/2)(1+\theta))(-iz)^{n(1-\alpha)/\alpha-1}.\end{aligned}$$

The result now follows since by Propositions 5.1 and 5.2, and (50) of Proposition 5.5, the function $\mathcal{G}$ is analytic in $\mathbb{C}\setminus[U_{\alpha,\theta}(1),\infty)$, together with the uniqueness of analytic continuation [23].

Proposition 5.6 above provides a series expansion for the function $\mathcal{G}$ for large values of $s\in\mathbb{C}$. Next, in the following result, we characterize the behavior of $\mathcal{G}$ in the vicinity of $U_{\alpha,\theta}(1)$.

Proposition 5.7. *For each $1<\alpha<2$ and $-1\le\beta\le 1$, we have that for each $0<\eta<2\pi$, $\eta\neq\pi$,*

$$\lim_{r\to 0} r\mathcal{G}(U_{\alpha,\theta}(1)+re^{i\eta}) = 0 \tag{57}$$

and

$$\left|r\mathcal{G}(U_{\alpha,\theta}(1)+re^{i\eta})\right| \le (1+\theta),\quad r>0. \tag{58}$$

Proof. First recall by (47) above that for $s\in\mathbb{C}$ with $\mathrm{Im}\ s\ge 0$ and $s\notin[U_{\alpha,\theta}(1),\infty)$, we may write

$$\mathcal{G}(s) = \int_0^{+\infty} e^{is\xi}(1-G((i\xi)^{(\alpha-1)/\alpha}))d\xi. \tag{59}$$

Next, recall that by (6) of Section 5 and the uniqueness of analytic continuation [23], it follows after a change-of-variables that for $\xi\in\mathbb{R}$ we may write,

$$\begin{aligned}&1-G((i\xi)^{(\alpha-1)/\alpha})\\ =\ &-\frac{1}{2}\exp(-i\xi U_{\alpha,\theta}(1))\int_0^{\infty}\exp(-i\xi u)dU_{\alpha,\theta}^{-1}(U_{\alpha,\theta}(1)+u).\end{aligned} \tag{60}$$

Hence, substituting (60) into (59), we obtain that for $s\in\mathbb{C}$ with $\mathrm{Im}\ s\ge 0$ and $s\notin[U_{\alpha,\theta}(1),\infty)$, we may write

$$\mathcal{G}(s) = -\frac{1}{2}\int_0^{+\infty} e^{i\xi(s-U_{\alpha,\theta}(1))}\left(\int_0^{\infty}\exp(-i\xi u)dU_{\alpha,\theta}^{-1}(U_{\alpha,\theta}(1)+u)\right)d\xi. \tag{61}$$

Now let $s = U_{\alpha,\theta}(1) + re^{i\eta}$ with $r > 0$ and $0 < \eta < \pi$. It then follows after making the change-of-variables $v = \xi r$ in (61) that we obtain

$$\begin{aligned} & r\mathcal{G}(U_{\alpha,\theta}(1) + re^{i\eta}) \\ = & -\frac{1}{2}\int_0^{+\infty} e^{ve^{i((\pi/2)+\eta)}} \left(\int_0^\infty \exp(-i(v/r)u)dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u)\right) dv. \end{aligned} \tag{62}$$

Moreover, note that since $0 < \eta < \pi$ we have that Re $e^{i(\pi/2+\eta)} < 0$. Also note that using Proposition 5.1, it follows by the Riemann-Lebesgue lemma [4] that for each $v > 0$,

$$\lim_{r\to 0}\int_0^\infty \exp(-i(v/r)u)dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u) \quad = \quad 0.$$

Hence, since by the triangle inequality and Proposition 5.1 we have for each $v > 0$,

$$\left|\int_0^\infty \exp(-i(v/r)u)dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u)\right| \quad \leq \quad 1+\theta,$$

it follows by (62) and the Bounded Convergence Theorem [24] that (57) holds for $0 < \eta < \pi$. Using the definition of $\mathcal{G}^-$ in (48), it may also be shown in a similar manner to the above that (57) holds in the case of $\pi < \eta < 2\pi$. This completes the proof of the claim (57).

