

Averaging out Inhomogeneous Newtonian Cosmologies: I. Fluid Mechanics and the Navier-Stokes Equation

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Abstract

The basic concepts and equations of classical fluid mechanics are presented in the form necessary for the formulation of Newtonian cosmology and for derivation and analysis of a system of the averaged Navier-Stokes-Poisson equations. A special attention is paid to the analytic formulation of the definitions and equations of moving fluids and to their physical content.

1 Introduction

In looking for adequate gravitational field equations applicable in the cosmological settings [1] there has been a number of attempts to average Einstein equations (for a review and references see [2], [3] and recent papers [4], [5]). Despite four decades of research in this field since the pioneering paper [6], there is still no definite consensus on the results and approaches to the averaging problem in general relativity and cosmology. It does reflect to a certain extent the fundamental character of this problem. The averaging problem in general relativity has actually originated in cosmology [1]-[3], [6]-[8], but the main questions remain unanswered. A definite and unambiguous picture of the averaged dynamics of the universe is not available as yet.

In view of the difficulties mentioned above, there have been some attempts to construct an averaged theory of the Newtonian self-gravitating fluid to describe on its basis various aspects of dynamics of homogeneous and inhomogeneous Newtonian cosmologies, such as backreaction, large-scale structure formation, scaling properties (for a review and references see [9], [10] and recent papers [11]-[13]). Studying the Newtonian gravity and cosmology is of importance for three main reasons:

(i) The physics of Newtonian self-gravitating configurations provides an adequate theoretical framework for modelling the structure formation, galaxy dynamics, clustering processes, etc. in cosmology [14];

(ii) It is well-known that Newtonian cosmology has very similar physics in many aspects with the general relativistic cosmology [15], [16];

(iii) Newtonian problems are simpler to deal with than the general relativistic ones and, given similarities and differences between them [17], [18], one can gain important intuition and experience to approach the similar general-relativistic cosmological problems..

In order to derive a system of the averaged Navier-Stokes-Poisson equations and to make use of the system for analysis of inhomogeneous Newtonian universes, it is significant to present the basic concepts and equations of classical fluid mechanics in the form necessary to approach this problem. In this presentation a special attention will be paid to the analytic formulation of the definitions and equations of moving fluids and their physical content.

The structure of this paper is as follows. Chapter 2 gives a brief account on the interconnection between fluid mechanics and Newtonian cosmology. In Chapter 3 some approaches to averaging inhomogeneous Newtonian cosmologies are reviewed and discussed. The main concepts regarding the fluid motion and the Eulerian and Lagrangian pictures of moving fluids are presented in Chapters 4. Chapter 5 presents the field description of moving fluids which provides a framework for an analytic formulation of fluid motion as the theory of the integrals of partial differential equations. The Euler expansion formula for the change of the volume of an infinitesimal element of fluid during its motion is proved in Chapter 6. The Reynolds transport theorem following from the Euler expansion formula for the change of an integral of a fluid quantity over an arbitrary fluid volume moving along the fluid paths is proved in Chapter 7. In Chapter 8 the fluids possessing the property of inertia are introduced and the equation of continuity for the fluid density is derived as a consequence of the fluid mass conservation. The Principle of conservation of linear momentum is formulated in Chapter 9 and the Cauchy equation of motion is obtained as a consequence of this conservation. The structure of the system of the Cauchy equation of motion and the equation of continuity for the description of fluid motion is also analyzed here. The kinematic quantities to characterize the change of a fluid element during its motion, namely, the expansion scalar, the shear tensor and the vorticity vector, are introduced and discussed in Chapter 10. Chapter 11 is devoted to analysis of the conservation of energy for fluids and formulation of the laws of thermodynamics for fluids with a special attention paid to the following energy transfer equations and the definition of the thermodynamical variables for the compressible and incompressible fluids. The next Chapter gives a discussion of the conservation of angular momentum and its relation with the structure of the fluid stress tensor. Chapter 13 provides a list of different forms of the Cauchy equation of motion in dependence on the structure of fluids. In particular, the Euler equation for the perfect fluid, the Stokes equation for the Stokean fluid and the Navier-Stokes equation of classical hydrodynamics for the Newtonian fluid are presented and discussed. The physical status of the Navier-Stokes equation as the macroscopic equation governing the dynamics of gases and liquids is also discussed.

Conventions and notations are as follows. All functions $f = f(x^i, t)$ are defined on 3-dimensional Euclidean space E^3 in the Cartesian coordinates $\{x^i\}$ with Latin space indices i, j, k, \dots running from 1 to 3, and t is the time variable. The Levi-Civita symbol ε_{ijk} is defined as $\varepsilon_{123} = +1$ and $\varepsilon^{123} = +1$ and δ^{ij} , δ_j^i and δ_{ij} are the Kronecker symbols. The symmetrization of indices of a tensor $T_{jk}^i = T_{jk}^i(x^l, t)$ is denoted by round brackets, $T_{(jk)}^i = \frac{1}{2}(T_{jk}^i + T_{kj}^i)$, and the antisymmetrization by square brackets, $T_{[jk]}^i = \frac{1}{2}(T_{jk}^i - T_{kj}^i)$. A partial derivative of T_{jk}^i with respect to a spatial coordinate x^i or time t is denoted either by comma, or by the standard calculus notation, $T_{jk,t}^i = \partial T_{jk}^i / \partial t$ and $T_{jk,l}^i = \partial T_{jk}^i / \partial x^l$, the fluid velocity is $u^i = u^i(x^j, t)$, and the material (total) derivative is $\dot{T}_{jk}^i \equiv dT_{jk}^i / dt = \partial T_{jk}^i / \partial t + u^l \partial T_{jk}^i / \partial x^l = T_{jk,t}^i + u^l T_{jk,l}^i$.

2 Fluid Mechanics and Newtonian Cosmology

The dynamics of a compressible Newtonian fluid evolving in its own Newtonian gravitational potential field is known to be governed by the Navier-Stokes-Poisson system (see [19], [20] and [21], [22] for the system of equations in the framework of the so-called 3+1 covariant formulation of general relativistic cosmology and references therein). It consists of the equation of continuity which states the conservation of mass of a fluid element, the Navier-Stokes equation which is the Newton second law for the element of fluid stating the conservation of the fluid element momentum, the Poisson equation for the fluid gravitational potential and an equation of state of the fluid. The fluid flow is described by its velocity, density, pressure, or temperature, and the molecular transport coefficients of dynamic and bulk viscosity and/or thermal conductivity. Generally, the evolution of a fluid element can be characterized by change of its kinematic characteristics: the expansion scalar of the fluid element characterizes the change of its volume, the shear tensor determines the distortion of the fluid element leaving its volume unchanged and the vorticity vector represents its rigid rotation. The Navier-Stokes equation and the equation of continuity can be therefore rewritten explicitly as a system of evolution equations for the expansion, the shear, the vorticity and the fluid density. In case of Newtonian cosmology the system of the Navier-Stokes-Poisson equations for a compressible Newtonian fluid without viscosity and heat transfer is assumed to describe the Newtonian universes, that is, possibly spatially infinite distributions of the evolving self-gravitating cosmological fluid governed by the Navier-Stokes-Poisson equations. For homogeneous and isotropic Newtonian universes only time-dependent functions of the expansion, the pressure and the density remain essential. The evolution equation for the expansion scalar then describes the dynamics of evolving homogeneous and isotropic Newtonian universes. This equation is known to have the same form as the Raychaudhuri equation in general relativity [23], [24] describing the evolution of the expansion scalar of a 4-dimensional flow of the cosmological fluid in homogeneous and isotropic general relativistic cosmology. It is equivalent to the Friedmann equation [25] for a scale factor of an expanding universe in both Newtonian and general relativistic cases. Therefore, studying Newtonian cosmology proved its importance for understanding some aspects of general relativistic cosmology

and revealed remarkable similarities between both theoretical approaches to studying the cosmological models of the real Universe [15], [16].

In case of inhomogeneous and anisotropic Newtonian universes, when the whole system of the equations must be considered, one can study the dynamics of a Newtonian universe which is smooth, homogeneous and/or isotropic, on large scales. In the general relativistic setting an inhomogeneous cosmological model which has the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time geometry [25]-[29] on average is generally believed to be more realistic than the standard FLRW model of our Universe [1]. An essential dynamical factor which must affect the character of dynamics of such an inhomogeneous universe is that it takes into account the effect of the backreaction of inhomogeneities on the global expansion of the universe (for a review and references see [30] and recent papers [12], [13], [31]). In case of Newtonian cosmology a similar setting is also of great interest for understanding the cosmological backreaction due to correlations of a self-gravitating cosmological fluid. To construct such a model a suitable averaging procedure must be applied to the Navier-Stokes-Poisson system, to bring about a set of equations for the averaged expansion, shear, vorticity, pressure, fluid density and gravitational potential together with their correlation functions involved.

