Fluid Mechanics in Membrane Filtration: A Simplified Analytical Approach

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Abstract
The filtration processes using membrane such as reverse osmosis, ultrafiltration, and microfiltration are commonly used in water desalination, paper, and food industry. In pressure driven membrane processes, fluid mechanics plays an important role and has great influence on the motion of suspended and dissolved solutes. In this paper a simplified analytical solution has been presented for the fluid mechanics of pressure driven membrane filtration for desalination of water. The solution is presented for wall Reynolds number, variable permeation flux, and incomplete solute rejection. Boundary conditions are used to numerically solve Navier Stokes and diffusion equations. Different plots for velocity, membrane concentration, and pressure variations have also been provided.

Keywords: *Concentration polarization, reverse osmosis, modeling, membranes, simulation.*

1. Introduction
Fluid flow analysis through porous media is linked to all engineers and scientists whose work involves transport phenomena. The transport may be mass, momentum or energy across porous media. Its wide range of applications cover various disciplines that makes fluid flow analysis of porous media to be an important domain in order to know the complete structure of the flow fields and to improve the design and the operation of membrane modules.
Reverse osmosis (RO) based desalination is one of widely use application of fluid flow in porous media. The process work by forcing the saline water from high concentration solute region to low concentration solute through semi-permeable membrane. The driving force is pressure that is accomplished by high pressure pump that overcome the osmotic pressure and force the water through membrane. It is a pressure driven process in which no heating or phase change is taking place. A schematic diagram describing the fluid velocity and pressure variations in RO process is shown in Fig. 1. The feed seawater is passed through high pressure pump, where its pressure increased from P_0 to P_H. This pressurized volume of feed is passed through semi-permeable membrane which separate the permeate (V_P) from saline water.

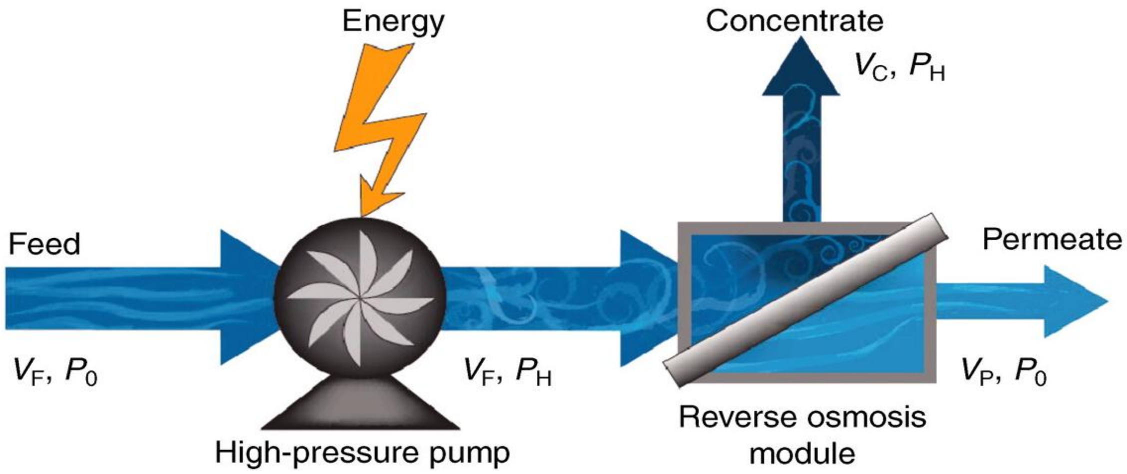

Figure 1. RO process.

The filtration process can be categorized as dead end or cross flow filtration. In dead-end filtration the fluid flows perpendicular to the membrane that allows permeate to pass through it and all the particles that are bigger than the

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pore size are retains on the membrane surface. The retained particles continue to increase on the surface of membrane and result in cake formation that hinders the purification potential of membrane and reduce filtration efficiency. The problem of cake formation is less with cross flow filtration where the fluid flow tangential to the membrane. The tangential velocity wash away most of the bigger particles on the membrane surface and prevent the formation of thicker layer of cake.