We now proceed to show that the claim (58) holds. Let $s = U_{\alpha,\theta}(1) + re^{i\eta}$ with $r > 0$ and $0 < \eta < \pi$. Then, using Propositions 5.1 and 5.2, it is straightforward to show that uniformly for $\{\xi \in \mathbb{C} : -\eta \leq \arg \xi \leq 0\}$, we have the convergence

$$e^{i\xi(s-U_{\alpha,\theta}(1))}\left(\int_0^\infty \exp(-i\xi u)dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u)\right) \quad \to \quad 0 \quad \text{as} \quad |\xi| \to \infty.$$

Moreover, again using Propositions 5.1 and 5.2, it follows that the inner integral in (61) converges uniformly for Im $\xi \leq 0$, and hence is analytic for Im $\xi < 0$. It therefore follows by (61) and using a standard argument involving Cauchy's integral theorem [23] that we may write

$$\begin{aligned} & \mathcal{G}(U_{\alpha,\theta}(1) + re^{i\eta}) \\ = & -\frac{e^{-i\eta}}{2}\int_0^{+\infty} e^{ir\xi}\left(\int_0^\infty \exp(\xi u e^{-i(\pi/2+\eta)})dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u)\right) d\xi. \end{aligned} \tag{63}$$

Now integrating-by-parts in (63) we obtain that

$$\begin{aligned} & e^{i(\pi/2+\eta)}2r\mathcal{G}(U_{\alpha,\theta}(1) + re^{i\eta}) \\ = & (1+\theta) + \int_0^{+\infty} e^{ir\xi}\frac{d}{d\xi}\left(\int_0^\infty \exp(\xi u e^{-i(\pi/2+\eta)})dU^{-1}_{\alpha,\theta}(U_{\alpha,\theta}(1)+u)\right) d\xi. \end{aligned} \tag{64}$$

Moreover, note that since $0 < \eta < \pi$ we have that Re $e^{i(\pi/2+\eta)} < 0$. Thus, passing the derivative underneath the integral sign in the above, a straightforward application of the triangle inequality together with Proposition 5.1 yields that

$$\left| \int_0^{+\infty} e^{ir\xi} \frac{d}{d\xi} \left(\int_0^{\infty} \exp(\xi u e^{-i(\pi/2+\eta)}) dU_{\alpha,\theta}^{-1}(U_{\alpha,\theta}(1)+u) \right) d\xi \right| \quad \leq \quad (1+\theta). \quad (65)$$

Combining (64) and (65), we now obtain (58) in the case of $0 < \eta < \pi$. Using the definition of $\mathcal{G}^-$ in (48), it may also be shown in a similar manner to the above that (58) holds in the case of $\pi < \eta < 2\pi$. Thus, claim (58) holds.

For our final result of this section, we have the following.

Proposition 5.8. *For each $x > U_{\alpha,\theta}(1)$, we have that*

$$\lim_{\delta \downarrow 0} \left(\mathcal{G}(x+i\delta) - \mathcal{G}(x-i\delta) \right) \quad = \quad -\pi (U_{\alpha,\theta^\star}^{-1})'(x).$$

Proof. First recall by Proposition 5.5 that for each $s \in \mathbb{C} \setminus [U_{\alpha,\theta}(1), \infty)$, we may write

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} i \int_{-\theta}^{1} \frac{1}{s - U_{\alpha,\theta}(\varphi)} d\varphi.$$

Hence, as in the proof of Proposition 5.3, we may use the change-of-variables $x = U_{\alpha,\theta}(\varphi)$ for $-\theta < \varphi \leq 1$, together with Proposition 5.1, in order to write

$$\mathcal{G}(s) \quad = \quad \frac{1}{2} i \int_{U_{\alpha,\theta}(1)}^{\infty} \frac{1}{x-s} (U_{\alpha,\theta}^{-1})'(x) dx, \quad s \in \mathbb{C} \setminus [U_{\alpha,\theta}(1), \infty). \quad (66)$$

The result now follows by (66) and the Sokhotski-Plemelj theorem (see Theorem I.1.6.1 of [10]).

6. Proofs of Series Expansions for $\mathcal{H}_1$ and $\mathcal{H}_2$

In Section 6.1 below we provide the proof of Proposition 3.2, and in Section 6.2 we provide the proof of Proposition 3.4.

