The Geometry of the Octonionic Multiplication Table

by
Peter Lloyd Killgore

A PROJECT

submitted to

Oregon State University

University Honors College

in partial fulfillment of
the requirements for the
degree of

Honors Baccalaureate of Science in Mathematics
(Honors Scholar)

Presented May 15th, 2015
Commencement June 2015

AN ABSTRACT OF THE THESIS OF

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Abstract approved:

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Tevian Dray

We analyze some symmetries of the octonionic multiplication table, expressed in terms of the Fano plane. In particular, we count how many ways the Fano plane can be labeled as the octonionic multiplication table, all corresponding to a specified octonion algebra. We show that only 28 of these labelings of the Fano plane are nonequivalent, which leads us to consider the automorphism group of the octonions. Specifically, we look at the case when the mapping between two labelings of the Fano plane is an automorphism. Each such automorphism is induced by a permutation, and we argue that only 21 such automorphisms exist. We give the explicit definition of all 21 automorphisms and determine the structure of the group they generate. Finally, we interpret our results in a geometric context, noting especially the connection to the 7-dimensional cross product.

Key Words: Octonions, Fano plane, group theory, geometry

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I understand that my project will become part of the permanent collection of Oregon State University, University Honors College. My signature below authorizes release of my project to any reader upon request.

Peter Lloyd Killgore, Author

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1 Introduction

1.1 Lower Dimensional Number Systems

The most studied object in mathematics is undoubtedly $\mathbb{R}$, the set of real numbers. We can describe the real numbers as 1-dimensional, which leads to the natural geometric interpretation of the real numbers as an infinite set of points on a line, a concept known as "the number line." A consequence of being 1-dimensional is that the real numbers can be ordered, a property that in turn makes the real numbers useful for measuring things such as quantity and distance.

One dimension higher than the real numbers are the complex numbers, which are numbers of the form

$$c = a + bi \tag{1}$$

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. Given this structure, the complex numbers can be represented geometrically as points in a plane, where we associate the x-axis with the real numbers and the y-axis with real multiples of i. Clearly, the real numbers are a subset of the complex numbers, specifically the set of those complex numbers with $b = 0$. Graphically, this set is the x-axis in the complex plane. The complex numbers cannot be ordered as the real numbers can. Despite the loss of this rather basic property, the complex numbers are still useful mathematical objects. They are essential for solving algebraic equations and play a large role in the study of spectral theory for matrices.

We now turn to an even larger number system: the quaternions. The quaternions are 4-dimensional. Three of the dimensions are imaginary. They were discovered by Sir William Rowan Hamilton on October 16th, 1843 when, in a moment of pure mathematical inspiration, he realized the governing equations relating the imaginary quaternionic units to each other [2]:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

In Hamilton's honor, the quaternions are denoted by $\mathbb{H}$. Like the complex numbers, the quaternions cannot be ordered. They also bear the property that multiplication over the quaternionic units is anticommutative. In fact, if one considers only the imaginary part of a quaternion, by which we mean the part that is not real, there is a complete isomorphism between multiplication on the imaginary quaternionic units and the 3-dimensional vector cross product. The reader is no doubt familiar with the mnemonic in Figure 1 for determining the cross product of any two of the 3-dimensional basis vectors, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. This same mnemonic can be used to determine products of the quaternionic imaginary units i, j and

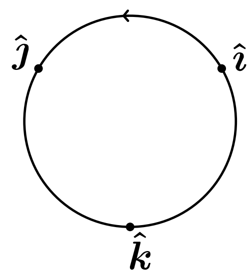

Figure 1: A mnemonic for both the quaternionic multiplication table and the 3-dimensional vector cross product.

k. That the quaternionic units and the basis vectors in three dimensions have the same names is no coincidence and is due to the fact that the foundations of vector and scalar computations are found in quaternionic algebra [2]. Even though quaternions have now been abandoned for vector computations, they still bear significance, notably in robotics and computer graphics [3].

1.2 The Cayley-Dickson Process

Recall at this point that we can think of a complex number as a point in the plane. Specifically, if we have a real axis and an imaginary axis, we can describe any $c \in \mathbb{C}$ by some pair of real numbers (a, b), where $c = a + bi$. We can then express $\mathbb{C}$ as

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i \tag{3}$$

where $\oplus$ is the direct sum and indicates taking all possible sums of all numbers $a \in \mathbb{R}$ with all numbers $b \in \mathbb{R}$ multiplied by i. Such a description of $\mathbb{C}$ emphasizes that the complex numbers are 2-dimensional.

Now, we know that in the quaternions, $ij = k$, as this is implied by (2). We can therefore express an arbitrary quaternion $h \in \mathbb{H}$ by

$$h = (a + bi) + (c + di)j. \tag{4}$$

Hence, we can describe the quaternions completely by

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j. \tag{5}$$

$\times$	i	j	k	$k\ell$	$j\ell$	$i\ell$	ℓ
i	-1	k	$-j$	$j\ell$	$-k\ell$	$-\ell$	$i\ell$
j	$-k$	-1	i	$-i\ell$	$-\ell$	$k\ell$	$j\ell$
k	j	$-i$	-1	$-\ell$	$i\ell$	$-j\ell$	$k\ell$
$k\ell$	$-j\ell$	$i\ell$	$-\ell$	-1	$-i$	$-j$	$-k$
$j\ell$	$k\ell$	ℓ	$-i\ell$	$-i$	-1	k	$-j$
$i\ell$	ℓ	$-k\ell$	$j\ell$	j	$-k$	-1	$-i$
ℓ	$-i\ell$	$-j\ell$	$-k\ell$	k	j	i	-1

Figure 2: The multiplication table for the units in the octonions.

This process by which we have built the complex numbers from the reals and the quaternions from the complex numbers is known as the Cayley-Dickson process. The Cayley-Dickson process can be repeated indefinitely to construct new number systems, each of which has twice the dimension of the previous system. However, only one additional repetition of the process yields another division algebra [6].

2 The Octonions

The octonions, denoted $\mathbb{O}$, form an 8-dimensional number system. We can build the octonions using the Cayley-Dickson process on the quaternions, obtaining

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell. \tag{6}$$

Thus, a number $z \in \mathbb{O}$ takes the form

$$z = x_1 + x_2 i + x_3 j + x_4 k + x_5 k\ell + x_6 j\ell + x_7 i\ell + x_8 \ell \tag{7}$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \mathbb{R}$, and $i, j, k, k\ell, j\ell, i\ell,$ and ℓ are each a distinct square root of -1, which we call imaginary basis units but in practice refer to simply as units. The product of any two units can be determined using the multiplication table given in Figure 2.