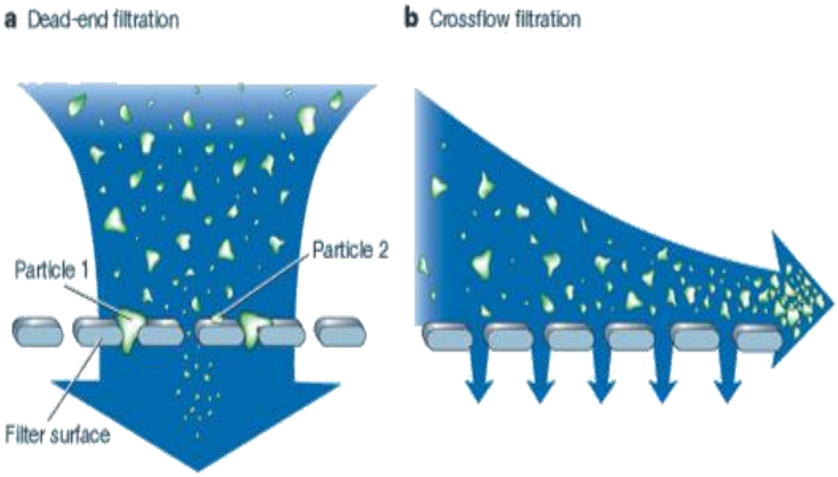


Figure 2. Filtration types

The concerned problem involve the cross flow filtration of the fluid in tabular membrane. The fluid enters at high pressure (P) and the outside tube pressure is P_o(atmospheric pressure).The pressure difference across the membrane is the driving force for the permeate to flow in outward direction. The permeate flow across membrane results in the decrease of flow rate along the tube. As fluid flows along the pipe, lower permeate flow is experienced due to the resistance caused by solute accumulation near the wall of membrane.
In this paper firstly comprehensive literature survey have been conducted for different approaches related to the solutions for problems related to porous media. Then using basic equations of flow and applying different boundary conditions, flow through membrane for water desalination have been analyzed. Simplified analytical solution of fluid flow in a cross flow tabular membrane have been presented for radial and axial velocity and pressure distribution along the membrane. Also, fluent environment have been used to determine pressure drop and velocity distribution.

2. Literature review

2.1 Porous pipe flow

There are two classes of theoretical solutions for laminar, in-compressible and steady flow with constant inlet and outlet as concluded by Terrill [1]. The one is fully developed flow for constant suction or injection. White Jr. [2] showed that this type of solution can be reduced to differential equation of ordinary nature at the suction as a function of Reynolds number. The entrance region solution is a second type of solution for which complete numerical approach is required to solve the problem of the velocity profile.
The pressure differential technique is used for the membrane filtration application for water desalination systems i.e. pressure gradient is main driving force for this process. As fluid strikes the porous media, smaller components compared to pores passes through media and components which are lager, are separated by membrane which forms layer across the membrane causing more resistance to the flow of fluid (this is called as concentration polarization).As polarization occurs there may be some change in transport properties of fluid or solute such as viscous effect. Berman [3] derived the expression for velocity components of fluid for flow through membrane (porous media) for small Reynolds number considering no slip condition at walls. Saffman [4] concluded that the velocity of slip is proportional to rate of shear.

$$u_{slip} = -\frac{\sqrt{k}}{\alpha}\frac{\partial u}{\partial y} + O(k) \qquad \textbf{(1)}$$

In all tangential flow membrane systems, $u_w << u_o$. Therefore axial axis diffusion will be small as compared to transverse axis diffusion so, that it can be neglected [5]. Importance information regarding viscosity and diffusion effects on transport properties is provided by Schmidt number, (Sc = u/D). For most applications for membrane filtration Schmidt number is order of 10^3 which is an indication of small fraction of concentration boundary layer and that the remaining entire channel is filled by the layer of momentum (For small region at entrance this is not true). Due to polarization across the permeable boundary there occurs non-uniform hydraulic resistance for solvent flow. Thus, the suction rate decreases along with the length of channel once a membrane structure is operational. As the magnitude of the coefficient of slip increases, shear rate at the porous media decreases and increase in concentration of polarization can be observed [5].

For free membrane surface negligible role is played by the velocity of slip (for permeability's of order of 10 – 17). However, for membranes coated with a porous deposit, slip velocity may be considered for flow modeling.

According to flow visualization study of Dybbs and Edwards for Reynolds number above a few hundred, flow of fluid in porous medium exhibits turbulent behavior. Antohe and Lage [6] derived two-equation turbulence model by taking Reynolds-averaging over volume-averaged for flow of macroscopic section. Masuoka by taking into consideration of averaging the transport equation for turbulence (turbulence two-equation model) derived an equation for turbulence of macroscopic transport. The micro turbulence model governing equations for flow are integrated over "control volume" for the purpose of obtaining a turbulence macroscopic model for a porous structure.

The inertial forces & effects of solid wall boundary on flow of fluid through porous medium are mostly neglected depending on Darcy law for purpose of mathematical formulation for porous media flow. For increased porosity and near wall these two effects become significant so the Darcys law becomes in-valid. Brinkman solved issue by introducing viscous term to the Darcy's law, to take effect of presence of solid boundary. While for inertial forces squared velocity term introduced to Darcys law by Muskat. The major drawback of Brinkman and Muskat work is that both solid boundary and inertial effects are not considered simultaneously [9]. For higher velocities the effects of inertial forces become high enough to take into account, so drag increases.

It is observed that in a porous wall flow the drop in pressure is related to flow velocity square which is because of inertia effects and linear combination of flow. Three resistances for flow must be taken into consideration while flow through the porous media is being analyzed: a) damping resistance which occurs because of the porous medium, b) resistance because of the viscous boundary, c) inertial forces resistance.

The increase in inertial effects occurs as the permeability becomes higher and the viscosity of fluid lowers. The change in velocity with time increase close to the wall, due to the boundary effect viscous resistance increases [9].

2.2 Coupling of Darcy andNavier stroke Equation

Cross flow membrane filtration is used to separate undesirable fine particles. It is different from pressure driven dead end filtration in a sense that it flow tangentially over a porous medium. Some part of fluid penetrates through the porous media and most of it flows out of the filtration assembly which is then recirculated. One of advantage is to provide the sufficient shear to force the particles and avoids blocking of the porous media.