The nonrelativistic kinematics and dynamics of fluid flows have been intensively studied since the eighteenth century (for a review and references see [32], [33]), which resulted in discovering the Navier-Stokes equation as the fundamental equation of classical hydrodynamics [34], [35]. This equation successfully describes the motion of various fluids and gases and other transport phenomena under a wide class of physical conditions including viscosity and heat transfer [20]. The existence of two different types of flow, called laminar and turbulent, has been established in the middle of the nineteenth century. The pioneering systematic investigations of turbulence by Rayleigh and Reynolds in the last decades of the nineteenth century led to understanding that the correlation functions of the fluid velocity and the fluid density must be recognized as the fundamental characteristics of turbulence (for a review and references see [19], [36]-[38]). The first system of differential equations for the correlation functions containing the single-point first and second moments of the fluid fields has been derived by Reynolds [39] using the time and space averagings of the system of the continuity and the Navier-Stokes equations for the case of incompressible fluids. An analytical formulation of the problem of turbulence has been given first by Keller and Friedmann [40] who developed a general method for obtaining the differential equations for the moments of arbitrary order by using the system of the continuity and the Navier-Stokes equations. The Reynolds equations are the first equations in the total infinite system of the Keller-Friedmann equations. Analysis of these systems of equations has resulted in remarkable progress in our understanding of the phenomenon of turbulence to establish the statistical nature of turbulence, to develop the theory of homogeneous and isotropic turbulence, to propose the hypothesis of locally isotropic turbulence and developed turbulence [19], [36]-[38]. However, despite the significant developments the problem of turbulence remains essentially unsolved since even such fundamental questions as which flow should be considered turbulent and which should not, what is a mechanism of the turbulent mixing, how the developed turbulence

occurs, etc. do not have satisfactory explanation as yet. The difficulties originate partly in the mathematics of the Navier-Stokes and Reynolds equations where the questions of existence, uniqueness, and smoothness of solutions are not mainly resolved [41]. On the other hand, theoretical approaches to formulate adequate physical models of real physical phenomena exhibiting turbulent processes are often very limited due to the absence of sufficient experimental evidence about a turbulent regime of interest [42].

3 Approaches to Averaging out Inhomogeneous Universes

As it has been pointed out above, the physics of Newtonian self-gravitating fluid configurations is widely employed for modelling cosmological gravitating structures and the Universe as a whole. The idea of Chaos, or Cosmic Turbulence, as the origin of our Universe goes back to Kepler, Descartes and Laplace [43]. Development of the hot Big Bang Model and attempts to understand the origin of galaxies has led to invoking the hydrodynamic turbulence theory in modelling the galaxy formation [44], [45] as opposed to the idea of the origin of galaxies due to the gravitational instability (see [14], [46] for discussion and references). It has been shown [47], in particular, that the physical conditions in the early universe could allow the Primeval Turbulence regime and the appearance of eddies to produce galaxies later on.

However, application of the Navier-Stokes-Poisson system and its suitably averaged version for analysis of the evolution of inhomogeneous and anisotropic Newtonian universes has not been accomplished in any considerable depth. Such important questions as backreaction, large-scale structure formation, scaling properties, dynamics of universe which is homogeneous and isotropic on average and other dynamical issues related to the character of the cosmological evolution when the gravitational correlation effects are taken into account are still open. In this setting the system of equations for the 3+1 covariant formulation of general relativistic cosmology has been hardly studied either.

A study of the averaged-over-ensemble self-gravitating Newtonian perfect fluid in the Eulerian formulation has been carried out first in [9] under assumption of the homogeneous isotropic turbulence [48]. Despite this strong requirement which leads to a considerable simplification of the system of the averaged Navier-Stokes-Poisson equations, the Newtonian cosmological models exhibit nontrivial dynamics. The evolution equation for the mean-square correlation function of vorticity has been derived and its solution has been shown to have a nontrivial behavior to blow up in a finite time for an expanding background in case of domination of the fragmentation of eddies (inhomogeneities), and to blow up always in a finite time for a contracting background. Thus, in dependence of a value of the initial vorticity the averaged evolution of an initially inhomogeneous Newtonian cosmological model can develop either a turbulence mixing regime, or a regime of the natural decay of inhomogeneities.

Analysis of the general relativistic Raychaudhuri equation has been attempted in [10] for the case of closed universes which are globally splitted into a topological product of

compact, orientable, differentiable 3-manifolds and the real line which represents time. These 3-manifolds can be naturally represented as a family of disjoint spacelike hypersurfaces parametrized by time, and there exists a future-oriented normal unit 4-vector field tangent to a normal congruence of curves which are orthogonal trajectories of the hypersurface family. A general relativistic analogue of the Reynolds transport theorem of fluid mechanics [32], [33], [49] for the rate of change of the integral of the expansion scalar of this 4-vector flow over a hypersurface moving with the flow has been derived. A consequent formula for the rate of change of the ratio of this integral to the volume of the corresponding hypersurface, which has been called the “average expansion”, was used to find out if this “average expansion” has a zero volume singularity characteristic to the FLRW models. As a result of this analysis it has been suggested that such a singularity can be avoided at all if during some epochs the universe is expanding highly nonuniformly.

This result, however, should not be misinterpreted. Since by its kinematic meaning a transport formula for a quantity is essentially an integral formulation of the corresponding differential law for this quantity, there is no new either kinematic, or dynamical information which can be gained by using such an integral formulation. So, for example, the conservation of the mass of fluid in a material volume moving with the fluid leads to the equation of continuity which is well-known to be a necessary and sufficient condition for a motion to conserve the mass of an arbitrary moving volume [32]. On the contrary, the averaged continuity equation obtained by the application of a properly defined space, time or ensemble average and the corresponding formulae for interchanging of operations of taking a derivative and an averaging is known to be a new differential dynamical equation, the continuity equation for the turbulent flow [19], [38]. Therefore, the conclusions derived in [10] on the basis of using a ratio of the integral of the expansion scalar over an arbitrary hypersurface moving along the normal 4-vector flow to the volume of this region as a space average of the expansion and using the transport formula for this ratio as a formula for interchanging of the time derivative and this “averaging” cannot be considered as a consistent averaging procedure. There is no new information in the derived integral formula as compared with the corresponding local differential relation which is the general relativistic Raychaudhuri equation.

Some recent approaches to construct an averaged theory of self-gravitating Newtonian fluid to study the dynamical properties of the homogeneous and inhomogeneous Newtonian universes, such as backreaction, large-scale structure formation, scaling properties (see [11]-[13] and references therein), have also made use of the similar ratio described above [10] as a “space average” of the expansion scalar and other kinematic quantities for the case of 3-dimensional cosmological fluid. A nonrelativistic version of the transport formula as in [10] which is known from the Reynolds transport theorem [32], [33], [49] modified by division by the region volume has been taken as a formula for interchanging of the time derivative and this “averaging” and a set of the transport relations for the kinematic quantities has been derived. This set of transport relations was regarded as the averaged equations for the kinematic quantities. In particular, the transport equation for the expansion scalar has been taken as the averaged Newtonian version of Raychaudhuri equation to analyze the character of the averaged dynamics. Again, such a procedure of

space averaging is not proper and all conclusions are achievable by analyzing the initial system of the differential equations for kinematic quantities.

4 The Eulerian and Lagrangian Pictures of Fluid Motion

The material fluid is considered to be a continuous physical substance distributed over a region of the 3-dimensional Euclidean space, which is assumed to possess a positive volume. A fluid flow, or a fluid motion, is represented mathematically by a one-parameter family of mappings of a 3-dimensional region filled with the fluid into itself, the mapping being usually assumed to be 1-1 and at least thrice continuously differentiable in all variables [32], [33], [50], [51] except possibly at certain singular surfaces, curves or points where a separate analysis is required. The real parameter t describing the mappings of fluid regions is identified with the time with its domain supposed to be $-\infty < t < +\infty$ where $t = 0$ is an arbitrary initial instant.

In order to describe the fluid motion analytically let us introduce a fixed rectangular coordinate system $\{x^i\}$, each coordinate triple of which is referred to as a position of an infinitesimal element of a fluid distribution, or configuration, often called a “fluid particle”. If one observes such a fluid particle moving with the fluid, it is easy to detect that the particle which was in the position $\{\xi^i\}$ has moved to position $\{x^i\}$ at a later time. Without loss of generality, one can take the first instance to be $t = 0$ and if the later time is t the fluid flow is represented by the transformation

$$x^i = x^i(\xi^j, t). \quad (1)$$

If ξ^i is fixed while t varies, Eq. (1) specifies the path of the fluid particle

$$x^i = x^i(t) \equiv x^i(\xi^j, t)|_{\xi^i=\text{const}}, \quad x^i(0) = \xi^i, \quad (2)$$

initially placed at ξ^i . If time t is fixed, Eq. (1) determines a transformation

$$x^i = x^i(\xi^j) \equiv x^i(\xi^j, t)|_{t=\text{const}}, \quad x^i(0) = \xi^i, \quad (3)$$

of the region initially occupied by the fluid into its position at time t . The transformation (1) is precisely the above-mentioned one-parameter family of mappings defining the fluid motion

$$x^i : x^i(0) = \xi^i \rightarrow x^i(\xi^j, t). \quad (4)$$

Since functions $x^i(\xi^j, t)$ are single-valued and thrice differentiable in all variables, the Jacobian of the transformation (1) does not vanish,

$$J = \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} = \det \left(\frac{\partial x^i}{\partial \xi^j} \right) \neq 0, \quad (5)$$

and there exists a set of inverse functions

$$\xi^j = \xi^j(x^i, t) \tag{6}$$

to define the initial positions of the fluid particle which is at any position $\{x^i\}$ at a moment of time t . This inverse transformation is also 1-1 and thrice differentiable in all variables. Due to Eq. (6) a fluid flow can be equivalently described by a set of initial positions of its particles in dependence of their positions in later times.