6.1. Proof of Proposition 3.2

Proof of Proposition 3.2 . First recall from (49) of Section 5.3 above the definition of the function $\mathcal{G}$ and also recall from Proposition 5.8 above that for each $x > U_{\alpha,\theta}(1)$, we have that

$$\lim_{\delta\downarrow 0}\ (\mathcal{G}(x+i\delta)-\mathcal{G}(x-i\delta)) \quad = \quad -\pi (U_{\alpha,\theta}^{-1})'(x).$$

Our goal now is to integrate the function $\mathcal{G}$ around a properly chosen contour in the complex plane in order to recover the integral on the righthand side of (17) .

We begin with the totally skewed to the left case. Specifically, let $1<\alpha<2$ and $\beta=-1$. Next, note that since by assumption we have that $0<\nu<\alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$, it follows that we may fix a pair of numbers

$$a>(\alpha-1)\alpha^{\alpha/(1-\alpha)} \ \text{ and } \ (\alpha-1)\alpha^{\alpha/(1-\alpha)}<b<2\pi/\nu^{\alpha/(\alpha-1)}. \tag{67}$$

Next, for each $R>(\alpha-1)\alpha^{\alpha/(1-\alpha)}$ and

$$0<r<\min((\alpha-1)\alpha^{\alpha/(1-\alpha)},R-(\alpha-1)\alpha^{\alpha/(1-\alpha)},b) \ \text{ and } \ 0<\delta<\min(b,r) \tag{68}$$

consider the contour $C_{r,\delta,R}$ in the complex plane depicted in Figure 1 below. Specifically,

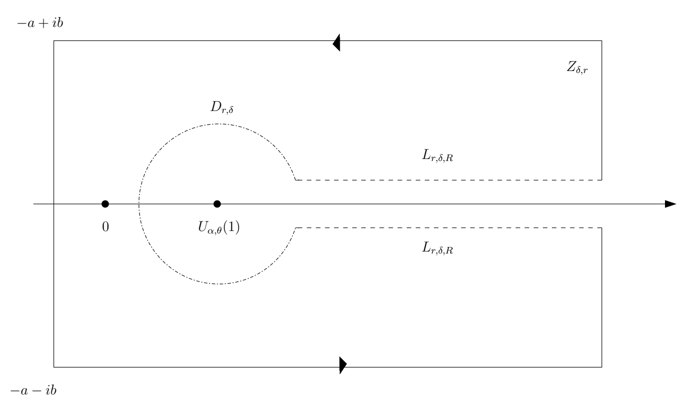

FIGURE 1: The contour $C_{r,\delta,R}$ in the case of $1<\alpha<2$ and $\beta=-1$.

we write $C_{r,\delta,R}=Z_{\delta,R}+L_{r,\delta,R}+D_{r,\delta}$, where

$$\begin{aligned} Z_{\delta,R} \quad = \quad & \{R+iy:\delta\le y\le b\}+\{x+ib:R\ge x\ge -a\} \\ & +\{-a+iy:b\ge y\ge -b\} \\ & +\{x-ib:-a\le x\le R\}+\{R+iy:-b\le y\le -\delta\}, \end{aligned} \tag{69}$$

and

$$L_{r,\delta,R} = \{x+i\delta : (\alpha-1)\alpha^{\alpha/(1-\alpha)} + (r^2-\delta^2)^{1/2} \le x \le R\}$$
$$+\{x-i\delta : R \ge x \ge (\alpha-1)\alpha^{\alpha/(1-\alpha)} + (r^2-\delta^2)^{1/2}\},$$

and

$$D_{r,\delta} = \{(\alpha-1)\alpha^{\alpha/(1-\alpha)} + re^{i\theta} : \arcsin(\delta/r) \le \theta \le 2\pi - \arcsin(\delta/r)\}. \quad (70)$$

Now note that by Proposition 5.5 and since by assumption $(\alpha-1)\alpha^{\alpha/(1-\alpha)} < b < 2\pi/\nu^{\alpha/(\alpha-1)}$, it is clear that the function

$$\frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s), \quad s \in \mathbb{C} \setminus (U_{\alpha,\theta}(1), +\infty), \; s \ne 2k\pi/\nu^{\alpha/(\alpha-1)}, \; k \in \mathbb{Z},$$

is analytic everywhere inside the closed contour $C_{r,\delta,R}$. Hence, using Cauchy's integral theorem [23] and the definition of $C_{r,\delta,R}$ given above, it follows that we may write

$$\oint_{L_{r,\delta,R}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds = -\oint_{Z_{\delta,R}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds \quad (71)$$
$$-\oint_{D_{r,\delta}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds.$$

We now take limits on both sides of the above, first as $\delta \to 0$, then as $r \to 0$, and finally as $R \to \infty$.