The linear momentum for free flow regime is expressed by Cauchy's equation that contains the viscous and convective terms. For incompressible highly viscous fluid flow analysis, due to high viscosity Reynolds number value is very small which means convective terms of Caucy's equation are neglected and it is converted to N-S Stokes equations for creeping flow [10].

In fluid flow across porous media, fluid flow analysis for free flow regime is not very difficult as this section can be analyze by using N-S equations. The area of concern is the accurate mathematical modeling of coupled free and porous regime. There are many model to analyze the couple regime but solution is not generalized. These models can provide useful information in certain specified cases but one have to develop his own model that justify the required desirable conditions.

Darcy's law is widely accepted for fluid flow analysis of incompressible viscous laminar flow in porous media with small porosity [11]. Therefore, coupled free and porous interface must be modeled by the combining Darcy and the N-S equations. Instead of widely accepted validity of Darcy's law for solving porous media problems, it faces severe criticisms because of the absence of the second-order derivatives that make it difficult to couple with NS equation. Other limitations involve the inability to specify no-slip boundary conditions at solid walls and its validity for the flow having Reynolds number close to unity. To overcome these limitations, lots of efforts are being made by various researchers that results in different modifications in conventional Darcy's law equations.

The substitution of alternative equation that have second-order derivatives in Darcy model is one of the modification that seems to resolves the issue related to differential order difficulty but we cannot use such an approach without experimental or theoretical justifications under specified conditions.

Nassehi[12] explains various methods of coupling NS-Darcy equations and also present the porous and cons of these methods. He focus the development of a scheme for viscous flow domains across permeable that help to resolve the issue of coupling the Darcy and Navier Stokes regimes under existing techniques. In this paper he develops a model for steady state laminar flow under isothermal consideration across a permeable wall. The results validate the viability of this model to be used for the analysis of coupled regime in cross flow filtration process [12].

Governing equations for steady state laminar flow under isothermal consideration across a permeable wall:

Continuity Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0 \qquad \textbf{(2)}$$

Expression for overall mass balance for an incompressible flow regime in polar coordinate is:

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \qquad \textbf{(3)}$$

Similarly for 2-D fluid flow analysis in Cartesian coordinate:

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\nabla\cdot\vec{u}=0 \quad \text{For steady incompressible fluid} \quad \frac{1}{\rho}\frac{D\rho}{Dt}+\nabla\cdot\vec{u}=0 \qquad \textbf{(4)}$$

So, $\nabla\cdot\vec{u}=0$ for 2-D steady incompressible fluid $\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$ **(5)**

Navier Stokes equations in r and z direction with incompressible axisymmetric flow with the assumption of no tangential velocity:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_r}{\partial r})+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right]+\rho g_r \qquad \textbf{(6)}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z \qquad \textbf{(7)}$$

Solution of theses governed equations is obtain by applying the following boundary conditions:
(1)Inlet feed stream velocity is known. (2)No-slip condition on solid walls & velocity surface velocity components are zero. (3) Porous regime (permeable wall) is modeled using Darcy's equations:

For cylindrical Coordinates: $\frac{\eta v_r}{K_r}+\frac{\partial p}{\partial r}=0 \quad \frac{\eta v_z}{K_z}+\frac{\partial p}{\partial z}=0$ **(8)**

For rectangular coordinates: $\frac{\eta v_x}{K_x}+\frac{\partial p}{\partial x}=0 \quad \frac{\eta v_y}{K_y}+\frac{\partial p}{\partial y}=0$ **(9)**

Above equations represent the filtrate viscosity and porous wall permeability coefficient.

Damaka[13] stated the boundary conditions by considering the fully develop flow profile at inlet and exit. He mention the studies explaining the fact that similarity assumption are not justified in every region as at the inlet fluid take some distance to fully develop its profile but for simplicity we neglect this effect up to a certain distance at the inlet due to high length to radius ratio of the tube. Hence, to justify the similarity assumption, one should consider sufficiently high L/R ratio in order to assume fully develop flow at the exit of porous tube**.**He also stated the conditions at the permeable wall, by considering the no slip condition Darcy law is used to get the wall suction velocity. Beavers and Joseph match the free and porous media flow at the coupling interface while considering the slip velocity. Singh and Laurence also did some theoretically analysis for slip velocity effect at the membrane surface. Schmitz and Prat experimental results shows that slip velocity has negligible effect at the membrane surface that justify the assumption of taking the axial velocity component equal to zero on permeable wall.

2.3 Flow in membrane

Recent studies have been made on membrane fouling as it is major issue and there is direct relationship between mass transfer and convective fluid flow. This issue can be resolved by considering solute, membrane and hydrodynamics properties while solving mass transfer, momentum and energy equations [14].
Arun et al. [14] studied the concentration polarization effect on membrane and found by their simulation results that due to low wall shear stress at membrane, concentration polarization effect is more.
Belfort [15] focused on dissolved solutes and suspended particles. He described about movement of different solutes in different mechanisms while discussing cross flow filtration. While describing hydrodynamics theories, he said that the colloidal foulants don't stick to membrane and move over the membrane with flow force so they are less resistant to the flow like other macromolecules which show the opposite behavior.