2.1 History of the Octonions

First called "octaves," the octonions were first discovered by John Graves [4]. However, Graves' discovery of the octonions was rather overshadowed by Arthur

Cayley's, despite his priority. Hamilton vouched that Graves had indeed discovered the octonions in December 1843, nearly two years prior to Cayley's publication in March 1845 in which he described the octonions [2]. Still, the fact that Cayley was the first to publish on the octonions gave him precedence in the mathematical community and led to the octonions being known as "Cayley numbers" [4]. In retrospect, both Graves and and Cayley are recognized for independently discovering the octonions. Interestingly enough, it was Hamilton who seems to have first noted one of the most peculiar and surprising properties of the octonions, namely the fact that they are nonassociative. As Baez points points out [2], Hamilton was the first to use the term associative and it is possible that the octonions played a significant role in clarifying this property since they lack it.

2.2 Important Properties of $\mathbb{O}$

Two especially noteworthy facts about multiplication over the octonions are that it is both non-associative and non-commutative. Below are a few examples that demonstrate these properties.

To see that multiplication is non-associative, observe that

$$j(i\ell) = k\ell, \tag{8}$$

but

$$(ji)\ell = -k\ell. \tag{9}$$

To see that multiplication is non-commutative, observe that

$$(i\ell)k = j\ell, \tag{10}$$

but

$$k(i\ell) = -j\ell. \tag{11}$$

The octonions are the largest of the normed division algebras [6]. In fact, the only possible dimensions for a division algebra are 1, 2, 4, and 8 [7]. Hence we see that corresponding to these choices of dimensionality, we have the real numbers, the complex numbers, the quaternions, and the octonions respectively. As noted by Okubo [8], the proof that the possible dimensions for a division algebra are limited to 1, 2, 4, and 8 is based in a topological argument and a pure algebraic proof has yet to be found.

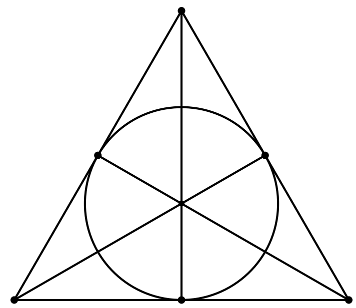

Figure 3: The 7-point projective plane, also known as the Fano geometry.

3 The Fano plane

Rather than the cumbersome and crowded multiplication table in Figure 2, a far more elegant multiplication table for the octonionic units can be constructed using the Fano plane. The Fano plane is a model for the Fano geometry, which is a finite geometry, that is, it contains finitely many objects. The axioms for the Fano geometry are given below [13]:

1. There exists at least one line.
2. There are exactly three points on every line.
3. Not all points are on the same line.
4. There is exactly one line on any two distinct points.
5. There is a least one point on any two distinct lines.

Given these axioms, a common model for the Fano geometry is that shown in Figure 3.[1]

[1]Since "line" is an undefined term in the axioms for the Fano geometry, we consider the circle in Figure 3 to be a line.

$$\begin{array}{ccc} \{i, j, k\} & \{i, \ell, i\ell\} & \{j, \ell, j\ell\} \\ \{k, \ell, k\ell\} & \{i\ell, k, j\ell\} & \{j\ell, i, k\ell\} \\ \{k\ell, j, i\ell\} & & \end{array}$$

Figure 4: The seven quaternionic triples of the octonions.

3.1 Multiplication Table of the Octonions

The multiplication table in Figure 2 reveals that multiplication is cyclic within those triples that are closed under multiplication. We refer to such triples as quaternionic triples, since, by taking any such triple and the real number 1, we can construct a number system isomorphic to the quaternions. Such a number system is known as a quaternionic subalgebra of the octonions [4]. The product of any two units in any quaternionic triple is the third unit in the triple, with the sign depending on the order of multiplication. We list the seven quaternionic triples in Figure 4.

3.2 Labeling the Fano Plane

The Fano plane can be labeled as a mnemonic for the octonionic multiplication table, as shown in Figure 5. To use the mnemonic, move cyclically through the units on a line in the direction of the arrow to determine the product of any two units. Moving against the arrow introduces a minus sign. For example,

$$(k\ell)j = i\ell \tag{12}$$

but

$$(i\ell)j = -k\ell. \tag{13}$$

When labeling the Fano plane as a mnemonic for octonionic multiplication, it must first be determined just how to construct a mnemonic that actually gives octonionic products. Such a construction is governed by the multiplicative structure described in Section 3.1. Specifically, the fact that quaternionic triples are closed under multiplication dictates that each triple be sent to a line in the Fano plane; triples cannot be broken or manipulated aside from being reordered. For example, if i and j are both on one line, the third unit cannot be ℓ; it must be k. If a different unit is chosen for the third point on the line containing i and j, it will break the cyclic nature of multiplication and thus yield a multiplication

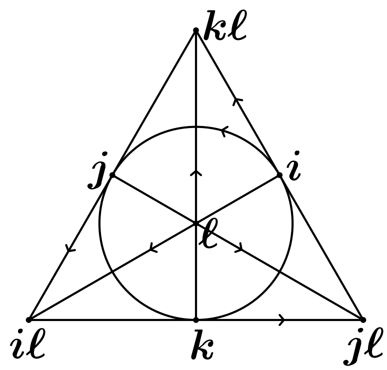

Figure 5: The Fano geometry labeled as to give products consistent with the multiplication table of the octonions.

table that does not represent the octonions. The fact that the octonionic units anticommute with each other determines how lines in the Fano plane must be oriented once the points have been labeled. If a line does not have the correct orientation, the multiplication table will give products with the wrong sign. The units can be listed in any order on a line in the Fano plane so long as all three units from a triple lie on one line in the Fano plane. It is only required that the line be oriented such that products have the correct sign. Since $3! = 6$, there are six different orderings for three units. The orderings for the triple $\{i, j, k\}$ are shown in Figure 6. Each triple is oriented so as to be consistent with the octonions.

Here we note some symmetries between the multiplication table of the octonions and the Fano plane: The Fano plane has seven lines and there are seven quaternionic triples in the octonionic multiplication table; Each line on the Fano plane contains three points and each quaternionic triple contains three octonionic units; Each point in the Fano plane is on three lines and each octonionic unit is in three quaternionic triples. These symmetries allow us to use the Fano plane as a mnemonic for the octonionic multiplication table, shown in Figure 5.

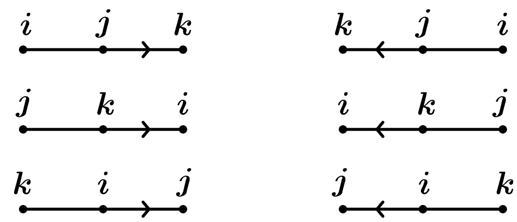

Figure 6: The properly oriented orderings for the triple $\{i, j, k\}$.