First note that by Proposition 5.8 and the Dominated Convergence Theorem [24], it is clear that

$$\lim_{R\to\infty}\lim_{r\to 0}\lim_{\delta\to 0} \oint_{L_{r,\delta,R}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds \quad (72)$$
$$= -\pi \int_{U_{\alpha,\theta}(1)}^{\infty} \frac{x}{\exp\left(\nu^{\alpha/(\alpha-1)}x\right)-1}\left(U_{\alpha,\theta}^{-1}\right)'(x)dx.$$

Next, note that by Proposition 5.6 we have that uniformly on its domain of definition, $|\mathcal{G}(s)| \to 0$ as $|s| \to \infty$, which implies that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\left(\int_{R+\delta i}^{R+bi} + \int_{R-bi}^{R-\delta i}\right) \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds = 0.$$

Hence, using the definition of the contour $Z_{\delta,R}$ in (69) we obtain that

$$\lim_{R\to\infty}\lim_{\delta\to 0} \oint_{Z_{\delta,R}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds \quad (73)$$
$$= \oint_{Z} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds,$$

where the contour Z is defined by

$$Z = \{x + ib : \infty > x \geq -a\} + \{-a + iy : b \geq y \geq -b\} + \{x - ib : -a \leq x < \infty\}. \tag{74}$$

Finally, note that using the definition of the contour $D_{r,\delta}$ in (70) together with Proposition 5.7, it is straightforward to show that

$$\lim_{r\to\infty}\lim_{\delta\to 0}\oint_{D_{r,\delta}} \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds = 0. \tag{75}$$

Now placing the identities (71), (72), (73) and (75) together, we obtain that

$$\int_{U_{\alpha,\theta}(1)}^{\infty} \frac{x}{\exp\left(\nu^{\alpha/(\alpha-1)}x\right)-1}(U^{-1}_{\alpha,\theta^\star})'(x)dx = \frac{1}{\pi}\oint_Z \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds. \tag{76}$$

We now analyze the contour integral on the righthand side of (76) above. First note that since $a, b > (\alpha-1)\alpha^{\alpha/(1-\alpha)}$, we have from Proposition 5.6 that the function $\mathcal{G}$ may be expressed as the uniformly convergent power series (53) along the contour Z. Moreover, it is clear from (53) that $\mathcal{G}$ is uniformly bounded along Z. Thus using the Dominated Convergence Theorem [24] we obtain from (76) that

$$\begin{aligned} &\frac{1}{i}\oint_Z \frac{s}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds \\ = &\frac{1}{2}(1+\theta)\oint_Z \frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}ds \\ &+\frac{1}{\pi}\frac{1}{\alpha}\frac{\alpha-1}{\alpha}\left(\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(\frac{n}{\alpha})\Gamma(n(\frac{\alpha-1}{\alpha}))}{\Gamma(n)}\sin(n(\pi/2)(1+\theta))\right. \\ &\left.\oint_Z \frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}(-s)^{n(1-\alpha)/\alpha}ds\right). \end{aligned} \tag{77}$$

Next, note that by Cauchy's residue theorem [23], it follows that

$$\frac{1}{2}(1+\theta)\oint_Z \frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}ds = \frac{1}{2}(1+\theta)2\pi i\nu^{\alpha/(1-\alpha)}. \tag{78}$$

Moreover, using the fact that $(\alpha-1)\alpha^{\alpha/(1-\alpha)} < b < 2\pi/\nu^{\alpha/(\alpha-1)}$ and making the change-of-variables $z = \nu^{\alpha/(\alpha-1)}s$ in the integral below, it follows by Riemann's [27] contour integral representation of the analytic continuation of the zeta function that for $n \geq 1$, we may write

$$\oint_Z \frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)} s\right) - 1} (-s)^{n(1-\alpha)/\alpha} ds \tag{79}$$
$$= -\nu^{\alpha/(1-\alpha)} \frac{2\pi i}{\Gamma(n(\frac{\alpha-1}{\alpha}))} \zeta(n(1-\alpha)/\alpha + 1)\nu^n.$$

Now placing (8), (17), (76), (77), (78) and (79) together, along with a little bit of algebra, we obtain (18) for the case of $\beta = -1$.