Thus, there are two equivalent ways to describe the fluid motion.

The Eulerian description by the spatial variables (x^i, t) *This picture presents the state of motion at each position $\{x^i\}$ such that throughout all time a given set of coordinates $\{x^i\}$ remains attached to a fixed position, serving as an identification for it - the spatial variables are called the Eulerian or spatial coordinates.*

The Lagrangian description by the material variables (ξ^i, t) *This picture of the motion chronicles the history of each fluid particle $\{\xi^i\}$ such that throughout all time a given set of coordinates $\{\xi^i\}$ remains attached to a single fluid particle, serving to uniquely identify it - the material variables are called the Lagrangian, material or convected coordinates ¹.*

It should be pointed out here that the Eulerian and Lagrangian descriptions have quite different physical interpretations when one would like to describe a moving fluid from the point of view of an observer carrying out measurements by a set of physical instruments. In the Eulerian picture an observer is always located in a particular position $\{x^i\}$ at a moment of time t and such an observer views moving fluid particles passing through and by the position. In the Lagrangian picture an observer is always moving with a fluid particle which was initially at a position $\{\xi^i\}$ and such an observer views the relative changes in the fluid comoving with the observer's particle.

5 The Field Description of Fluid Motion

Although the fluid flow is completely determined either by the transformation (1) in the Eulerian description, or its inverse transformation (6) in the Lagrangian description, it is important also to be able to describe the state of motion at a given position during the course of time. To achieve this goal one should characterize the motion of fluid in terms of objects defined at a position $\{x^i\}$ at a time t . A simplest object to be considered for this role is the instant fluid velocity u^i , functions of which can be defined in both descriptions, $u^i(x^j, t)$ and $u^i(\xi^j, t)$.

Let us consider a quantity f which characterizes a fluid in the framework of the Eulerian or Lagrangian descriptions. Generally, any such a quantity f is a function of spatial variables $f = f(x^i, t)$ and it is also a function of material variables $f = f(\xi^i, t)$

¹Though the descriptions of the fluid motion by the spatial and material coordinates are usually called Eulerian and Lagrangian, correspondingly, both of them are known to be due to Euler [32], [50].

due to

$$f = f[x^i(\xi^j, t), t] \quad \text{or} \quad f = f[\xi^i(x^j, t), t], \quad (7)$$

the functions (7) being related by Eqs. (1) and (6) considered as the laws of the change of the variables. Geometrically, $f = f(x^i, t)$ is the value of f felt by the fluid particles instantaneously located at the position $\{x^i\}$ and $f = f(\xi^i, t)$ is the value of f experienced at time t by the fluid particle initially located at the position $\{\xi^i\}$. Then the change in quantity f when the fluid moves can be characterized by the two different time derivatives

$$\frac{\partial f}{\partial t} \equiv \frac{\partial f(x^i, t)}{\partial t} \Big|_{x^i=\text{const}} \quad \text{and} \quad \frac{df}{dt} \equiv \frac{\partial f(\xi^i, t)}{\partial t} \Big|_{\xi^i=\text{const}}. \quad (8)$$

The partial derivative $\partial f/\partial t$ gives the rate of change of f with respect to a fixed position $\{x^i\}$, while the material, or convective, derivative df/dt measures the rate of change of f with respect to a moving fluid particle.

The material derivative of the position of a fluid particle defined by the second Eq. (8) for $f = x^i$ is called its velocity u^i

$$u^i(\xi^j, t) = \frac{dx^i}{dt} \equiv \frac{\partial x^i(\xi^j, t)}{\partial t} \Big|_{\xi^i=\text{const}}. \quad (9)$$

The notion of the fluid velocity plays a fundamental role in the field formulation of fluid mechanics. Indeed, in most physical settings, when the actual motion of a fluid particle, that is either $x^i = x^i(\xi^j, t)$ or $\xi^j = \xi^j(x^i, t)$, is not given explicitly, it is more convenient to deal with the Eulerian form of the fluid velocity, $u^i = u^i(x^j, t)$. Given a vector field of fluid velocity $u^i = u^i(x^j, t)$, which can be actually measured in laboratory, defined in a fluid 3-region U for a time interval Δt , one can determine the functions $x^i = x^i(\xi^j, t)$ by solving the initial value problem for the system of the first order ordinary differential equations

$$\frac{dx^i}{dt} = u^i(x^j, t), \quad x^i(0) = \xi^i. \quad (10)$$

A unique solution $x^i = x^i(\xi^j, t)$ to Eqs. (10) is known (see, for instance, [52], [53]) to exist always on $U \times \Delta t$ for at least once continuously differentiable in all variables and bounded functions $u^i = u^i(x^j, t)$, and it depends continuously on the initial values $\{\xi^i\}$, to completely characterize the fluid motion (1). Therefore, the description of fluid motion in terms of the fluid velocity is equivalent to the Eulerian or Lagrangian pictures:

The field description by the fluid velocity in the spatial variables $u^i(x^j, t)$ *This picture presents the state of a fluid in motion by its velocity field specifying the velocity $u^i = u^i(x^j, t)$ of a fluid particle at each position $\{x^i\}$ at each time instance t , such that throughout all time a given set of velocity components $\{u^i\}$ remains attached to a fixed position, serving as an identification for a single fluid particle in dependence of its initial position $\{\xi^i\}$ - the velocity variables are called the field variables².*

²Euler and d'Alembert are considered to be the first to discover the field description of moving fluids. However, it was Euler who put forward the idea of formulating a field theory of fluid motion as the theory of the integrals of partial differential equations [32], [50].

The physical interpretation of the field description of fluid motion represents an observer as located in a particular position $\{x^i\}$ at a moment of time t to measure the velocities of the fluid particles moving by the observer and which were initially at $\{\xi^i\}$.

The field description of moving fluid has a fundamental significance for the proper analytic treatment of processes occurring in physical continuous media under a wide class of physical conditions. It is the field description that provides the possibility of an analytic formulation of fluid kinematics and dynamics by finding a set of partial differential equations governing the fluid motion.

With the definitions of the time derivatives (8) one can calculate the material derivative of any quantity $f(x^j, t)$ as

$$\begin{aligned} \frac{df}{dt} &= \left[\frac{\partial}{\partial t} f(\xi^i, t) \right]_{|\xi^i=\text{const}} = \\ & \left\{ \frac{\partial}{\partial t} f[x^i(\xi^j, t), t] \right\}_{|\xi^i=\text{const}} = \frac{\partial f}{\partial t} \Big|_{x^i=\text{const}} + \frac{\partial f}{\partial x^i} \left[\frac{\partial x^i(\xi^j, t)}{\partial t} \right]_{|\xi^i=\text{const}} \end{aligned} \quad (11)$$

that can be written in the following form by using the fluid velocity $u^i = u^i(x^j, t)$ defined by Eq. (9):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i}. \quad (12)$$

This formula relates the material and spatial derivatives through the velocity field in the field picture of fluid motion and it expresses the rate of change in $f(x^j, t)$ with respect to a moving fluid particle located at a position $\{x^i\}$ at time t . An important application of the material derivative (12) is the definition of the fluid acceleration $a^i = a^i(x^j, t)$,

$$a^i = \frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j}, \quad (13)$$

as the rate of change of the velocity of fluid particle experienced during its motion. Due to (13) the fluid acceleration a^i is represented only in terms of the fluid velocity u^i .

6 The Euler Expansion Formula

The Jacobian J (5) of the transformation (1) represents the dilatation of the infinitesimal volume of a fluid particle as it moves along its path (2). Indeed, considering (1) as the coordinate transformation from the initial coordinates $\{\xi^i\}$ at time $t = 0$ of a fluid particle to the particle coordinates $\{x^i\}$ at later time t , the change of the particle's infinitesimal volume is given by $dV = JdV_0$ where $dV = dx^1 dx^2 dx^3$ is the particle volume at the position $\{x^i\}$ and $dV_0 = d\xi^1 d\xi^2 d\xi^3$ is the initial particle volume. Because the fluid motion is continuous and (1) is continuously differentiable and therefore invertible, the fluid dilatation, or expansion, dV/dV_0 , is positive and bounded

$$J = \frac{dV}{dV_0}, \quad 0 < J < \infty. \quad (14)$$

Since the infinitesimal volume of a fluid particle undergoes, in general, changes and distortions during its motion, the rate of change of the dilatation characterizes such changes. The formula for the dilatation change had been found by Euler [54] (see [32], [33], [50], [51] for discussion).