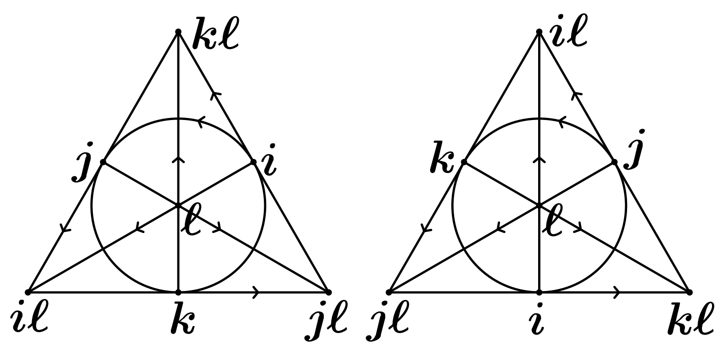

Figure 7: An example of two labelings from the same congruence class.

3.3 Counting Labelings

Surely the labeling of the Fano plane in Figure 5 is not unique. In fact, given the seven lines, each with an orientation, and the seven octonionic imaginary units, it is clear that there are many possible ways to label the Fano plane. There are seven points on which to place units. Therefore, there are $7! = 5040$ ways to label the points in the Fano plane. Since each line can be oriented two different ways, there are 2^7 orientations of the Fano plane. That being the case, it would appear that there are

$$2^7 \cdot 7! = 645120 \tag{14}$$

different ways to label the Fano plane. However, not all of these labelings are truly distinct. We consider two labelings to be equivalent if one can be transformed into the other by a series of reflections, rotations, or a combination thereof. Any given labeling can be rotated three ways and reflected through three axes of symmetry. An example of clockwise rotation of a labeling is shown in Figure 7. Once equivalence among tables has been accounted for, there still remains the question of just how many of the remaining tables correspond to the octonions. The table shown in Figure 8 is proof enough that not all labelings correspond to the octonions. This table gives the following products:

$$(k\ell)i = j\ell, \tag{15}$$

$$i(j\ell) = k\ell, \tag{16}$$

$$(j\ell)(k\ell) = i. \tag{17}$$

On the octonions, these products all have a minus sign:

$$(k\ell)i = -j\ell, \tag{18}$$

$$i(j\ell) = -k\ell, \tag{19}$$

$$(j\ell)(k\ell) = -i. \tag{20}$$

Rather than first determining how many different multiplication tables exist and then determining which multiplication tables correspond to the octonions, the goal is to determine a method for building all tables that correspond to the octonions and then count the number of such tables.

We here make an important clarification in how we are counting tables. Schray and Manogue [12] state that there are 480 octonionic multiplication tables. Taking into account our notions of equivalence, we will argue that there are 28 different tables. However, since we are counting the number of nonequivalent ways to label

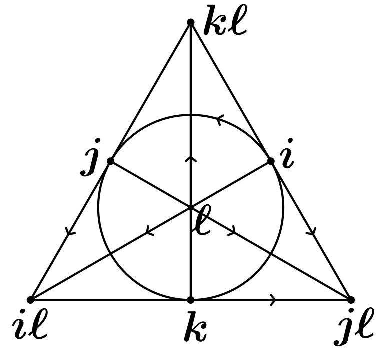

Figure 8: An example of a labeling that does not correspond to the octonionic multiplication table.

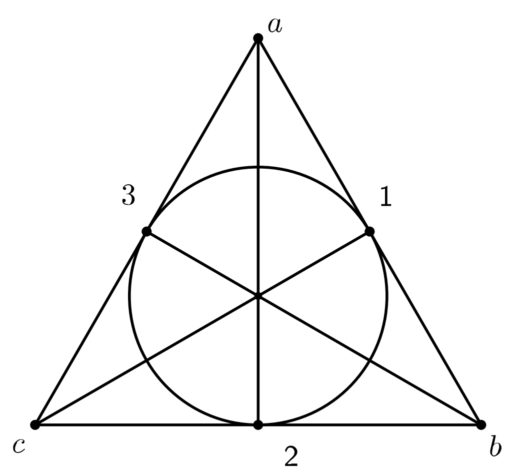

Figure 9: The Fano plane labeled for referencing for our process of constructing the octonionic multiplication table.

the Fano plane as the octonionic multiplication table, we are in effect counting the number of tables that correspond to a given octonionic algebra, such as the one described by the table in Figure 5. The 480 tables Manogue and Schray [12] count actually represent different algebras, each of which is isomorphic to the octonionic algebra we have described. There is therefore no contradiction between our 28 tables and the 480 tables of Schray and Manogue [12].

In order to label the Fano plane as a multiplication table for the octonions, we first consider labeling only the perimeter. We clarify our process by referencing Figure 9, which has the vertices and segments named. Since quaternionic triples must go to lines in the Fano plane, we pick one of the seven quaternionic triples to place on a perimeter line of the Fano plane. Without loss of generality, suppose we send the triple to segment 1. Each triple can be ordered six ways. Hence, we have

$$7 \cdot 6 = 42 \tag{21}$$

choices for how to pick and order the first triple. Next, a second triple must be picked to go on one of the adjacent sides of the Fano plane. Again, without loss of generality, suppose the second triple goes to segment 2. This second triple must share exactly one unit with the first triple that was chosen. Specifically, it must share the unit on vertex b. We know from Section 3.2 that each unit is contained in exactly three triples. Therefore, since the first triple already contains the shared unit, there are two possible choices for the second triple. Since the placement of the shared unit is fixed, the number of orderings of the second triple is restricted. The only possible variation is that the placement of the other two units can be swapped. Therefore, there are two possible orderings for the second triple. An example is shown in Figure 10. Given seven choices for the first triple, six ways to order the first triple, two choices for the second triple, and two ways to order the second triple, there are

$$7 \cdot 6 \cdot 2 \cdot 2 = 168 \tag{22}$$

ways to pick the first two triples to go on the border of the Fano plane.

Picking the third triple to go on the perimeter of the Fano is the most significant step in this process, mainly because there is only one way it can be picked. The third triple goes on the remaining perimeter segment of the Fano plan, which intersects the segments containing both the first and the second triple. As a consequence, the third triple must share exactly one unit with the first triple and one unit with the second triple. Specifically, the third triple must share the unit on vertex a with the first triple and the unit on vertex c with the second triple.

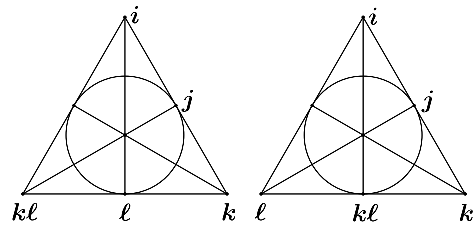

Figure 10: The second triple can be ordered in two distinct ways

Therefore, two units in the third triple are predetermined. However, given the cyclic nature of multiplication within triples, there can be only one triple that contains any two distinct units. Moreover, any two distinct units are contained in exactly one quaternionic triple. Therefore, there exists a triple containing the units on vertices a and c and this triple is unique. This leads us to the fact that there is only one way the third triple can be chosen. Hence, there are

$$7 \cdot 6 \cdot 2 \cdot 2 = 168 \tag{23}$$

ways to label the perimeter.