The proof of Proposition 3.2 for the case of $1 < \alpha < 2$ and $-1 < \beta \leq 1$ follows in a similar manner to the case of $1 < \alpha < 2$ and $\beta = -1$ above. Indeed, the only significant difference is that one must now redefine the contour $C_{r,\delta,R}$ as depicted in Figure 2 below. We omit the remainder of the proof for the case of $1 < \alpha < 2$ and $-1 < \beta \leq 1$ for the sake of brevity.

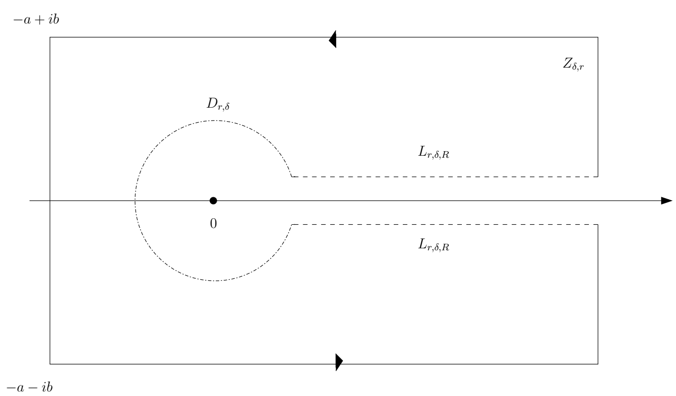

FIGURE 2: The contour $C_{r,\delta,R}$ in the case of $1 < \alpha < 2$ and $-1 < \beta \leq 1$.

As noted in Section 3, one may also prove Proposition 3.2 in the Gaussian case of $\alpha = 2$ and $\beta = 0$. Moreover, the proof is simpler in this case. Specifically, using the definition of $U_{\alpha,\theta}$ in (5), it follows after some algebra that if $\alpha = 2$ and $\beta = 0$, then we may write

$$\mathcal{H}_1(\nu) \quad = \quad \frac{1}{\nu} - \frac{1}{2\pi}\nu \int_{1/4}^{\infty} \frac{1}{\exp(\nu^2 x) - 1} \frac{1}{\sqrt{x - 1/4}} dx, \quad \nu > 0.$$

One may then proceed in a similar manner to the proof of Proposition 3.2 but without the need of the detailed analysis of $U_{\alpha,\theta}^{-1}$ provided in Section 5. We also mention that another method for proving Proposition 3.2 in the Gaussian case of $\alpha = 2$ and $\beta = 0$, recently suggested by Janssen [14], is to express the function $\mathcal{H}_1$ in terms of Lerch's transcendent.

6.2. Proof of Proposition 3.4

Proof of Proposition 3.4 . We proceed in a similar manner to the proof of Proposition 3.2 in the case of $1 < \alpha < 2$ and $\beta = -1$. First note that since by assumption $0 < \nu < \alpha(2\pi/(\alpha-1))^{(\alpha-1)/\alpha}$, we may fix a pair of numbers a, b satisfying (67). Next, for each $R > (\alpha-1)\alpha^{\alpha/(1-\alpha)}$ and r, δ satisfying (68), consider the closed contour $C_{r,\delta,R}$ in the complex plane depicted in Figure 1 above. Our goal now is to integrate the function $\mathcal{G}$ around the closed contour $C_{r,\delta,R}$ in order to recover the integral on the righthand side of (24).

Note that by Proposition 5.5 and since by assumption $(\alpha-1)\alpha^{\alpha/(1-\alpha)} < b < 2\pi/\nu^{\alpha/(\alpha-1)}$, it is clear that the function

$$\frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s), \quad s \in \mathbb{C} \setminus (U_{\alpha,\theta}(1), +\infty), \; s \neq 2k\pi/\nu^{\alpha/(\alpha-1)}, \; k \in \mathbb{Z},$$

is analytic everywhere in the closed contour $C_{r,\delta,R}$, with the exception of a simple pole at $s = 0$ which, by Proposition 5.5 and (39) in the proof of Proposition 3.3, has residue

$$\mathcal{G}(0)\frac{1}{\nu^{\alpha/(\alpha-1)}} \quad = \quad -\frac{1}{2}i\int_{-\theta}^{1}\frac{1}{U_{\alpha,\theta}(\varphi)}d\varphi\frac{1}{\nu^{\alpha/(\alpha-1)}} \quad = \quad -\frac{1}{\alpha-1}i\frac{1}{\nu^{\alpha/(\alpha-1)}}.$$