Theorem 1 (The Euler expansion formula) *The rate of change of the fluid dilatation J (14) following a fluid particle path is given by*

$$\frac{dJ}{dt} = J \frac{\partial u^i}{\partial x^i}, \quad \text{or} \quad \frac{d}{dt} \ln J = \frac{\partial u^i}{\partial x^i}. \quad (15)$$

Proof. A derivation of the Euler formula is straightforward by differentiation of the Jacobian determinant, $dJ = J \partial \xi^i / \partial x^j d(\partial x^i / \partial \xi^j)$,

$$\frac{dJ}{dt} = J \frac{\partial \xi^j}{\partial x^i} \frac{d}{dt} \left(\frac{\partial x^i}{\partial \xi^j} \right) = J \frac{\partial \xi^j}{\partial x^i} \frac{\partial u^i}{\partial \xi^j} = J \frac{\partial \xi^j}{\partial x^i} \frac{\partial u^i}{\partial x^k} \frac{\partial x^k}{\partial \xi^j} = J \frac{\partial u^i}{\partial x^i}. \quad (16)$$

QED

Since the Euler expansion formula (15) establishes the divergence of the fluid velocity as a measure of the fluid dilatation, or expansion, the former is also called the fluid expansion and denoted as $\theta = \partial u^i / \partial x^i$. The Euler formula (15) emphasizes the fundamental role of the fluid velocity in the field description of fluid motion. When a fluid is moving without change in volume for all fluid particles at all position $\{x^i\}$ throughout all time t ,

$$\frac{dJ}{dt} = 0 \quad \Leftrightarrow \quad \frac{\partial u^i}{\partial x^i} = 0, \quad (17)$$

such a fluid is called incompressible. If a fluid is not incompressible,

$$\frac{dJ}{dt} \neq 0 \quad \Leftrightarrow \quad \frac{\partial u^i}{\partial x^i} \neq 0, \quad (18)$$

is called compressible.

7 The Transport Theorem of Reynolds

An important kinematical theorem follows from the Euler expansion formula (15). It is called the Reynolds transport theorem [49] and it concerns the rate of change not an infinitesimal volume of a fluid particle, but any integral of any quantity f which is a function of spatial variables $f = f(x^i, t)$ over an arbitrary closed fluid region, or volume, $\Sigma(t)$ moving with the fluid [32], [33], [50], [51]. Let us consider a volume integral of $f(x^i, t)$ over $\Sigma(t)$,

$$F(t) = \int_{\Sigma(t)} f(x^i, t) dV, \quad (19)$$

which is a well-defined function $F = F(t)$ of time.

Theorem 2 (The Reynolds transport theorem) *The rate of change of the volume integral (19) following the fluid particle paths is given by*

$$\frac{d}{dt} \int_{\Sigma(t)} f dV = \int_{\Sigma(t)} \left(\frac{df}{dt} + f \frac{\partial u^k}{\partial x^k} \right) dV. \quad (20)$$

Proof. It is straightforward by transformation of (19) into the integral over the fixed initial volume $\Sigma_0 = \Sigma(0)$ in the material coordinates $\{\xi^i\}$

$$\frac{d}{dt} F(t) = \frac{d}{dt} \int_{\Sigma_0} f[x^i(\xi^j, t), t] J dV_0 = \int_{\Sigma_0} \left\{ \frac{d}{dt} f[x^i(\xi^j, t), t] J + f[x^i(\xi^j, t), t] \frac{dJ}{dt} \right\} dV_0 \quad (21)$$

which immediately brings (20) by applying the Euler expansion formula (15). **QED**

By substitution (12) for the material derivative df/dt in the integrand of (20), one can obtain the Reynolds transport theorem in the following form

$$\frac{d}{dt} \int_{\Sigma(t)} f dV = \int_{\Sigma(t)} \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial x^k} (f u^k) \right) dV. \quad (22)$$

There is yet another useful form of the last formula obtained by application of the Gauss theorem

$$\frac{d}{dt} \int_{\Sigma(t)} f dV = \int_{\Sigma(t)} \frac{\partial f}{\partial t} dV + \int_{\partial \Sigma(t)} f u^k dS_k \quad (23)$$

where the 2-surface element dS_k is a vector equal to the infinitesimal 2-square and normal to the surface $\sigma(t) = \partial \Sigma(t)$ enclosing the volume $\Sigma(t)$. The formula (23) admits an immediate physical interpretation as expressing the rate of change of a quantity f within a moving fluid volume as the integral of the rate of change in this quantity itself at a point plus the net flow of f over the surface bounding the volume.

One particular case of the Reynolds transport theorem is of importance. For the constant quantity $f = \text{const}$ the Reynolds transport theorem (20) gives a formula for the change of value of the volume $V_{\Sigma}(t)$ of an arbitrary moving fluid region $\Sigma(t)$,

$$V_{\Sigma}(t) = \int_{\Sigma(t)} dV. \quad (24)$$

Corollary 1 (The change of volume) *The rate of change of the value of the volume $V_{\Sigma}(t)$ (24) of an arbitrary moving fluid region $\Sigma(t)$ is determined by the formula*

$$\frac{d}{dt} V_{\Sigma}(t) = \int_{\Sigma(t)} \frac{\partial u^k}{\partial x^k} dV = \int_{\partial \Sigma(t)} u^k dS_k. \quad (25)$$

On the basis of the Euler formulas [50], [55] for the transformation of the infinitesimal line element of a fluid particle path between a spatial line element dx^i and a material line element $d\xi^i$,

$$dx^i = d\xi^k \frac{\partial}{\partial \xi^k} x^i(\xi^j, t) \quad \text{and} \quad d\xi^i = dx^k \frac{\partial}{\partial x^k} \xi^i(x^j, t), \quad (26)$$

and the Nanson formulas [50], [56] for the transformation of the infinitesimal surface elements for a 2-dimensional surface made of fluid particle paths between a spatial surface element dS_i and a material surface element $dS_{(0)i}$,

$$dS_k \frac{\partial}{\partial \xi^i} x^k(\xi^j, t) = J dS_{(0)i} \quad \text{and} \quad dS_{(0)k} \frac{\partial}{\partial x^i} \xi^k(x^j, t) = J dS_i, \quad (27)$$

one can derive the corresponding line and surface integral transport relations [33], [50], [51] similar to the Reynolds transport theorem.

The Reynolds transport theorem is an integral statement essentially equivalent to the Euler expansion formula. The same true for the corresponding line and surface integral theorems and the Euler and the Nanson formulae (26), (27).

As it has been discussed in Introduction, the Reynolds transport theorem has been recently utilized for a definition of a space averaging over fluid volumes. This issue will be discussed in [57].

8 The Equation of Continuity

The definition of the fluid velocity (9), the system of partial differential equations (10) for determination of the equation of motion of fluid $x^i = x^i(\xi^j, t)$, the Euler expansion formula (15) and the Reynolds transport theorem (20) constitute the kinematics of moving fluid. The formulation of kinematics does not require any hypotheses about the structure and the nature of fluids as physical substances. To establish the differential equations governing the fluid dynamics, one must formulate such physical axioms on the basis of experimental evidence.

As it has been mentioned above, in the framework of the field description one must characterize the motion of fluids in terms of objects defined at a position $\{x^i\}$ at a time t . The fluid velocity u^i has been shown to be the simplest quantity to formulate the kinematics of fluid. Let us now further consider fluids which possess the property of inertia. Such a fluid can be endowed with density function $\rho = \rho(x^i, t)$ of the physical dimension [mass/volume], which enables one to define the mass $M(t)$ of the fluid held within the region $\Sigma(t)$

$$M(t) = \int_{\Sigma(t)} \rho dV. \quad (28)$$

The fluid mass (28) for any region $\Sigma(t)$ is assumed to be always positive that means positivity of the density function,

$$\rho(x^i, t) > 0, \quad (29)$$

everywhere. The following physical principle is taken to hold for the fluid.

The Principle of conservation of mass *The mass of fluid $M(t)$ (28) in an arbitrary region $\Sigma(t)$ does not change following the fluid particle paths*

$$\frac{d}{dt}M(t) = 0 \quad \text{or} \quad \frac{d}{dt} \int_{\Sigma(t)} \rho dV = 0. \quad (30)$$

The Principle of conservation of mass immediately brings about a differential relation for the density function $\rho(x^i, t)$ that is called the equation of continuity [32], [33], [50], [51] discovered first by Euler [54].