Arun et al. [14] performed simulations to analyze pressure, flow and concentration profiles in spacer-filled & open cross flow channels. They derived equation for concentration polarization and discussed about separation performance parameters.
While discussing pressure variations in open and spacer-filled channels, they found that in open channel there is pressure drop due to less momentum loss at stationary points (at entrance and exit). The spacer filled channel showed more pressure drop than open channel.
Their velocity distribution discussion shows that wall shear rates are more at inlet point and less at exit point and their analytical wall shear rates were more accurate at low Reynolds number.
Kotzev[16] performed study of mass transfer in a tube membrane. They considered both navier-stroke and diffusion rate equation with certain boundary conditions to come up with results. They found that to carry out ultrafiltration process it is necessary that membrane permeability should be less than 1.
They assumed uniform permeation flux in their case and their results also showed that permeation flux is nearly negligible in longitudinal direction. They showed by their results that permeation flux decreases due concentration polarization and membrane fouling. Their results also showed that solute concentration on the membrane surface, at the steady ultrafiltration performance, increases with Reynoldsnumber but it not has dominate influence on ultrafiltration performance.

3. Solution approach

The solution approach involve the consideration of two different flow domains. Free flow domain in a pipe is solved by Navier stokes equation and flow through porous media is solved by Darcy law , that linked instant discharge rate across porous media, fluid viscosity and the pressure drop over given length. To accomplish the desired solution, following steps are followed:

- The governing equations (Continuity, Momentum) are written in their complete form.
- Different flow considerations (steady, laminar, incompressible, viscous) are applied and governing equations are reduced to their simplest form.
- Free and porous domain equations are solved analytically to find velocity and pressure distribution.
- Suitable boundary conditions are applied to obtain the constants in the governed equations.
- Numerical solution is obtained by using fluent.

3.1 Problem formulation

The problem which has been analyzed in this paper involves steady, laminar, and incompressible, axisymmetric fluid flow in a circular channel that enters the membrane (porous media) at high pressure. This problem is solved keeping in mind the application of membrane in water desalination systems. As solute particles present in the sea water (salts etc.) cannot penetrate the porous wall, its concentration increases at membrane liquid interface (solute accumulation) which causes increase in resistance to the outer flow through the membrane leading to reduced velocity of flow. For pipe N-S equation is used for velocity distribution while velocity of flow at porous media is obtained using Darcy Law. Diffusion rate is analyzed using diffusion equation. The geometry of the flow under consideration is given in figure below: The geometry of the flow under consideration is given in Fig. 3.

Assumptions:

We are taking L/R ratio very large so the flow can be considered fully developed along the pipe including inlet and outlet.

- Transport properties such as k, μ, ρ are constant.
- Pressure driven flow.
- Porous wall is homogenous.
- Small Reynolds number.
- No slip conditions at membrane surface.
- No rotational flow.

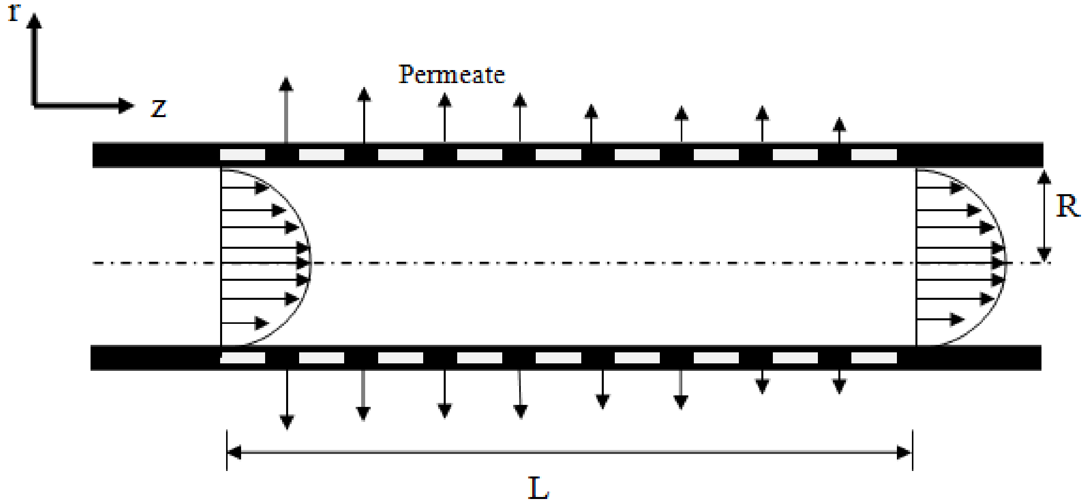


Figure 3. Geometry for flow channel.

