Derivation of Taylor Series Expansion

Objective:

Given $f(x)$, we want a power series expansion of this function with respect to a chosen point x_o, as follows:

$$f(x) = a_0 + a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 + \cdots \qquad (1)$$

(Translation: find the values of a_0, a_1, a_2, … of this infinite series so that the equation holds.)

Method:

The general idea will be to process both sides of this equation and choose values of x so that only one unknown appears each time.

To obtain a_o: Choose $x=x_o$ in equation (1). This results in
$a_0 = f(x_o)$

To obtain a_1: First take the derivative of equation (1)

$$\frac{d}{dx} f(x) = a_1 + 2a_2(x - x_o) + 3a_3(x - x_o)^2 + 4a_4(x - x_o)^3 + \cdots \qquad (2)$$

Now choose $x=x_o$.

$$a_1 = \left.\frac{df}{dx}\right|_{x=x_o}$$

To obtain a_2: First take the derivative of equation (2)

$$\frac{d^2}{dx^2} f(x) = 2a_2 + 3\cdot 2a_3(x - x_o) + 4\cdot 3a_4(x - x_o)^2 + 5\cdot 4a_5(x - x_o)^3 + \cdots \qquad (3)$$

Now choose $x=x_o$.

$$a_2 = \frac{1}{2}\left(\left.\frac{d^2 f}{dx^2}\right|_{x=x_o}\right)$$

To obtain a_3: First take the derivative of equation (3)

$$\frac{d^3}{dx^3} f(x) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x - x_o) + 5\cdot 4\cdot 3a_5(x - x_o)^2 + 6\cdot 5\cdot 4a_6(x - x_o)^3 + \cdots \qquad (4)$$

Now choose $x=x_o$.

$$a_3 = \frac{1}{3\cdot 2}\left(\left.\frac{d^3 f}{dx^3}\right|_{x=x_o}\right) = \frac{1}{3!}\left(\left.\frac{d^3 f}{dx^3}\right|_{x=x_o}\right)$$

…

To obtain a_k: First take the kth derivative of equation (1) and then choose $x=x_o$.

$$a_k = \frac{1}{k!}\left(\left.\frac{d^k f}{dx^k}\right|_{x=x_o}\right)$$

Summary:

The taylor series expansion of *f(x)* with respect to x_o is given by:

$$f(x) = f(x_o) + \left(\frac{df}{dx}\right)_{x=x_o}(x - x_o) + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\right)_{x=x_o}(x - x_o)^2 + \cdots + \frac{1}{k!}\left(\frac{d^k f}{dx^k}\right)_{x=x_o}(x - x_o)^k + \cdots$$

Generalization to multivariable function:

$$\begin{aligned} f(x,y,z) = A &+ a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 + \cdots \\ &+ b_1(y - y_o) + b_2(y - y_o)^2 + b_3(y - y_o)^3 + \cdots \\ &+ c_1(z - z_o) + c_2(z - z_o)^2 + c_3(z - z_o)^3 + \cdots \end{aligned} \qquad (5)$$

Using similar method as described above, using partial derivatives this time,

$$A = f(x_o, y_o, z_o)$$

$$a_k = \frac{1}{k!}\left(\left.\frac{\partial^k f}{\partial x^k}\right|_{x=x_o,\, y=y_o,\, z=z_o}\right)$$

$$b_k = \frac{1}{k!}\left(\left.\frac{\partial^k f}{\partial y^k}\right|_{x=x_o,\, y=y_o,\, z=z_o}\right)$$

$$c_k = \frac{1}{k!}\left(\left.\frac{\partial^k f}{\partial z^k}\right|_{x=x_o,\, y=y_o,\, z=z_o}\right)$$

(Note: the procedure above does not guarantee that the infinite series converges. Please see Jenson and Jeffreys, *Mathematical Methods in Chemical Engineering*, Academic Press, 1977, for a thorough discussion on how to analyze the convergence of the resulting series.)