Theorem 3 (The equation of continuity) *The equation of continuity*

$$\frac{d\rho}{dt} + \rho \frac{\partial u^j}{\partial x^j} = 0 \quad (31)$$

is necessary and sufficient for the mass of any region following the fluid particle paths to be conserved (30).

Proof. Using the Reynolds transport theorem (20) for the fluid mass $M(t)$ (28) gives the rate of change of the mass as

$$\frac{d}{dt}M(t) = \int_{\Sigma(t)} \left[\frac{d\rho}{dt} + \rho \frac{\partial u^k}{\partial x^k} \right] dV. \quad (32)$$

Necessity follows immediately upon substitution of the equation of continuity (31) into (32). Sufficiency follows from the mass conservation (30) and (32) by taking into account that the fluid region $\Sigma(t)$ is arbitrary. **QED**

Using the formula for the material derivative (12) gives another form of the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}(\rho u^j) = 0. \quad (33)$$

An important formula for the rate of change of an integral of a product of a quantity $f(x^i, t)$ by the density function $\rho(x^i, t)$

$$[\rho f](t) = \int_{\Sigma(t)} \rho(x^i, t) f(x^j, t) dV. \quad (34)$$

is due to the Reynolds transport theorem (20) and the equation of continuity (31).

Corollary 2 For any quantity $f(x^i, t)$ the rate of change of the volume integral of the product $\rho(x^i, t)f(x^j, t)$ (34) following fluid particle paths is given by

$$\frac{d}{dt} \int_{\Sigma(t)} \rho f dV = \int_{\Sigma(t)} \rho \frac{df}{dt} dV. \quad (35)$$

The equation of continuity holds for any fluid satisfying the Principle of conservation of mass (30). It should be pointed out here that though the fluid mass is not a kinematic quantity the derivation of the equation of continuity is fully due to the fluid kinematics. Given a fluid distribution with the density $\rho(x^i, t)$, one can consider a problem of determination of the velocity field $u^i(x^j, t)$ for the fluid motion which preserves the mass of its each region. In such a setting the equation of continuity (31) is the first order partial differential equations to be solved to find the functions of the fluid velocity. This equation is not sufficient in general to find 3 unknown functions of the fluid velocity components $u^i(x^j, t)$. The dynamics of fluids cannot be therefore reconstructed only on the basis of the equation of continuity and the fluid density cannot serve as the physical reason for fluid motion.

9 The Cauchy Equation of Fluid Motion

The fluid dynamics considers the physical forces, external and internal, to act upon fluid particles to cause their motion, and, as a result, a motion of the fluid. The forces acting on a moving fluid particle may be of two kinds. Internal, or contact, forces are those acting on a fluid particle through its bounding surface, for example, such as fluid pressure. External, or body, forces are those acting throughout a medium, for example, such as gravity or the electromagnetic force. Due to the Cauchy stress principle [58] over any closed surface $\partial\Sigma(t)$ there exists a distribution of a stress vector $s^i = s^i(x^j, t; n^k)$ depending on the normal vector n^k to the surface, whose action and moment are equivalent to the actual internal forces exerted by the fluid outside the region enclosed by this surface upon the fluid inside [32], [51], [59]. Such internal forces per unit surface are represented by a vector $s_{(n)}^i = s_{(n)}^i(x^j, t)$ directed along an outward normal to the fluid particle surface. The external forces per unit mass are represented by a vector $F^i = F^i(x^j, t)$ which is assumed to be known as functions of position $\{x^i\}$ and time t and perhaps also the state of the motion of the fluid. In accordance with the Cauchy stress principle the following dynamical principle relates the linear momentum $P^i(t)$ of an arbitrary fluid region $\Sigma(t)$

$$P^i(t) = \int_{\Sigma(t)} \rho u^i dV \quad (36)$$

with the corporate action of the stress vector $s_{(n)}^i$ and the external forces F^i .

The Principle of conservation of linear momentum *The rate of change of linear momentum $P^i(t)$ (36) of an arbitrary fluid region $\Sigma(t)$ is equal to the total resultant force*

acting on the fluid volume within the region

$$\frac{d}{dt}P^i(t) = \frac{d}{dt} \int_{\Sigma(t)} \rho u^i dV = \int_{\Sigma(t)} \rho F^i dV + \oint_{\partial\Sigma(t)} s_{(n)}^i dS. \quad (37)$$

By means of the formula (35) the Principle of conservation of linear momentum can be written in the following form:

$$\int_{\Sigma(t)} \rho \frac{du^i}{dt} dV = \int_{\Sigma(t)} \rho F^i dV + \oint_{\partial\Sigma(t)} s_{(n)}^i dS. \quad (38)$$

In the representation (38) an arbitrary moving region $\Sigma(t)$ can be replaced, without loss of generality, by a fixed region $\Sigma(t_0)$. An important theorem regarding the stress vector s^i holds [32], [33], [51].

Theorem 4 (The local equilibrium of the stress forces) *For any fluid moving with the conservation of linear momentum (37) the stress forces are in the local equilibrium*

$$\lim_{\Sigma(t_0) \rightarrow 0} \frac{1}{D^2} \oint_{\partial\Sigma(t_0)} s_{(n)}^i dS = 0. \quad (39)$$

Here D is the diameter of the region $\Sigma(t_0)$. The volume $V_{\Sigma}(t_0)$ of the region $\Sigma(t_0)$ is $V_{\Sigma}(t_0) \simeq D^3$ and the square $S_{\partial\Sigma}(t_0)$ of the region surface $\partial\Sigma(t_0)$ is $S_{\partial\Sigma}(t_0) \simeq D^2$.

Proof. This result is derived by division of both sides of (38) by D^2 and taking limit of $\Sigma(t_0) \rightarrow 0$. Then due to the boundedness of both integrands for the volume integrals the formula (38) gives (39). **QED**

Analysis of the structure of the stress vector on the basis of its local equilibrium (39) gives the following result [32], [33], [51], [58].

Corollary 3 (The Cauchy formula for the stress vector) *The stress vector $s_{(n)}^i = s_{(n)}^i(x^j, t)$ is a linear function of the normal vector $n^i = n^i(x^j, t)$ to the surface $\partial\Sigma(t)$*

$$s_{(n)}^i = T^i_j n^j. \quad (40)$$

Here the coefficients of the representation (40) form a tensor $T^{ij} = T^{ij}(x^k, t)$ which is called the fluid stress tensor.

In general the second rank fluid stress tensor $T^{ij}(x^k, t)$ has no algebraic symmetry in its indices [32], [60].

As a result of the Cauchy formula (40) one derives the Cauchy equation of motion [32], [33], [51], [58].

Theorem 5 (The Cauchy equation of motion) *The Cauchy equation of motion*

$$\rho \frac{du^i}{dt} = \rho F^i + \frac{\partial T^{ij}}{\partial x^j} \quad (41)$$

is necessary and sufficient for the linear momentum of any region following the fluid particle paths to be conserved.

Proof. The equation (41) follows from the observation that the Cauchy formula for the stress tensor (40) and the Gauss theorem enable one to rewrite the Principle of conservation of linear momentum (38) as

$$\int_{\Sigma(t)} \rho \frac{du^i}{dt} dV = \int_{\Sigma(t)} \left(\rho F^i + \frac{\partial T^{ik}}{\partial x^k} \right) dV. \quad (42)$$

Now necessity follows immediately upon substitution of the equation of motion (41) into (42). Sufficiency can be easily proved from the conservation of linear momentum (42) by taking into account that the fluid region $\Sigma(t)$ is arbitrary. **QED**

The Cauchy equation of motion is the most general equation valid for any fluid motion which preserves the linear momentum of its each region regardless of the form of the fluid stress tensor. From the point of view of Newtonian mechanics the Cauchy equation of motion is the Newton second law of motion stating that the acceleration $a^i(x^j, t)$ (13) of a fluid particle with the mass ρdV is due to the resulting force of an external force $\rho F^i dV$ and an internal force $(\partial T^{ij} / \partial x^j) dV$ acting on the particle. In such a setting the equation of motion (41) is a system of the first order partial differential equations to be solved to find the functions of the components of the fluid velocity $u^i(x^j, t)$. The equation involves the density function $\rho(x^i, t)$ which satisfies the equation of continuity (31). Thus,

The Cauchy equation of motion for the fluid velocity u^i

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \rho F^i + \frac{\partial T^{ij}}{\partial x^j}, \quad (43)$$

The equation of continuity for the fluid density ρ

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j} (\rho u^j) = 0, \quad (44)$$

can be considered as a system of four first order partial differential equations for four unknown functions of the components of the fluid velocity $u^i(x^j, t)$ and the fluid density $\rho(x^i, t)$ provided the external forces $F^i(x^j, t)$ and the fluid stress tensor $T^{ij}(x^k, t)$ are given together with the initial and boundary conditions for the velocity and density. The dynamics of a fluid is expected to be reconstructed on the basis of solving this system.

Upon finding the fluid velocity $u^i(x^j, t)$, the actual equation motion of the fluid particles, either $x^i = x^i(\xi^j, t)$, or $\xi^j = \xi^j(x^i, t)$, is to be found by solving (10).