Once the perimeter is labeled, it remains to label the point in the center of the Fano plane. There are six points on the perimeter of the Fano plane which have already been labeled. Since there are seven units that must be placed on the Fano plane, only one unit is left unused. Therefore, the point in the center of the Fano plane must be labeled with the one unit that has not yet been assigned to a point. The question is whether or not this unit completes the unfinished triples that lie on axes of symmetry of the Fano plane. In fact, it does.

Theorem 3.1. *Given our process for labeling the Fano plane thus far, the remaining unit to be placed on the center point completes the triples on the axes of symmetry.*

Proof. Suppose the perimeter of the Fano plane has been labeled with three distinct triples in the manner we have described, as in Figure 11, where e_1, e_2, e_3, e_4, e_5, and e_6 are all octonionic units. Then, we have used six of the seven octonionic units and the only point in the Fano plane that has not been labeled is the point in the center of the circle. Call our unused unit e_7. Assume e_7 does not complete one of the triples on an axes of symmetry of the Fano plane. Call this

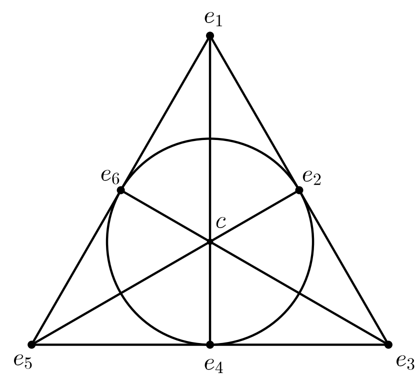

Figure 11: e_1, e_2, e_3, e_4, e_5, and e_6 correspond to octonionic units chosen such that the perimeter of the Fano plane has been labeled with three distinct triples, each of which shares exactly one unit with the other two.

triple T and denote the missing unit by c. Without loss of generality, suppose T is the triple on the segment whose endpoints are labeled with e_5 and e_2. Since all the other octonionic units have been placed on the Fano plane, c must lie on the perimeter of the Fano plane. Clearly c is not e_5 or e_2 since c is missing from T and both e_5 and e_2 are contained in T. Therefore, c must be one of $e_1, e_3, e_4,$ or e_6. Without loss of generality, suppose c is e_6 or e_1. If c is e_6, then, due to the closure of quaternionic triples under multiplication, e_1 must equal e_2. This implies that $e_3 = -1$, a contradiction to how we labeled the perimeter of the Fano plane. Similarly, if c is e_1, e_6 must equal e_2, implying that $e_4 = -1$, which is again a contradiction. Since assuming there exists a triple on an axis of symmetry that e_7 does not complete leads to a contradiction, we have that e_7 complete all the triples on axes of symmetry of the Fano plane. □

Due to the symmetries we have noted between the multiplication table of the octonions and the construction of the Fano plane, the three units on the circle also form a triple.

Theorem 3.2. *Given our process for labeling the Fano plane thus far, the three units on the circle in the Fano plane form a quaternionic triple.*

Proof. Recall that the Fano plane has seven points and there are seven octonionic units that must be placed on the Fano plane. Similarly, the Fano plane contains seven lines and there are seven quaternionic triples. Lastly, just as any point in the Fano plane lies on exactly three lines, any octonionic unit is contained in exactly three quaternionic triples. Given how we have labeled the Fano plane, each unit on the circle is contained in a triple on the perimeter and in a triple on an axis of symmetry of the Fano plane. Therefore, each unit on the circle is contained in two triples. The other four units are already contained in three triples each though. Moreover, six completed triples have already been placed on the Fano plane, specifically those placed on the perimeter lines and on the axes of symmetry. Therefore, there is one triple yet to be accounted for, and all three of the units on the circle still need to be placed in one triple each. It must then be the case then that the three units on the circle indeed make up a triple. □

Using our process for labeling the Fano plane, we have placed all seven quaternionic triples on lines in the Fano plane. It remains to provide an orientation for each line of the Fano plane. Each line has two possible orientations. However, due to the the non-commutativity of multiplication on the octonions, once the points on a line have been labeled, there is only one orientation that will give the products defined on the octonions. It has already been seen in Figure 6 that for any ordering of a triple, there is an orientation of the line on which it lies that yields octonionic products. Once all the points in the Fano plane have been labeled, the orientation of any one line has no effect on any other line. Therefore, each line can be orientated such that the entire Fano plane yields the products defined on the octonions. Moreover, since any ordering of a triple has only one orientation that will produce the desired products, there is only one orientation for the Fano plane as a whole that will make a given labeling an octonionic multiplication table. Therefore, since there is exactly one orientation for each line that will produce octonionic products, we have from (23) that there are 168 ways to label the Fano plane such that it can be used as a mnemonic for the octonionic multiplication table.

3.4 Accounting for Equivalence

We have shown that there are at most 168 different ways to label the Fano plane so that it can be used as a mnemonic for the multiplication table of the octonions. We now introduce the idea of equivalent and nonequivalent labelings.

Definition 3.1. *Given two labelings of the Fano plane as a multiplication table for the octonions, we consider the labelings to be equivalent if there is a rotation,* *reflection, or a composition of the two that will transform one labeling into the other.*

Definition 3.2. *If two labelings of the Fano plane as a multiplication table for the octonions are not equivalent, they are said to be nonequivalent.*

Given our definition of equivalence, it turns out we have been counting some labelings multiple times. The Fano plane can be rotated in one of three ways, one of which is the identity rotation. Since the Fano plane has three axes of symmetry, it can also be reflected through any of these three axes. Each rotation, reflection, and any composition of a rotation and a reflection defines an equivalence relation between two labelings of the Fano plane. The question then is, given any particular labeling, how many ways can the labeling be rotated or reflected, thus obtaining an equivalent labeling? Obviously, there are three equivalent labelings due to rotation. Accounting for the reflections is slightly trickier since reflecting is a binary choice. It turns out that all three reflections can be generated by reflecting the table through any axis of symmetry and then rotating that labeling, an example of which is shown for one labeling in Figure 12. Table 1 can be reflected across its three axes of symmetry to obtain tables 2, 3, and 4. However, Tables 2, 3, and 4 are all rotations of each other!

So, given a table, it can be reflected through an axis of symmetry giving two equivalent tables. Each table can then be rotated one of three ways. This accounts for all rotations and all reflections of a table. Hence, for any given table, there are $3 \cdot 2 = 6$ equivalent ways of labeling the table. Therefore, we can divide 168 by 6 and this will account for all equivalent tables. Since $168/6 = 28$, there are 28 distinct ways of labeling the Fano plane so that it can be used as a mnemonic for the octonionic multiplication table.

4 Automorphisms on the Octonions

The result that there are 28 distinct ways of labeling the Fano plane so that it can be used as a multiplication table for the octonions leads us to consider how to send one labeling to another. For example, how can we move back and forth between Labeling 1 and Labeling 2, shown in Figure 13? To answer this question, some group theory is necessary.

4.1 Necessary Group Theory

We first establish some definitions.