Continuity Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0 \quad \textbf{(10)}$$

As no rotational motion, v_θ is zero.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial}{\partial z}(v_z)=0 \quad \textbf{(11)}$$

Expression for overall mass balance for an incompressible flow regime in polar coordinate is:

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \quad \textbf{(12)}$$

Momentum Equation:
r-Momentum:

$$\frac{\partial v_r}{\partial t}+(V.\nabla)v_r-\frac{1}{r}v_\theta^2=-\frac{1}{\rho}\frac{\partial p}{\partial r}+g_r+v\left\{\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right\} \quad \textbf{(13)}$$

For steady irrotational flow and neglecting the gravity in above equation:

$$(V.\nabla)v_r=-\frac{1}{\rho}\frac{\partial p}{\partial r}+v\left\{\nabla^2 v_r-\frac{v_r}{r^2}\right\} \quad \textbf{(14)}$$

$$V.\nabla=v_r\frac{\partial}{\partial r}+v_z\frac{\partial}{\partial z} \quad \textbf{(15)}$$

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{\partial^2}{\partial z^2} \quad \textbf{(16)}$$

Putting Eq. (15) and (16) in Eq. (13):

$$(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z})=-\frac{1}{\rho}\frac{\partial p}{\partial r}+v\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_r}{\partial r})+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}\right\} \quad \textbf{(17)}$$

z-Momentum:

$$\frac{\partial v_z}{\partial t}+(V.\nabla)v_z-\frac{1}{r}v_\theta^2=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g_z+v(\nabla^2 v_z) \quad \textbf{(18)}$$

Putting Eq. (15) and (16) in Eq. (18):

$$v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r})+\frac{\partial^2 v_z}{\partial z^2}\right\} \quad \textbf{(19)}$$

At Porous wall:
Using Darcy law:

$$V = -\frac{k}{\mu}(\nabla \cdot p - \rho g) \quad \textbf{(20)}$$

Neglecting the gravity effect:

$$V = -\frac{k}{\mu}(\nabla \cdot p) \quad \textbf{(21)}$$

Where,
K=intrinsic permeability
In z-direction:

$$v_z = -\frac{k}{\mu}(\frac{\partial p}{\partial z}) \quad \textbf{(22)}$$

In r-direction:

$$v_r = -\frac{k}{\mu}(\frac{\partial p}{\partial r}) \quad \textbf{(23)}$$

For the porous zone
As we are treating the free flow domain across pipe and porous flow in membrane separately, we need to introduce the source term to our standard equations that gives us the model for complete flow regime. This source term include the viscous and inertial resistance term.

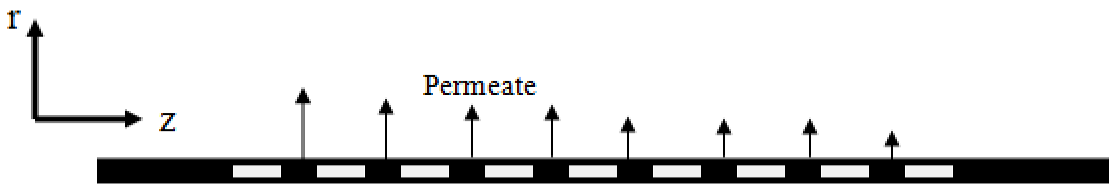
$$s_i = \frac{\mu}{\alpha} v_i + C\,\frac{1}{2}\,\rho\, v\ \ v_i \quad \textbf{(24)}$$

α = permeability
$1/\alpha$ = Viscous resistance
C = inertial resistance factor
The pressure drop across the porous zone is given by the equation below.

$$\Delta p/l = \frac{\mu}{k} v_r + C\frac{1}{2}\,\rho\,(v_r)^2 \quad \textbf{(25)}$$

$\Delta p/l$=pressure drop per unit length
$\mu = Fluid\ viscosity$
$\frac{1}{k}$=viscous resistance
C=inertial resistance

Boundary conditions:



At r=R
V_z=0 (No flow in z (axial) direction in porous wall)

$$v_r = -\frac{k}{\mu}(\frac{\partial p}{\partial r}) \quad \textbf{(26)}$$

Where,

$$\frac{\partial p}{\partial r} = \frac{p - p_o}{\Delta x} \quad \textbf{(27)}$$

$p - p_o$ is pressure difference inside and outside of membrane and Δx is thickness of membrane.

Diffusion rate:
For steady flow we have:

$$v_z \frac{\partial c}{\partial z} + v_r \frac{\partial c}{\partial r} = D\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}) + \frac{\partial^2 c}{\partial z^2}\} \quad \textbf{(28)}$$

Where,

D=diffusion coefficient

C=concentration

For porous wall: $\frac{\partial c}{\partial z} = 0$, So Eq. (28) becomes:

$$v_r \frac{\partial c}{\partial r} = D\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r})\} \quad \textbf{(29)}$$

Integrating w.r.t "r":

$$v_r c = D\frac{\partial c}{\partial r} \quad \textbf{(30)}$$

As,

$$u(z,R) = v_r \quad \textbf{(31)}$$

$$u(z,0) = 0 \quad \textbf{(32)}$$

So as a function of z and R we can write above equation for diffusion:

$$v_r(z,R)c(z,R) = D\frac{\partial c(z,R)}{\partial r} \quad \textbf{(33)}$$

Introducing rejection coefficient φ

$$\varphi v_r(z,R)C(z,R) = D\frac{\partial c(z,R)}{\partial r} \quad \textbf{(34)}$$