The Cauchy equation of motion (41) or (43) can be written in another form by introduction of the total acceleration vector $A^i = A^i(x^j, t)$ which represents the combined effect of the inertial force acting on a fluid particle through its acceleration $a^i = a^i(x^j, t)$ (13) and the effect of the external force F^i ,

$$A^i = \frac{du^i}{dt} - F^i \equiv a^i - F^i. \quad (45)$$

The Cauchy equation of motion then reads

$$\rho A^i = \frac{\partial T^{ij}}{\partial x^j} \quad (46)$$

which is transparent from the physical point of view stating that the total acceleration of a fluid particle is determined by action of the internal force on the fluid particle.

10 The Kinematics of a Moving Fluid Particle

A fluid particle moving in accordance with the equations (43) and (44) is subjected to changes in its shape, the value of its volume and it generally rotates with respect to its initial, or any other positions $\{x^i\}$ at a time t , taken as a reference point. Since it is the motion of a fluid particle under influence of the internal and external forces (43) that affects its state, such changes are due to its changing velocity in dependence of a position and a moment of time. Let us consider the tensor $\partial u_i / \partial x^j$ constructed from the fluid velocity vector $u_i = \delta_{ij} u^j$, which describes the spatial variations in the velocity of a fluid particle. This is a second rank tensor has no algebraic symmetry in its indices and therefore it can be generally decomposed into the following form:

$$\frac{\partial u_i}{\partial x^j} = \theta_{ij} + \omega_{ij}, \quad (47)$$

where the symmetric tensor $\theta_{ij} = \theta_{ij}(x^k, t)$ called the deformation or rate of strain tensor is defined as

$$\theta_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right), \quad \theta_{ij} = \theta_{ji}, \quad (48)$$

and the antisymmetric tensor $\omega_{ij} = \omega_{ij}(x^k, t)$ called the vorticity tensor is defined as

$$\omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right), \quad \omega_{ij} = -\omega_{ji}. \quad (49)$$

In many case it is convenient to use the vorticity vector ω^i ,

$$\omega^i = \frac{1}{2} \varepsilon^{ijk} \omega_{jk}, \quad (50)$$

instead of the vorticity tensor ω_{ij} . It is easy to show that $\omega_{ij}\omega^j = 0$.

The geometrical meaning of the deformation and the vorticity tensors are given by the following theorem [32], [33], [51] due to Cauchy [61] and Stokes [35].

Theorem 6 (The Cauchy-Stokes decomposition theorem) *An arbitrary instantaneous state of a fluid particle moving along its path may be resolved at each position $\{x_{(P)}^i\}$ as a superposition of a uniform velocity of translation $u_{i(P)} = u_i(x_{(P)}^k, t)$, a dilatation $\partial [x^i x^j \theta_{(P)ij}] / \partial x^k$ along three mutually perpendicular axes and a rigid rotation $\varepsilon_{ijk} x^j \omega_{(P)}^k$ of these axes,*

$$u_i(x^k, t) = u_{i(P)} + \frac{1}{2} \frac{\partial}{\partial x^i} [x^j x^k \theta_{(P)jk}] + \frac{1}{2} \varepsilon_{ijk} x^j \omega_{(P)}^k + \mathcal{O}(x^2). \quad (51)$$

Here $\theta_{(P)ij} = \theta_{ij}(x_{(P)}^k, t)$ is the deformation tensor at $\{x_{(P)}^i\}$, $\omega_{(P)}^k = \omega_{(P)}^k(x_{(P)}^i, t)$ is the vorticity vector at $\{x_{(P)}^i\}$.

Proof. Let us consider the velocity field $u^i(x^k, t)$ of a moving fluid particle in a neighborhood of its position $\{x_{(P)}^i\}$ at a time t . Since the velocity field is continuous and differentiable, one has a Taylor expansion of the velocity functions near $\{x_{(P)}^i\}$ which takes the following form with using the decomposition (47)

$$u_i(x^k, t) = u_{i(P)} + \theta_{(P)ik} x^k + \omega_{(P)ik} x^k + \mathcal{O}(x^2). \quad (52)$$

The second term in (52) can be written as

$$\theta_{(P)ik} x^k = \frac{1}{2} \frac{\partial}{\partial x^i} [x^j x^k \theta_{(P)jk}]. \quad (53)$$

This term represents a velocity field normal at each point to the quadric surface $x^j x^k \theta_{(P)jk} = \text{const}$ which contains the point $\{x_{(P)}^i\}$. Since a symmetric tensor also possesses three mutually perpendicular eigenvectors, the eigenvalues of the deformation tensor θ_{ij} measure the rates of extension per unit length of a fluid particle at $\{x_{(P)}^i\}$ in directions of the eigenvectors which can be always taken as three basis vectors for the velocity field (53). By replacing the vorticity tensor by its vector (50) the last term in (52) it takes the form

$$\omega_{(P)ik} x^k = \frac{1}{2} \varepsilon_{ijk} x^j \omega_{(P)}^k \quad (54)$$

with its immediate interpretation as representing a rigid rotation with an angular velocity $\omega_{(P)}^k/2$. **QED**

Thus, the deformation tensor θ_{ij} determines the distortion of the element of fluid occupied by a fluid particle along the principal axes defined by the eigenvectors of the tensor θ_{ij} , namely a change in the fluid particle's volume and shape. The vorticity tensor ω_{ij} , or vector ω^i , determines a rigid rotation of a fluid particle with keeping its volume and shape the same. The rotation axis is defined by the direction of the vorticity vector ω^i .

There are two important results following from the Cauchy-Stokes theorem.

Corollary 4 (The rate of change of the fluid path arc) *The rate of change in the squared element of arc dx^i , $ds^2 = \delta_{ik}dx^i dx^k$, following a fluid particle path is given*

$$\frac{d}{dt}(ds^2) = 2\theta_{ik}dx^i dx^k. \quad (55)$$

Corollary 5 (The rigid motion) *For the rigid motion of a fluid when*

$$\theta_{ij}(x^k, t) = 0 \quad (56)$$

everywhere, the fluid velocity $u^i(x^k, t)$ is determined everywhere by

$$u^i = \frac{1}{2}\varepsilon_{ijk}x^j\omega^k + \text{const.} \quad (57)$$

It is convenient to work sometimes with the trace-free deformation tensor which is called the shear tensor $\sigma_{ij} = \sigma_{ij}(x^k, t)$,

$$\sigma_{ij} = \theta_{ij} - \frac{1}{3}\delta_{ij}\theta, \quad \delta^{ij}\sigma_{ij} = 0, \quad (58)$$

where $\theta = \theta(x^k, t)$ is the scalar of expansion,

$$\theta = \delta^{ij}\theta_{ij} \equiv \frac{\partial u^i}{\partial x^i}. \quad (59)$$

The shear tensor determines the distortion of a fluid particle along the principal axes defined by its eigenvalues leaving the volume of the particle unchanged. The magnitude of shear σ is determined by $\sigma = |\sigma^2|^{1/2}$,

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij}, \quad \sigma^2 \geq 0, \quad (60)$$

where $\sigma^{jl} = \delta^{ij}\delta^{kl}\sigma_{ik}$. One can easily prove that $\sigma = 0$ if and only if $\sigma_{ij} = 0$.

The scalar of expansion θ determines the change of the volume of a fluid particle keeping its (spherical) form unchanged. It is useful to define a representative scalar $R = R(x^k, t)$ of a fluid particle by the equation

$$\frac{1}{R} \frac{dR}{dt} = \frac{1}{3}\theta. \quad (61)$$

If the expansion scalar vanishes everywhere (17), the fluid is incompressible. If the expansion scalar (59) does not vanish identically for a fluid distribution (18), the fluid is compressible.

The magnitude of vorticity ω is determined by $\omega = |\omega^2|^{1/2}$,

$$\omega^2 = \frac{1}{2}\omega^{ij}\omega_{ij}, \quad \omega^2 \geq 0, \quad (62)$$

where $\omega^{jl} = \delta^{ij}\delta^{kl}\omega_{ik}$ and it is easy to show that $\omega^2 = \omega^i\omega_i$. One can prove that $\omega = 0$ if and only if either $\omega_{ij} = 0$ or $\omega^i = 0$.