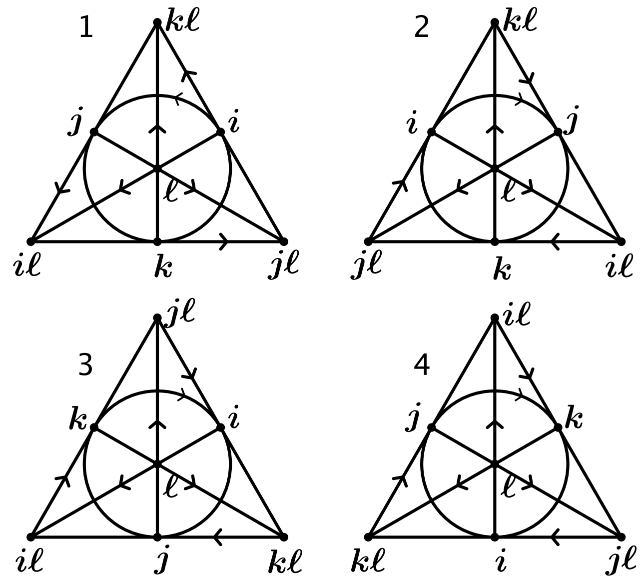

Figure 12: The table in the upper left can be reflected through an axis of symmetry to obtain any of the other three tables. The other three tables are rotations of each other.

Definition 4.1. *An algebra is a vector space V endowed with a product defined on the vectors.* [2]

Definition 4.2. *An algebra automorphism is a vector space isomorphism from an algebra V to itself that preserves the multiplicative structure of V. Specifically, an automorphism ϕ must satisfy $\phi(ab) = \phi(a)\phi(b)\ \forall\ a, b \in V$.*

There are a few properties of automorphisms that are especially important. Firstly, the composition of two automorphisms is an automorphism. Secondly, automorphisms are linear, that is, they respect addition and (real) scalar multiplication. More explicitly, if ϕ is an automorphism,

$$\phi(ab + c) = a\phi(b) + \phi(c) \tag{24}$$

for all $a \in \mathbb{R}$ and all $b, c \in \mathbb{O}$. Lastly, all automorphisms map the identity element to itself [10]. The fact that $\phi(-1) = -1$ immediately follows from linearity. For

[2] For a more detailed treatment of vector spaces and algebras, see [5].

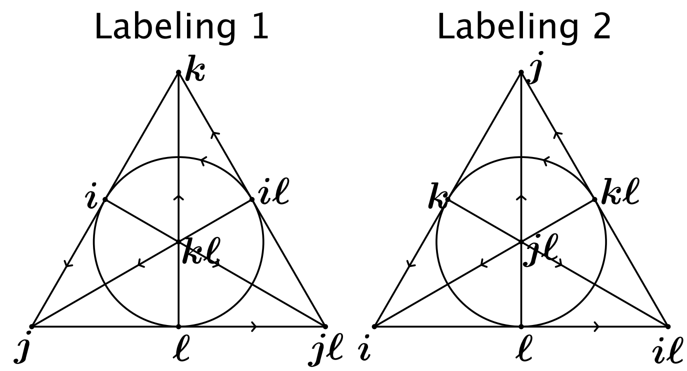

Figure 13: Labelings 1 and 2 of the Fano plane as the octonionic multiplication table. These labelings are nonequivalent

this reason, when giving the explicit definition of a particular automorphism, we leave out the action of the automorphism on 1 and -1, since it is trivial.

The set $\{1, i, j, k, k\ell, j\ell, i\ell, \ell\}$ is a basis of the octonions, that is, any octonion x can be written as a linear combination of the elements of $\{1, i, j, k, k\ell, j\ell, i\ell, \ell\}$, where the coefficients are real. Moreover, since all automorphisms map the identity to itself, linearity immediately implies that any automorphism acts trivially on all elements of $\mathbb{R}$. Therefore, we can define an automorphism on the octonions by specifying its action on $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$.

4.2 Automorphisms and Nonequivalent Labelings

We can express a mapping between two nonequivalent labelings of the Fano plane as the octonionic multiplication table using a permutation. Due to linearity, some permutations will in fact induce an automorphism. For example, we can change Labeling 1 in Figure 13 into Labeling 2 with the permutation

$$f = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k & i & j & j\ell & i\ell & k\ell & \ell \end{pmatrix}, \tag{25}$$

where the top line is the the domain and the bottom line is the range. The permutation f in (25) induces an automorphism. We can change Labeling 2 into

Labeling 1 with the permutation

$$f^{-1} = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & k & i & i\ell & k\ell & j\ell & \ell \end{pmatrix}, \tag{26}$$

which also induces an automorphism. When a permutation induces an automorphism, we refer to the permutation as an automorphism. Since the composition of two automorphisms of an automorphism, this leads us to consider the group of automorphisms on the octonions.

4.3 The Automorphism Group on $\mathbb{O}$

Consider the following automorphisms:

$$\theta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i\ell & i & \ell & k & j\ell & j & k\ell \end{pmatrix} \tag{27}$$

$$\alpha = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k\ell & k & \ell & j & i\ell & i & j\ell \end{pmatrix} \tag{28}$$

The automorphism θ transforms the labeling of the Fano plane in Figure 5 into Labeling 1 in Figure 13 and α transforms the labeling of the Fano plane in Figure 5 into Labeling 2 in Figure 13.

There are three categories of automorphisms we can consider. First, are those like α and θ which simply permute $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$. We can also consider those automorphisms which, rather than simply permuting $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$ also involve sign changes, such as

$$g = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ -k\ell & k & -\ell & -j & i\ell & -i & j\ell \end{pmatrix}. \tag{29}$$

Lastly, we could also consider automorphisms that send the basis units to linear combinations of basis units that do not have integer coefficients. We could, for example, send i to $j\cos\omega + k\sin\omega$ for some angle ω.

We now define a notion of positivity that both α and θ posses.

Definition 4.3. *An automorphism ϕ has the positivity property if ϕ is induced by a permutation.*

All automorphisms that only permute $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$ have the positivity property. To emphasize this fact, we refer to these automorphisms as *positive* *acting automorphisms*. The group of positive acting automorphisms, to which we now turn our attention, is related to the permutation group on seven elements. Specifically, the group of positive acting automorphisms can be loosely thought of as the intersection of S_7, the permutation group on seven elements, with G_2, the automorphism group of the octonions.[3] The permutation group S_7 is of finite order, containing 7! elements. The automorphism group of the octonions is an infinite group [4]. Since S_7 is finite, the intersection of S_7 with G_2 must also be finite. We will show that their intersection contains exactly 21 elements.

Definition 4.4. *Denote by $Aut_+(\mathbb{O})$ the group of positive acting automorphisms on the octonions, which we loosely consider to be $S_7 \cap G_2$.*

Next, we have some basic definitions from group theory that will allow us to study and understand $\mathrm{Aut}_+(\mathbb{O})$.