Now using Cauchy's residue theorem [23] and the definition of $C_{r,\delta,R}$, it follows taking limits first as $\delta \to 0$, then as $r \to 0$, and finally as $R \to \infty$ that we obtain in a similar manner to (71) through (76) of Proposition 3.2 that we may write

$$\begin{aligned}\frac{1}{2}\int_{U_{\alpha,\theta}(1)}^{\infty}\frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}x\right)-1}\left(U_{\alpha,\theta^\star}^{-1}\right)'(x)dx \quad &= \quad -\frac{1}{\alpha-1}\frac{1}{\nu^{\alpha/(\alpha-1)}} \\ &+\frac{1}{2\pi}\oint_Z\frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds,\end{aligned} \tag{80}$$

where the contour Z is as defined in (74).

We now proceed to analyze the contour integral on the righthand side of (80) above. Similar to the analysis in (77) through (79) of Proposition 3.2, it follows from assumption (67) on a, b, Proposition 5.6, the Dominated Convergence Theorem [24] and Riemann's [27] contour integral representation of the analytic continuation of the zeta function that we may write

$$\begin{aligned} & \frac{1}{i}\oint_Z \frac{1}{\exp\left(\nu^{\alpha/(\alpha-1)}s\right)-1}\mathcal{G}(s)ds \qquad (81) \\ = & -\frac{\pi}{2}(1+\theta)i \\ & -\frac{1}{\pi}\frac{1}{\alpha}\frac{\alpha-1}{\alpha}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\frac{n}{\alpha})\Gamma(n(\frac{\alpha-1}{\alpha}))}{\Gamma(n)}\sin(n(\pi/2)(1+\theta))\right. \\ & \left.\frac{\alpha}{\alpha-1}\frac{1}{n}\frac{2\pi i}{\Gamma(n\left(\frac{\alpha-1}{\alpha}\right))}\zeta(n(1-\alpha)/\alpha)\nu^n\right). \end{aligned}$$

Now placing (8), (24), (80) and (81) together, along with some algebra, we obtain (25) as desired.

One may also prove Proposition 3.4 in the Gaussian case of $\alpha = 2$ and $\beta = 0$. Specifically, using the definition of $U_{\alpha,\theta}$ in (5), it follows after some algebra that if $\alpha = 2$ and $\beta = 0$, then we may write

$$\mathcal{H}_2(\nu) = \frac{1}{\nu^2} - \frac{1}{4\pi}\int_{1/4}^{\infty}\frac{1}{\exp(\nu^2 x)-1}\frac{1}{x}\frac{1}{\sqrt{x-1/4}}dx, \quad \nu > 0.$$

In order to prove Proposition 3.4, one may then proceed in a similar manner to the above but without the need of the results in Section 5.

7. Application to a Problem Arising in Queueing Theory

We now demonstrate how Theorem 3.1 may used to obtain approximations of the limiting waiting time distribution for the $GI/D/N$ queue. The precise setup for the $GI/D/N$ queue is as follows. We assume that the system is initially empty at time 0. Next, let $\{A_j, j \geq 1\}$ be a sequence of non-negative, i.i.d random variables with mean 1. The kth arrival to the system then occurs at time

$$\tau_{\lambda,k} = \frac{1}{\lambda}\sum_{j=1}^{k} A_j, \quad k \geq 1,$$

where $\lambda > 0$ is the arrival rate to the system. Each customer arriving to the system has a service time that is deterministic and equal to $1/\mu$, where $\mu > 0$. The number of servers in the system is allowed to vary with the arrival rate λ and is denoted by N_λ. Customers arriving to the system are served on a first-come-first-served basis.

Moreover, we assume that the system is non-idling in the sense that customers are never waiting in the queue while there are servers idle.