11 The Conservation of Energy

On the basis of the Principles of conservation of mass (28) and linear momentum (37) one can establish the transfer equation [32], [51] for the kinetic energy of an arbitrary moving fluid region $\Sigma(t)$

$$\mathcal{K}(t) = \frac{1}{2} \int_{\Sigma(t)} \rho \delta_{ij} u^i u^j dV. \quad (63)$$

Theorem 7 (The energy transfer equation) *For any region $\Sigma(t)$ following the particle paths of the fluid which conserves mass (28) and linear momentum (37) the rate of change of its kinetic energy (63) is equal to the rate at which work is being done on the region by external forces reduced by the work which is being done to change the volume and the shape of the region and by the work which is being done to rotate the region*

$$\frac{d}{dt} \mathcal{K}(t) = \int_{\Sigma(t)} \rho \delta_{ij} u^i F^j dV + \oint_{\partial \Sigma(t)} \delta_{ij} u^i s_{(n)}^j dS - \int_{\Sigma(t)} \theta_{ij} T^{(ij)} dV - \int_{\Sigma(t)} \omega_{ij} T^{[ij]} dV. \quad (64)$$

Proof. The energy transfer equation (64) can be easily derived by using (35), the Cauchy equation of motion (41) and a decomposition of the fluid stress tensor T^{ij} into its symmetric and antisymmetric parts to produce the last term in accordance with the definitions of the fluid deformation tensor θ_{ij} (47) and vorticity tensor ω_{ij} (49). **QED**

The last two terms involving the interaction of the symmetric and antisymmetric parts of the fluid stress tensor with the deformation and vorticity tensors can be considered as a “dissipation” part. This part of the power in (64) may well be recoverable, but the rest must be accounted for as heat.

Considering now a class of chemically inert fluids without diffusion the total energy $\mathcal{T}(t)$ of an arbitrary moving fluid region $\Sigma(t)$ is defined as a sum of its kinetic energy $\mathcal{K}(t)$ (63) and its internal energy $\mathcal{E}(t)$,

$$\mathcal{T}(t) = \mathcal{K}(t) + \mathcal{E}(t) = \frac{1}{2} \int_{\Sigma(t)} \rho \delta_{ij} u^i u^j dV + \int_{\Sigma(t)} \rho E dV, \quad (65)$$

where $E = E(x^i, t)$ is the specific internal energy. The thermodynamical interpretation of E is different for the compressible and incompressible fluids, (18) and (17) correspondingly. Indeed, for the compressible fluid its density is a function, $\rho(x^i, t) \neq \text{const}$, due to the continuity equation (31). Then it is naturally to consider the specific internal energy E as a thermodynamical state variable satisfying

The Postulate of thermodynamical state for the compressible fluids

$$TdS = dE + pd\tau, \quad \tau = \frac{1}{\rho}. \quad (66)$$

Here T is the absolute temperature, S is the specific entropy, p is the pressure, and τ is the specific fluid volume. Due to this equation of the thermodynamic state (66) the fluid pressure p is a thermodynamical variable and there exists in general an equation of state

$$p = p(\rho, S) \quad \text{or} \quad \rho = \rho(p, T). \quad (67)$$

The equation of state is usually chosen or derived on the basis of a phenomenological macroscopic and/or a microscopic physical model of the fluid substance.

For the incompressible fluid $\rho(x^p, t) = \text{const}$ due to the continuity equation (31) and the pressure cannot be a thermodynamical variable. In this case the hydrodynamic pressure is not defined as a function of the state of the fluid. Taking one or another definition of pressure here, one should remember that it may not be necessarily the physical pressure measured in experiments. The freedom in choice of the pressure is usually utilized to simplify the equations of motion.

The Postulate of thermodynamical state for the incompressible fluids

$$TdS = dE, \quad \rho \equiv \text{const}. \quad (68)$$

In general, due to possible internal dissipation processes a fluid can allow a heat transfer which is described by a heat flux vector $q^i = q^i(x^j, t)$ with the physical dimension of energy per unit area per unit time. The fundamental postulate of conservation of total energy reads as follows [32], [33], [51].

The First law of thermodynamics for fluids (The Postulate of conservation of total energy) *The rate of change of total energy $\mathcal{T}(t)$ (65) of an arbitrary fluid region $\Sigma(t)$ is equal to the rate at which work is being done on the region plus the rate at which heat is conducted into the region*

$$\frac{d}{dt} [\mathcal{K}(t) + \mathcal{E}(t)] = \int_{\Sigma(t)} \rho \delta_{ij} u^i F^j dV + \oint_{\partial\Sigma(t)} \delta_{ij} u^i s_{(n)}^j dS - \oint_{\partial\Sigma(t)} q^i dS_i. \quad (69)$$

The following fundamental theorem establishing the total energy equation now can be proved.

Theorem 8 (The total energy equation) *The total energy equation*

$$\rho \frac{dE}{dt} = \theta_{ij} T^{(ij)} + \omega_{ij} T^{[ij]} - \frac{\partial q^i}{\partial x^i} \quad (70)$$

is necessary and sufficient for the First law of thermodynamics for fluids (69) to be satisfied for any region which follows the fluid particle paths as that the energy transfer equation (64) for the rate of change of its kinetic energy holds.

Proof. Using the definition of the total energy (65) contained in an arbitrary fluid region $\Sigma(t)$, Eq. (35), the Gauss theorem and the kinetic energy transfer equation (64), the First law of thermodynamics for fluids (69) takes the form

$$\frac{d}{dt}\mathcal{E}(t) = \int_{\Sigma(t)} \rho \frac{dE}{dt} dV = \int_{\Sigma(t)} \left(\theta_{ij} T^{(ij)} + \omega_{ij} T^{[ij]} - \frac{\partial q^i}{\partial x^i} \right) dV. \quad (71)$$

Now the necessity of the total energy equation (70) to satisfy Eq. (71) is evident. The sufficiency can be easily seen from (71) as a result of arbitrariness of the moving fluid region $\Sigma(t)$. **QED**

The total energy (70) in the case of a symmetric fluid stress tensor, $T^{[ij]} = 0$, is due to Neumann [62]. Further assumptions on the form of the fluid stress tensor T^{ij} , the heat flux vector q^i and the equation of state (67) relating the thermodynamical variables should be made to apply the First law of thermodynamics for fluids (69) for the resolution of the energy balance for a particular moving fluid.

The fundamental postulate of the production of entropy can be formulated for moving fluids in the following form [32], [33], [51].

The Second law of thermodynamics for fluids (The Postulate of increasing entropy) *The rate of increase of entropy of an arbitrary fluid region $\Sigma(t)$ should not be less than the heat conducted into the region divided by the temperature at which the heat transfer occurs*

$$\frac{d}{dt} \int_{\Sigma(t)} \rho S dV \geq - \oint_{\partial\Sigma(t)} \frac{1}{T} q^i dS_i. \quad (72)$$

Theorem 9 (The local increase of entropy) *The following inequality for the local increase of entropy*

$$\rho \frac{dS}{dt} \geq - \frac{1}{T} \frac{\partial q^i}{\partial x^i} + \frac{q^i}{T^2} \frac{\partial T}{\partial x^i} \quad (73)$$

is necessary and sufficient for the Second law of thermodynamics for fluids (72) to be satisfied for any region which follows the fluid particle paths with the conservation of its mass (28).

Proof. Using (35) and the Gauss theorem, the Second law of thermodynamics for fluids (72) takes the form

$$\int_{\Sigma(t)} \left(\rho \frac{dS}{dt} + \frac{1}{T} \frac{\partial q^i}{\partial x^i} - \frac{q^i}{T^2} \frac{\partial T}{\partial x^i} \right) dV \geq 0. \quad (74)$$

Now the necessity of the inequality (73) for the local increase of entropy (72) to ensure the integral inequality (74) follows because an integral of a function everywhere nonnegative is nonnegative. The sufficiency can be easily seen from (74) as a result of arbitrariness of a moving fluid region $\Sigma(t)$, which forces the integrand to be nonnegative everywhere. **QED**

12 The Fluid Stress Tensor and the Conservation of Angular Momentum

In general, the second rank tensor $T^{ij}(x^k, t)$ has no algebraic symmetry in its indices. From a physical point of view that means that a fluid can be generally subjected to extraneous torques due to action of the external forces $F^i(x^j, t)$. Such fluids are called polar and they are capable of transmitting extraneous stress couples and carrying the body torques [32], [51], [60]. If a fluid can only admit the torques arising from the moments of direct forces, it is called nonpolar. Thus, certain assumptions concerning the nature of the forces exerted across fluid surface elements will specify the properties of the stress tensor. Let us accept the following postulate [32], [60] about the fluid stress tensor.