Definition 4.5. *By the notation g^p, we mean $\overbrace{g \circ g \circ \ldots g}^{p\ times}$. By the order of an element g of a group G, we mean the smallest positive integer p such that g^p is the identity automorphism.*

Definition 4.6. *Let G be a group. The order of G is the number of elements of G.*

Both α and θ are elements of $\mathrm{Aut}_+(\mathbb{O})$. Therefore, by Lagrange's theorem, the order of both α and θ divide the order of $\mathrm{Aut}_+(\mathbb{O})$ [10]. Composing θ with itself shows that the order of θ is 3. In the same manner, the order of α is found to be 7. The order of $\mathrm{Aut}_+(\mathbb{O})$ must therefore be divisible by 3 and by 7. Since the least common multiple of 3 and 7 is 21, the order of $\mathrm{Aut}_+(\mathbb{O})$ must be at least 21. We therefore consider the set $S_{\alpha,\theta} = \{a \circ b : a \in \langle\alpha\rangle,\ b \in \langle\theta\rangle\}$, where $\langle\alpha\rangle$ denotes the cyclic group generated by α and $\langle\theta\rangle$ denotes the cyclic group generated by θ. Since the order of α is 7, it follows that the order of $\langle\alpha\rangle$ is 7. Similarly, it follows that the order of $\langle\theta\rangle$ is 3. The set $S_{\alpha,\theta}$ must then have 21 elements. It remains to determine whether or not this set forms a group with respect to function composition.

[3]Defining the group of positive acting automorphisms on the octonions as the intersection of S_7 with G_2 is a somewhat imprecise definition since there is no universal set U such that S_7 and G_2 are both subsets of U. Due to linearity though, we can write a positive acting automorphism using permutation notation. For this reason, we think of the group of positive acting automorphisms as $S_7 \cap G_2$

4.4 Proving Closure of $S_{\alpha,\theta}$

To show closure of $S_{\alpha,\theta}$ under function composition, we turn to the computer algebra program *Mathematica* [14]. We also make substantial use of a package developed for *Mathematica* that allows the user to do computations involving octonions [1].

Theorem 4.1. *The order of* $Aut_+(\mathbb{O})$ *is 21 and the group itself is* $S_{\alpha,\theta}$ *with function composition.*

Proof. We first prove that the order of $\mathrm{Aut}_+(\mathbb{O})$ is 21. If we are going to construct an automorphism that only permutes the basis elements of $\mathbb{O}$, we can start by mapping i to one of the seven imaginary basis units. Hence, we have seven choices of where to map i. Next, we must map j to a basis unit. However, rather than six choices, we only have three. This is the case because, given any imaginary basis unit $\phi(i)$, there are only three other imaginary basis units, call them $\phi_m(j)$ with $m = 1, 2, 3$, such that the sign of $\phi(k) = \phi(i)\phi(j)_m$ is positive. To see this, refer to Figure 5 and note that for each of the imaginary basis units, there are only three other units such that the product of the first with the second is positive. Hence, we have seven choices of where to map i but only three choices of where to map j. Given that i and j have already been mapped to their respective basis units, there is only one choice of where to map k. We can complete our automorphism by specifying where to map ℓ. Now, since we are constructing an automorphism, we have the following relationships:

$$\phi(k) = \phi(i \cdot j) = \phi(i)\phi(j) \tag{30}$$

$$\phi(i) = \phi(j \cdot k) = \phi(j)\phi(k) \tag{31}$$

$$\phi(j) = \phi(k \cdot i) = \phi(k)\phi(i). \tag{32}$$

So, $\phi(i), \phi(j)$, and $\phi(k)$ form a quaternionic triple. Now, since we are only allowed to permute the imaginary basis units to construct our automorphism, we require that there be no minus sign on $\phi(i)\phi(\ell)$, $\phi(j)\phi(\ell)$ and $\phi(k)\phi(\ell)$. We now prove a lemma that will help complete the proof that the order of $\mathrm{Aut}_+(\mathbb{O}) = S_7 \cap G_2$ is 21.

Lemma 4.1. *Given any quaternionic triple,* T*, there is only one imaginary basis unit* e_m *such that the sign of* $e \cdot e_m$ *is positive* $\forall\ e \in T$.

Proof. The proof of Lemma 4.1 is accomplished by exhaustion using Figure 5. We need only consider each of the quaternionic triples individually and see which of the four remaining units has the property that multiplying on the left by any unit

$$\{i, j, k\} \to \ell$$
$$\{i, \ell, i\ell\} \to k\ell$$
$$\{j, \ell, j\ell\} \to i\ell$$
$$\{k, \ell, k\ell\} \to j\ell$$
$$\{i\ell, k, j\ell\} \to i$$
$$\{j\ell, i, k\ell\} \to j$$
$$\{k\ell, j, i\ell\} \to k$$

Figure 14: For each quaternionic triple T, there is one unit e_m such that the sign of $e \cdot e_m$ is positive $\forall\ e \in T$.

from the triple gives a product with a plus sign. In Figure 14, we give the seven quaternionic triples, each paired with the unique unit that has positive products when multiplied on the left by each unit in the triple.

□

Now, since $\phi(i), \phi(j)$, and $\phi(k)$ form a quaternionic triple, Lemma 4.1, implies there is only one possible choice of where to map ℓ if the sign on $\phi(i)\phi(\ell),\ \phi(j)\phi(\ell)$ and $\phi(k)\phi(\ell)$ is to be positive. Choosing where to map ℓ determines where each of $i\ell, j\ell$ and $k\ell$ must be mapped. Hence, we have that if we only permute the imaginary basis units with our automorphism, there are $7 \cdot 3 \cdot 1 = 21$ ways to construct our automorphism. Therefore, the order of $\mathrm{Aut}_+(\mathbb{O}) = S_7 \cap G_2$ is 21.

It now remains to prove closure of $S_{\alpha,\theta}$. Since $S_{\alpha,\theta} \subseteq \mathrm{Aut}_+(\mathbb{O})$ and the order of $S_{\alpha,\theta}$ is equal to the order of $\mathrm{Aut}_+(\mathbb{O})$, closure of $S_{\alpha,\theta}$ leads us to the conclusion that $S_{\alpha,\theta} = \langle\alpha, \theta\rangle = \mathrm{Aut}_+(\mathbb{O})$, where $\langle\alpha, \theta\rangle$ is the subgroup of G_2 generated by the elements α and θ. We therefore use the notations $\langle\alpha, \theta\rangle$ and $\mathrm{Aut}_+(\mathbb{O})$ interchangeably. In Mathematica, after loading the octonion package, we define functions f2 $=$ θ, and f3 $= \alpha$, and assign to a variable 'basis' the list $\{0, i, j, k, k\ell, j\ell, i\ell, \ell\}$.[4] Computing f2[basis] tells us the action of f2 $= \theta$ on $\{i, j, k, k\ell, j\ell, i\ell, \ell\}$. We next use Mathematica to compute the output of the 21