Now, for each $k \geq 1$, let $W_{\lambda,k}$ denote the waiting time of the kth customer to arrive to the system. Then, assuming that $\rho = \lambda/(N_\lambda \mu) < 1$, it follows by the results of Kiefer and Wolfowitz [20] that $W_{\lambda,k} \Rightarrow W_\lambda$ as $k \to \infty$. Moreover, following the analysis of Jelenković, Mandelbaum and Momčilović [19], one has the equality in distribution

$$W_\lambda \quad \stackrel{d}{=} \quad \max\left(W_\lambda + \frac{1}{\mu} - \frac{1}{\lambda} \sum_{j=1}^{N_\lambda} A_j, 0 \right). \tag{82}$$

Suppose now that the i.i.d. sequence $\{A_j, j \geq 1\}$ lies in the normal domain of attraction of the stable law $S_\alpha(1,0,1)$ for some $1 < \alpha < 2$. That is, there exists a constant $\varsigma > 0$ such that

$$\frac{1}{k^{1/\alpha}} \sum_{j=1}^{k} (A_j - 1) \quad \Rightarrow \quad \varsigma S_\alpha(1,0,1) \quad \text{as} \quad k \to \infty. \tag{83}$$

Note that since the sequence of random variables $\{A_j, j \geq 1\}$ is non-negative, we have by necessity that the α-stable random variable appearing on the righthand side of (83) is totally skewed to the right. Now let

$$N_\lambda \quad = \quad \left\lceil \frac{\lambda}{\mu} + \kappa \left(\frac{\lambda}{\mu} \right)^{1/\alpha} \right\rceil, \quad \kappa > 0. \tag{84}$$

The fact that $\kappa > 0$ implies by [20] that the limiting waiting time distribution W^λ is well-defined. Hence, let

$$\widehat{W}_\lambda \quad = \quad \mu^{1/\alpha} \lambda^{(\alpha-1)/\alpha} W^\lambda, \quad \lambda > 0,$$

and set

$$M \quad = \quad \sup_{k \geq 0} S_k,$$

where $S_0 = 0$ and S_k for $k \geq 1$ is the kth partial sum of i.i.d. α-stable random variables with distribution $\varsigma S_\alpha(-1,0,1) - \kappa$. We then have the following result which may be proven in a similar manner to Theorem 1 and Corollary 1 of Jelenković, Mandelbaum and Momčilović [19] or Theorem 1 of Janssen, van Leeuwaarden and Zwart [17], both of whom treated the Gaussian case of $\mathrm{Var}(A_1) < \infty$.

Theorem 7.1. *As $\lambda \to \infty$, it follows that*

1. $\widehat{W}_\lambda \Rightarrow M$,
2. $\mathbb{P}(\widehat{W}_\lambda = 0) \to \mathbb{P}(M = 0)$.

Proof. Note that using (82) above we may write

$$\widehat{W}_\lambda \quad \stackrel{d}{=} \quad \max\left(\widehat{W}_\lambda + \widehat{X}_\lambda - \kappa, 0\right),$$

where

$$\widehat{X}_\lambda \quad \stackrel{d}{=} \quad \left(\frac{\lambda}{\mu}\right)^{-1/\alpha} \sum_{j=1}^{N_\lambda}(1 - A_j) + \mu^{1/\alpha}\lambda^{(\alpha-1)/\alpha}\left(\frac{1}{\mu} - \frac{N_\lambda}{\lambda}\right) + \kappa.$$

However, note that by (83) above and the definition of N_λ in (84), we have that $\widehat{X}_\lambda \Rightarrow \varsigma S_\alpha(-1, 0, 1)$ as $\lambda \to \infty$. Moreover, by Theorem 1 of Owen [26] and (3.18) of Billingsley [3], it follows that the sequence $\{(\widehat{X}_\lambda)^+, \lambda > 0\}$ is uniformly integrable. Hence, by Theorem X.6.1 of Asmussen [2], we have that Item 1 above holds. The proof of Item 2 above follows in the same manner as the proof of Item 2 of Theorem 2 of Janssen, van Leeuwaarden and Zwart [17].

Now note that M is equal in distribution to the all-time maximum of an α-stable random walk with increments that are distributed according to $\varsigma S_\alpha(-1, 0, 1) - \kappa$. Hence, we may use Theorem 3.1 above in order to obtain approximations to $\mathbb{P}(\widehat{W}_\lambda = 0)$ for λ large.

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