The Boltzmann Postulate *The fluid stress tensor $T^{ij} = T^{ij}(x^k, t)$ is symmetric*

$$T^{ij} = T^{ji}. \quad (75)$$

One can prove an important theorem [32], [51], [58] that fluids with a symmetric stress tensor $T^{ij}(x^k, t)$ (75) conserve the angular momentum $L_i(t)$ of an arbitrary fluid region $\Sigma(t)$

$$L_i(t) = \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j u^k dV. \quad (76)$$

The Principle of conservation of angular momentum *The rate of change of angular momentum $L_i(t)$ (76) of an arbitrary fluid region $\Sigma(t)$ is equal to the torque due to the total resultant force acting on the fluid volume within the region*

$$\frac{d}{dt} L_i(t) = \frac{d}{dt} \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j u^k dV = \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j F^k dV + \oint_{\partial \Sigma(t)} \varepsilon_{ijk} x^j s_{(n)}^k dS. \quad (77)$$

Theorem 10 (The conservation of angular momentum) *For any fluid moving with conservation of mass (28) and linear momentum (37) the Boltzmann postulate for the fluid stress tensor (75) is the necessary and sufficient condition for the fluid to conserve also the angular momentum (77) for any region following the fluid particle paths.*

Proof. By using (35) and the Cauchy equation of motion (41) one can show that the rate of change of the angular momentum (76) can be expressed in the following form:

$$\begin{aligned} \frac{d}{dt}L_i(t) &= \frac{d}{dt} \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j u^k dV = \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j \frac{d}{dt} u^k dV = \\ &= \int_{\Sigma(t)} \rho \varepsilon_{ijk} x^j F^k dV + \oint_{\partial\Sigma(t)} \varepsilon_{ijk} x^j s_{(n)}^k dS - \int_{\Sigma(t)} \varepsilon_{ijk} T^{jk} dV. \end{aligned} \quad (78)$$

Since $\varepsilon_{ijk} T^{jk}(x^l, t) \equiv 0$ if the fluid stress tensor is symmetric, the Boltzmann postulate (75) necessarily brings the Principle of conservation of angular momentum (77). Sufficiency of the Boltzmann postulate follows from (77) and (78) due to arbitrariness of the fluid region $\Sigma(t)$ when the vanishing of the last integral in (78) requires $\varepsilon_{ijk} T^{jk}(x^j, t) \equiv 0$ everywhere.

QED

The interpretation of the components of a symmetric fluid stress tensor is well-known (see for example, [32], [33], [51]).

13 The Navier-Stokes Equation and Its Physical Status

The different physical hypotheses about the nature of moving fluids and forces causing their motion bring different analytical forms for the fluid stress tensor. As a result, historically various forms of the Cauchy equation of motion (43) have been discovered to describe the dynamics of various fluids. By restriction to the class of fluids without heat transfer,

$$q^i(x^j, t) = 0, \quad T = \text{const}, \quad (79)$$

there are three forms of the equations of motion of interest and relevance here.

The Euler equation of motion for the perfect fluid $T^{ij} = -\delta^{ij} p$

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \rho F^i - \delta^{ij} \frac{\partial p}{\partial x^j} \quad (80)$$

[32], [51], [54]. Here $p = p(x^j, t)$ is the fluid pressure which depends on the thermodynamical state of fluid. It can be positive, $p > 0$, which tends to compress a fluid particle, and negative, $p < 0$, which results in an expansion of a fluid particle. To close the system of the four first order partial differential equations (44) and (80) for four unknown functions of the components of the fluid velocity $u^i(x^j, t)$ and the density $\rho(x^i, t)$ provided the external forces $F^i(x^j, t)$ and the initial and boundary conditions for the velocity and density are given, one must specify in addition to the system the of state (67) which does not depend on temperature due to (79)

The equation of state

$$p = p(\rho) \quad \text{or} \quad \rho = \rho(p). \quad (81)$$

The Stokes equation of motion for the Stokean fluid $T^{ij} = (-p + \alpha)\delta^{ij} + \beta\theta^{ij} + \gamma\delta_{kl}\theta^{ik}\theta^{jl}$

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \rho F^i - \delta^{ij} \frac{\partial p}{\partial x^j} + \frac{\partial}{\partial x^j} (\beta\theta^{ij}) + \delta_{kl} \frac{\partial}{\partial x^j} (\gamma\theta^{ik}\theta^{jl}) \quad (82)$$

[32], [35], [51]. Here $\alpha = \alpha(x^i, t)$, $\beta = \beta(x^i, t)$ and $\gamma = \gamma(x^i, t)$ are the scalar functions which depend on the thermodynamical state of fluid as well as on the principal invariants of the symmetric deformation tensor $\theta^{ij} = \theta^{ij}(x^k, t)$, $\theta^{ij} = \delta^{ik}\delta^{jl}\theta_{kl}$ (48). They describe definite types of viscous response of the fluid. For example, a choice of p , α , β and γ so that T^{ij} is linear in θ^{ij} provides the classical Cauchy-Poisson law of viscosity. The Stokean fluid is the most general type of a nonelastic fluid. To close the system of Eqs. (44), (81) and (82) for four unknown functions of the components of the fluid velocity $u^i(x^j, t)$ and density $\rho(x^i, t)$ provided the external forces $F^i(x^j, t)$ and the initial and boundary conditions for the velocity and the density are given, one must specify in addition the viscosity functions $\alpha(x^i, t)$, $\beta(x^i, t)$ and $\gamma(x^i, t)$.

The Navier-Stokes equation of motion of classical hydrodynamics for the Newtonian fluid $T^{ij} = \delta^{ij}(\lambda\theta - p) + 2\mu\theta^{ij}$

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \rho F^i - \delta^{ij} \frac{\partial p}{\partial x^j} + \delta^{ij} \frac{\partial}{\partial x^j} (\lambda\theta) + 2 \frac{\partial}{\partial x^j} (\mu\theta^{ij}) \quad (83)$$

[32], [34], [35], [51]. Here $\lambda = \lambda(x^i, t)$ and $\mu = \mu(x^i, t)$ are the scalar functions of the first and second coefficients of viscosity. For the case of constant viscosity coefficient, $\lambda(x^i, t) = \text{const}$ and $\mu(x^i, t) = \text{const}$, the Navier-Stokes equations becomes much simpler

$$\rho \left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = \rho F^i - \delta^{ij} \frac{\partial p}{\partial x^j} + (\lambda + \mu) \delta^{ij} \frac{\partial^2 u^k}{\partial x^j \partial x^k} + \mu \delta^{kl} \frac{\partial^2 u^i}{\partial x^k \partial x^l}. \quad (84)$$

Its further simplification for the case of incompressible fluid (17) has been especially thoroughly studied in connection with the phenomenon of fluid and gas turbulence [19], [36]-[38], [41], [48].

The Newtonian fluid is a linear Stokean fluid with the Cauchy-Poisson law of viscosity. For a Newtonian fluid its stress tensor depends linearly on the rate of fluid deformation. It is generally believed that the Newtonian fluid is an adequate physical model when the fluid deformations are relatively small in comparison with, say, the ratio of some reference speed and reference length. However, this physical hypothesis is not well understood as yet

[32]. It is not to be derived from experiments, nor can it be proved by abstract modelling. The Navier-Stokes equation (83) is the dynamical equation of classical hydrodynamics. Since their discovery they successfully describe various regimes of fluid motion including turbulence for a wide class of physical gaseous and liquid substances.

A closed system of the equations of classical hydrodynamics without heat transfer includes Eqs. (44), (81) and (84) for four unknown functions of the components of the fluid velocity $u^i(x^j, t)$ and the fluid density $\rho(x^i, t)$ provided the external forces $F^i(x^j, t)$, the viscosity coefficient functions $\lambda(x^i, t)$ and $\mu(x^i, t)$ and the initial and boundary conditions for the velocity and the density are given.

The mathematical difficulties in solving the Navier-Stokes equation are well known, see [41] for a discussion and further references. Because the 3-dimensional Navier-Stokes equation is not a well-posed partial differential system, no theorem of existence of smooth, physically reasonable solutions is not known as yet. These problems originate in the nonlinear structure of the Euler equation (80) and the Navier-Stokes equation (84) due to the presence of the so-called inertial term $u^j \partial u^i / \partial x^j$. These equations are among the very few equations of mathematical physics for which the nonlinearity arises not from the physical attributes of the system, such as a physical interaction, but rather from the kinematical aspect of fluid motion. Another source of the difficulties for the Navier-Stokes equation is due to the viscosity term which brings the second order operator into the first order equations, which involves the singular perturbations.

As far as the physical status of the Navier-Stokes equation is concerned it must be regarded as the macroscopic equation to be able to describe the dynamics of fluids considered as continuous physical substances. The flow of fluids at the microscopic level is governed by the statistical mechanics of fluids. The appropriate statistics is given by the solution of the Boltzmann equation which represents the evolution of the distribution function in dependence on positions and velocities of colliding particles as a result of thermal excitation. The collisions are described by the so-called collision integral which, in principle, contains all information about interaction between particles and it forms the collision operator. One must evaluate this operator to obtain a set of macroscopic equations for one or another physical configuration of interacting (colliding) particles. Only for the case of dilute gases when one is limited to the case of the evolution of a single particle distribution with only binary collisions taken into account it has been possible to find a satisfactory solution given by Chapman [63] and Enskog [64]. It has been shown that the Boltzmann equation with this collision operator gives the equation of the conservation of mass and the equation of the conservation of linear momentum which is the Navier-Stokes equation for an incompressible fluid [65]. Although such a derivation has been carried for dilute gases, a derivation for liquid fluids when the binary collisions are almost of negligible significance and all liquid physics is due to collisions between particle clusters is still an open problem. In this situation the Navier-Stokes equation (84) is taken as a reasonable model for liquid flows at least for the simple constant viscosity.

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