[4]We use this list as a variable instead of $\{1, i, j, k, k\ell, j\ell, i\ell, \ell\}$, because it yields the correct output in a readable form whereas using $\{1, i, j, k, k\ell, j\ell, i\ell, \ell\}$ does not. We acknowledge that this is a kludge.

```
myList = {};
myList = Append[myList, {Map[VPrint, NestList[f3, basis, 6]]}];
AppendTo[myList, {Map[VPrint, NestList[f2, f2[basis], 1]]}];
myList = Flatten[myList, 2];
AppendTo[myList, VPrint[f3[f2[f2[basis]]]]];
AppendTo[myList, VPrint[f3[f3[f2[f2[basis]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f2[f2[basis]]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f2[f2[basis]]]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f3[f2[f2[basis]]]]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f3[f3[f2[f2[basis]]]]]]]]]];
AppendTo[myList, VPrint[f3[f2[basis]]]];
AppendTo[myList, VPrint[f3[f3[f2[basis]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f2[basis]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f2[basis]]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f3[f2[basis]]]]]]]];
AppendTo[myList, VPrint[f3[f3[f3[f3[f3[f3[f2[basis]]]]]]]]];
myList
```

Figure 15: The code used to compute the outputs of the 21 elements of $S_{\alpha,\theta}$.

elements of $S_{\alpha,\theta}$ using the code in Figure 15. The code for defining the 21 automorphisms in *Mathematica* is given in Figure 18.

By construction, all powers of θ and α and all compositions of the form $\alpha^i \circ \theta^j$ for integers i, j are in $S_{\alpha,\theta}$. Therefore, in order to show that $S_{\alpha,\theta}$ forms a group under function composition, we need only show that $\theta \circ \alpha$ is one of these 21 automorphisms, all of which are listed in Figure 16. This will prove that the 21 elements close under function composition since any composition of the 21 elements of $S_{\alpha,\theta}$ that is not known to be in the group by its construction can be broken into compositions of $\alpha \circ \theta$ with $\theta \circ \alpha$. *Mathematica* confirms that $\theta \circ \alpha = \upsilon$ since the command θ`[`α`[basis]] ==` υ`[basis]` returns `True`. Therefore, the 21 elements of $S_{\alpha,\theta}$ close under function composition. Hence, $S_{\alpha,\theta} = \langle \alpha, \theta \rangle = \mathrm{Aut}_+(\mathbb{O})$. □

4.5 Automorphisms and the Orientation of the Fano Plane

Here, we offer an alternate proof that $|\mathrm{Aut}_+(\mathbb{O})| = 21$, which in turn leads to an interesting corollary.

Proof. Suppose we wish to label the Fano plane with the imaginary basis units of

$$\epsilon = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i & j & k & k\ell & j\ell & i\ell & \ell \end{pmatrix} \qquad \alpha = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k\ell & k & \ell & j & i\ell & i & j\ell \end{pmatrix}$$

$$\beta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & \ell & j\ell & k & i & k\ell & i\ell \end{pmatrix} \qquad \gamma = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k & j\ell & i\ell & \ell & k\ell & j & i \end{pmatrix}$$

$$\delta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ \ell & i\ell & i & j\ell & j & k & k\ell \end{pmatrix} \qquad \zeta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j\ell & i & k\ell & i\ell & k & \ell & j \end{pmatrix}$$

$$\eta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i\ell & k\ell & j & i & \ell & j\ell & k \end{pmatrix} \qquad \theta = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i\ell & i & \ell & k & j\ell & j & k\ell \end{pmatrix}$$

$$\iota = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & i\ell & k\ell & \ell & j\ell & i & k \end{pmatrix} \qquad \kappa = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k & i & j & j\ell & i\ell & k\ell & \ell \end{pmatrix}$$

$$\lambda = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ \ell & k\ell & k & i\ell & i & j & j\ell \end{pmatrix} \qquad \mu = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j\ell & j & \ell & i & k\ell & k & i\ell \end{pmatrix}$$

$$\nu = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i\ell & k & j\ell & k\ell & j & \ell & i \end{pmatrix} \qquad \xi = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i & \ell & i\ell & j & k & j\ell & k\ell \end{pmatrix}$$

$$o = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k\ell & j\ell & i & k & \ell & i\ell & j \end{pmatrix} \qquad \rho = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ i & k\ell & j\ell & \ell & i\ell & k & j \end{pmatrix}$$

$$\sigma = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k\ell & j & i\ell & j\ell & i & \ell & k \end{pmatrix} \qquad \tau = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & k & i & i\ell & k\ell & j\ell & \ell \end{pmatrix}$$

$$\upsilon = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ k & \ell & k\ell & i & j & i\ell & j\ell \end{pmatrix} \qquad \phi = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ \ell & j\ell & j & k\ell & k & i & i\ell \end{pmatrix}$$

$$\psi = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j\ell & i\ell & k & j & \ell & k\ell & i \end{pmatrix}$$

Figure 16: The 21 positive acting automorphisms on the octonions.

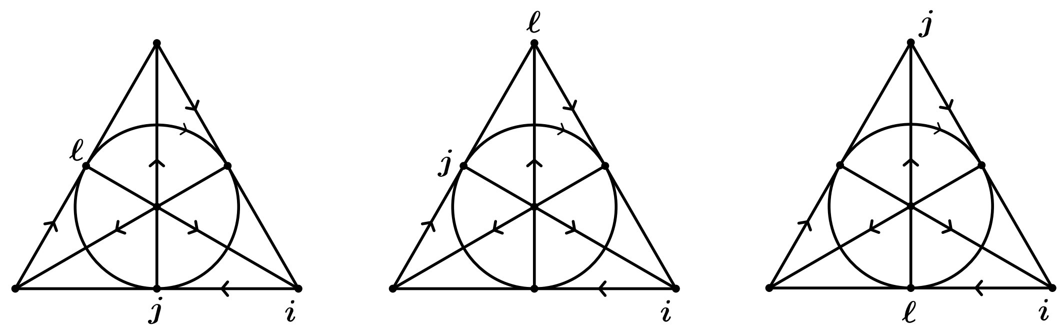

Figure 17: Once i has been placed, there are three points where j can go and one place ℓ can go after j has been placed.

the octonions in order to obtain a mnemonic for octonionic multiplication after we have fixed an orientation of the Fano plane. This is equivalent to changing a given labeling into another labeling with an automorphism. We can map i to any of the seven points. However, there are only three points to which we can now map j since we have already chosen an orientation. Now, there is only one point to which we can map k. Also, there is only one point to which we can map ℓ if we do not allow the use of minus signs when labeling the Fano plane. In Figure 17, we show all three cases of where to place j and ℓ given a fixed orientation and a fixed placement of i.