Where,

$$\varphi = 1 - \frac{C_p}{C_w} \quad \textbf{(35)}$$

C_p = solute concentration in permeate

C_w = solute concentration in retentate

3.2 Introduction of non-dimensionless numbers

Now we are interested to convert our problem in dimensionless form. To make dimensionless form, let us define following variables:

$$v_r^* = \frac{v_r}{v_{zo}}, \quad r^* = \frac{r}{2R}, \quad z^* = \frac{z}{2R}, \quad v_z^* = \frac{v_z}{v_{zo}}$$

$$P^* = \frac{(P_o - P)}{\rho(v_{zo}^2)}, \quad K^* = \frac{K}{2eR}$$

Introducing above defined variables into Eq. (17):

$$v_r^*\frac{\partial v_r^*}{\partial r^*} + v_z^*\frac{\partial v_z^*}{\partial z^*} = -\frac{\partial p^*}{\partial r^*} + \frac{1}{\text{Re}}\left\{\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*\frac{\partial v_r^*}{\partial r^*}) + \frac{\partial^2 v_r^*}{\partial (z^*)^2} - \frac{v_r^*}{(r^*)^2}\right\} \quad \textbf{(36)}$$

Where

$$\text{Re} = \frac{\rho v_{z,o}(2R)}{\mu} \quad \textbf{(37)}$$

From Eq. (19):

$$(v_r^*\frac{\partial v_z^*}{\partial r^*} + v_z^*\frac{\partial v_z^*}{\partial z^*}) = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\text{Re}}\left\{\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*\frac{\partial v_z^*}{\partial r^*}) + \frac{\partial^2 v_z^*}{\partial z^{*2}}\right\} \quad \textbf{(38)}$$

Similarly at porous wall:

$$v_r * v_{z,o} = -\frac{k*(2\Delta x R)}{\mu} \frac{p*\rho v_{z,o}^2}{\Delta x} \quad \textbf{(39)}$$

3.2.1 Non-dimensionless boundary conditions

At Inlet:

When z*=0 , v_r *=0

At r*=0 (Centerline):

$$\frac{\partial}{\partial r*}(v_z*) = 0 \quad \textbf{(40)}$$

At r=0.5(porous wall):

$$v_z* = 0 \quad \textbf{(41)}$$

At exit section:

When z*=L /2R , v_r *=0

When r*=0

$$\frac{\partial}{\partial z*}(v_z*) = 0 \quad \textbf{(42)}$$

When r*=0.5

V_r*=0 **(43)**

4. Solving analytical equations

Let:

Flow Parameters

- Water with kinematic viscosity $v = 10^{-6} m^2/s$
- Mean centerline velocity at inlet U_o = 0.05 m/s
- Reynolds number $U_o\ D/v$ =250
- Assumption for fully developed
- Diameter of Pipe= 30mm
- Length of pipe= 1.5m
- $\cdot K = 4.54x10^{-3}$
- Pressure at inlet of Pipe= 200 KPa

At the porous wall v_z=0 so

$$0 = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z}\right) \quad \textbf{(44)}$$

Integrating w.r.t z we have

$$p = -\frac{\mu}{k} z + c \quad \textbf{(45)}$$

At z=0 Pressure is equal to inlet pressure so we can write after using this boundary condition and putting value of C.

$$p = -\frac{\mu}{k} z + p_i \quad \textbf{(46)}$$

For radial velocity distribution we have

$$v_r = -\frac{k}{\mu}\left(\frac{p - p_o}{\Delta x}\right) \quad \textbf{(47)}$$

Equation 46, will give the pressure at the wall along the axial direction as shown in Fig. 4.It shows that the pressure is decreasing along the length of the wall. Because of this decrease in pressure, there will be slight variations in the radial velocity along the length flowing through the membrane.The variations of radial velocity along the length of the membrane is shown in Fig. 5. As it can be observed the velocity decreases along the length because of decrease in driving force (pressure difference).

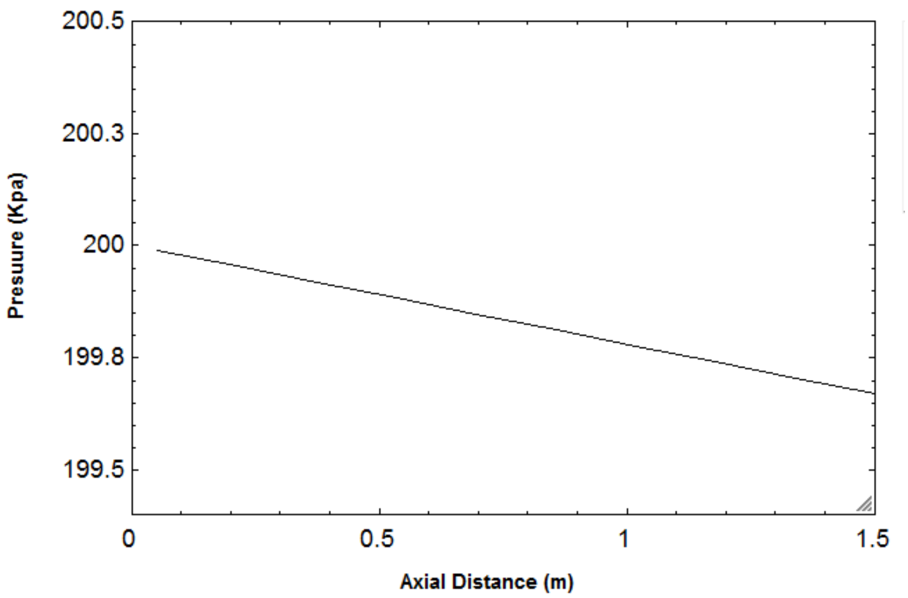


Figure 4. Pressure variations along the length at wall.

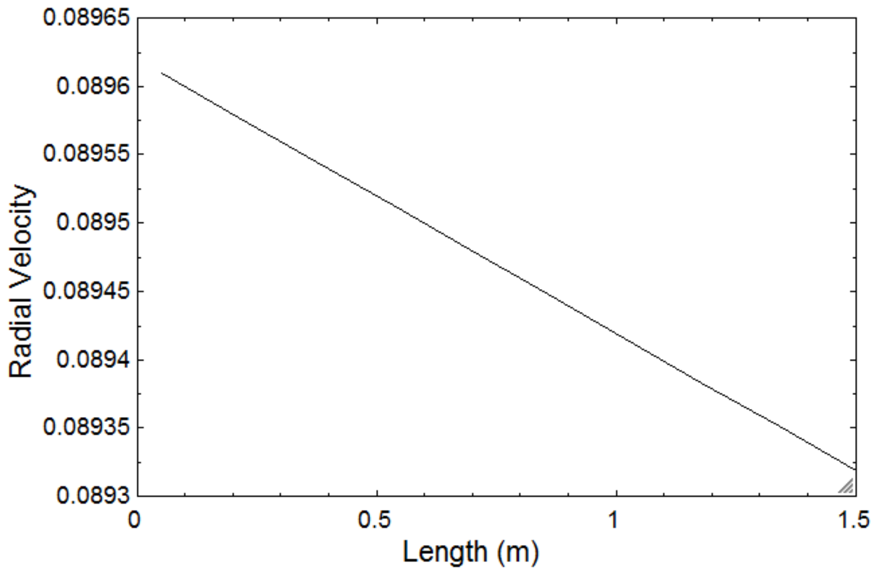


Figure 5. Radial velocity variations along the length.

For numerical solution the Eqs. (48 – 53) can be solved by using finite difference method. The approach used here is central difference method. Applying this approach to above equations, we have two set of linear algebraic equations as follows:

$$A_j V^*_{zi,j-1} + B_j V^*_{zi,j} + C_j V^*_{zi,j+1} + \frac{dp^*}{dz^*} = D_j \quad \textbf{(48)}$$

$$A_i V^*_{ri-1,j} + B_j V^*_{ri,j} + C_j V^*_{r+1,j} + \frac{dp^*}{dz^*} = D_i \quad \textbf{(49)}$$

$$Q_1 = 2\Pi \sum_{j=1}^{N-1} \frac{\overset{*}{v}_{z1,j}\overset{*}{r}_j + \overset{*}{v}_{z1,j+1}\overset{*}{r}_{j+1}}{2} \Delta r \qquad \textbf{(50)}$$

$$Q_1 = 0 \qquad \textbf{(51)}$$

$$Q_{j+1} = Q_j + 2\Pi \frac{\overset{*}{v}_{z1,j}\overset{*}{r}_j + \overset{*}{v}_{z1,j+1}\overset{*}{r}_{j+1}}{2} \Delta r - 2\Pi \frac{\overset{*}{v}_{zM,j}\overset{*}{r}_j + \overset{*}{v}_{zM,j+1}\overset{*}{r}_{j+1}}{2} \Delta r \qquad \textbf{(52)}$$

$$Q_{j+1} = 2\Pi \sum_{i=1}^{M-1} \frac{\overset{*}{v}_{ri,j+1} + \overset{*}{v}_{i+1,j+1}}{2} \overset{*}{v}_j \Delta r \qquad \textbf{(53)}$$

4.1 Flow rates

The outlet flow rate:

$$Q_o = 2\pi \int_0^R v_z r dr \qquad \textbf{(54)}$$

$$Q_o = 2\pi \int_0^R \frac{1}{4\mu} \frac{\partial p}{\partial z} (r^2 - R^2) r dr \qquad \textbf{(55)}$$

$$Q_o = -\frac{\pi R^4}{24\mu} \left(\frac{\partial p}{\partial z}\right) \qquad \textbf{(56)}$$

The flow rate through the permeate is given as:

$$Q_p = 2\pi R \int_0^L v_r d_z \qquad \textbf{(57)}$$

$$Q_p = 2\pi R \int_0^L \frac{k}{\mu} \left(\frac{p - p_o}{\Delta x}\right) d_z \qquad \textbf{(58)}$$

P is a function of z.

So the inlet flow rate is:

$$Q_{in} = Q_p + Q_o \qquad \textbf{(59)}$$

5. Simulation results

Mesh quality is given by orthogonal quality, which ranges from 0-1, 0 corresponds to low and 1 represents high quality. Aspect ratio= 9.95(Non-uniformity of mesh grids around the pipe profile). The mesh geometry created in fluent environment is shown in Fig. 6.

Figure 7 shows the variations of pressure along axial distance while Fig. 8 and 9 is representing variations of velocity along radial and axial distance, respectively.

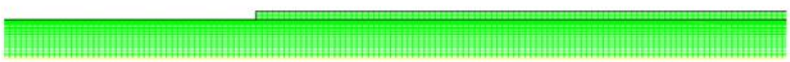


Figure 6. Meshed geometry.

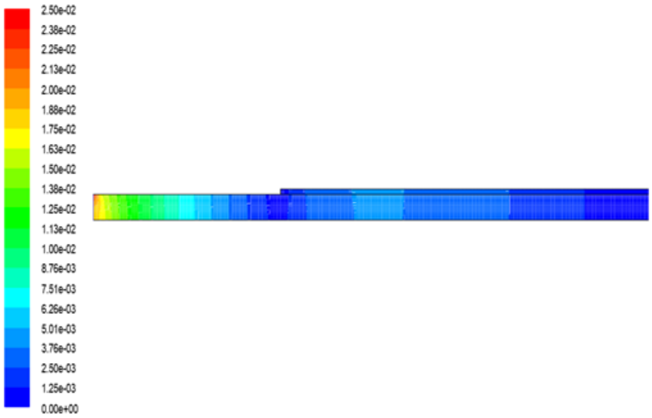


Figure 7. Pressure variations along the length.

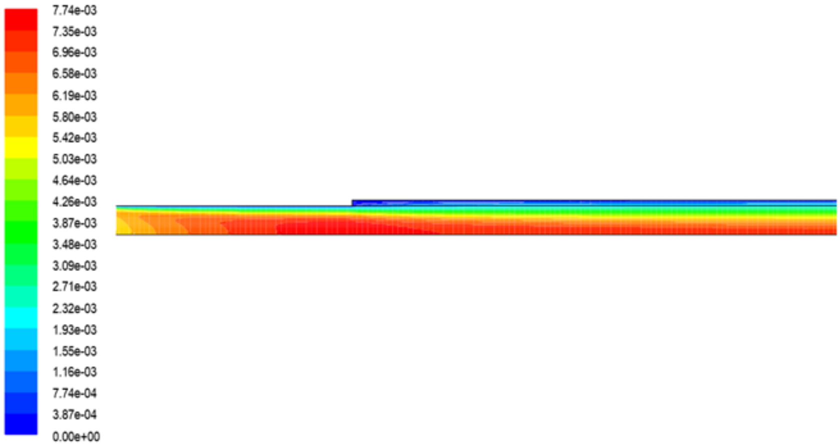


Figure 8. Velocity profile w.r.t radial distance.

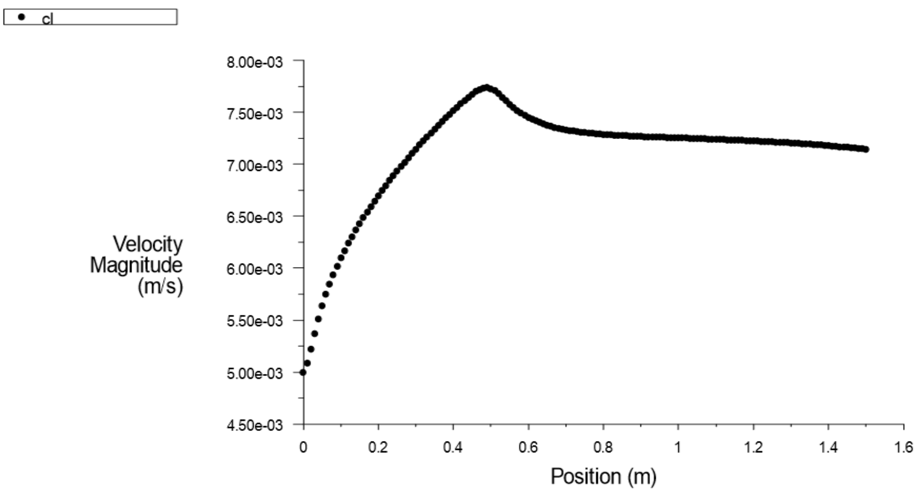


Figure 6. Velocity distribution along the length.

6. Discussions and Conclusions

The flow through porous media (membrane) has been analyzed by considering a variable suction velocity along the length of the pipe. The diameter of pipe is assumed 30 mm and length is 1.5 m. The permeability of the membrane is taken as 4.5 X 10^{-3}. Reynolds number is taken 250 which is in the range of laminar flow. Fig. 4 shows the variations of pressure along the length at porous wall. It can be observed that, along the length of membrane there is decrease in pressure. Also, along the length concentration polarization effect increases and pressure decreases. As pressure difference inside and outside the porous wall is main driving force for permeate to flow through the membrane, this pressure difference decreases along the length which consequently causes a decrease in permeate. Same effects for pressure and permeate velocity were observed by fluent.

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