Since we can map i to one of seven points, j to one of three points, and after this each unit can only be mapped to one point of the Fano plane there are only $7 \cdot 3 = 21$ ways to define an automorphism that only permutes the imaginary basis units of the octonions. □

As a corollary, we have the result that the elements of $\mathrm{Aut}_+(\mathbb{O})$ correspond to mappings between labelings of the Fano plane where the orientation of the Fano plane is fixed. For example, note that in Figure 12, the mapping from Table 1 to any of Table 2, 3, or 4 is not an automorphism. For example, the mapping from Table 1 to Table 2, defined by

$$h = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & i & k & k\ell & i\ell & j\ell & \ell \end{pmatrix} \tag{33}$$

is not an automorphism because

$$h(k) = k = i \cdot j \neq h(i) \cdot h(j) = j \cdot i = -k. \tag{34}$$

To make h an automorphism, we need to as minus signs in the appropriate places, obtaining

$$h = \begin{pmatrix} i & j & k & k\ell & j\ell & i\ell & \ell \\ j & i & -k & -k\ell & i\ell & j\ell & \ell \end{pmatrix}. \tag{35}$$

However, the automorphism κ maps Table 2 to Table 3, Table 3 to Table 4, and Table 4 to Table 2.

4.6 Determining the group Structure of $\mathrm{Aut}_+(\mathbb{O})$

In order to determine the group structure of $\mathrm{Aut}_+(\mathbb{O})$, we first define all 21 of the automorphisms in $\mathrm{Aut}_+(\mathbb{O})$ in *Mathematica* with the code given in Figure 18. We next create a table in *Mathematica* that is equal to the result of using the `OutputForm[]` command on the output of the code in Figure 19 and create a list `greek`, which contains all the outputs of the the 21 automorphisms. Finally, using the code in Figure 20, we can obtain the group table of $\mathrm{Aut}_+(\mathbb{O})$, which is given in Figure 21.

Given the definition of $S_{\alpha,\theta} = \langle \alpha, \theta \rangle = \mathrm{Aut}_+(\mathbb{O})$, every automorphism in Figure 16 can be written in the form $\alpha^p\theta^q$ for some $p, q \in \mathbb{Z}$. The lists `myList` in Figure 15 and `greek` in Figure 20 are the same. Since the function outputs in the list `greek` are listed such that they correspond to the order in which functions are listed in the list `myAutomorphisms` in Figure 18, we can simply read off the $\alpha^p\theta^q$ form of all our automorphisms from Figure 15. The $\alpha^p\theta^q$ form for all the automorphisms is given in Figure 22.

The group $\mathrm{Aut}_+(\mathbb{O})$ is completely described by the fact that it has an element α of order 7 and element θ of order 3, which satisfy

$$\alpha\theta = \theta\alpha^2. \tag{36}$$

This is the case because there are only two groups of order 21 up to isomorphism, one of which is abelian [9], [11]. The group $\mathrm{Aut}_+(\mathbb{O})$ is clearly non-abelian. This leads us directly to the conclusion that $\mathrm{Aut}_+(\mathbb{O})$ is isomorphic to the other group of order 21. This group is the subgroup of S_7 generated by the permutations[5] $(2, 3, 5)(4, 7, 6)$ and $(1, 2, 3, 4, 5, 6, 7)$. For more detail on this group, see [9], [11].

[5] We assume the reader is familiar with cycle notation for permutations. For an explanation, see [10].

```
ϵ[{x1_, x2_, x3_, x4_, x5_, x6_, x7_, x8_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
α[{x1_, x5_, x4_, x8_, x3_, x7_, x2_, x6_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
β[{x1_, x3_, x8_, x6_, x4_, x2_, x5_, x7_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
γ[{x1_, x4_, x6_, x7_, x8_, x5_, x3_, x2_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
δ[{x1_, x8_, x7_, x2_, x6_, x3_, x4_, x5_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ζ[{x1_, x6_, x2_, x5_, x7_, x4_, x8_, x3_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
η[{x1_, x7_, x5_, x3_, x2_, x8_, x6_, x4_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
θ[{x1_, x7_, x2_, x8_, x4_, x6_, x3_, x5_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ι[{x1_, x3_, x7_, x5_, x8_, x6_, x2_, x4_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
κ[{x1_, x4_, x2_, x3_, x6_, x7_, x5_, x8_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
λ[{x1_, x8_, x5_, x4_, x7_, x2_, x3_, x6_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
μ[{x1_, x6_, x3_, x8_, x2_, x5_, x4_, x7_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ν[{x1_, x7_, x4_, x6_, x5_, x3_, x8_, x2_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ξ[{x1_, x2_, x8_, x7_, x3_, x4_, x6_, x5_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
o[{x1_, x5_, x6_, x2_, x4_, x8_, x7_, x3_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ρ[{x1_, x2_, x5_, x6_, x8_, x7_, x4_, x3_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
σ[{x1_, x5_, x3_, x7_, x6_, x2_, x8_, x4_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
τ[{x1_, x3_, x4_, x2_, x7_, x5_, x6_, x8_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
υ[{x1_, x4_, x8_, x5_, x2_, x3_, x7_, x6_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ϕ[{x1_, x8_, x6_, x3_, x5_, x4_, x2_, x7_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
ψ[{x1_, x6_, x7_, x4_, x3_, x8_, x5_, x2_}]:={x1, x2, x3, x4, x5, x6, x7, x8}
myAutomorphisms={ϵ, α, β, γ, δ, ζ, η, θ, ι, κ, λ, μ, ν, ξ, o, ρ, σ, τυ, ϕ, ψ}
```

Figure 18: The code for defining the 21 automorphisms in Mathematica and assigning them all to a list.

```
groupTable = Table[Composition[myAutomorphisms[[i]], myAutomorphisms[[j]]][basis],
{i, 1, Length[myAutomorphisms]}, {j, 1, Length[myAutomorphisms]}]
```

Figure 19: The code that generates the outputs of all possible compositions of the elements of $\mathrm{Aut}_+(\mathbb{O})$.

```
TableForm[Table[Flatten[Position[greek, table[[1, i, j]]]][[1]], {i, 1, 21},
{j, 1, 21}]]/.{1 → ϵ, 2 → α, 3 → β, 4 → γ, 5 → δ, 6 → ζ, 7 → η, 8 → θ,
9 → ι, 10 → κ, 11 → λ, 12 → μ, 13 → ν, 14 → ξ, 15 → o, 16 → ρ, 17 → σ, 18 → τ,
19 → υ, 20 → ϕ, 21 → ψ}
```

Figure 20: The code that creates the group table of $\mathrm{Aut}_+(\mathbb{O})$.

$\circ$	ϵ	α	β	γ	δ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	ρ	σ	τ	υ	ϕ	ψ
ϵ	ϵ	α	β	γ	δ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	ρ	σ	τ	υ	ϕ	ψ
α	α	β	γ	δ	ζ	η	ϵ	ρ	κ	λ	μ	ν	ξ	o	ι	σ	τ	υ	ϕ	ψ	θ
β	β	γ	δ	ζ	η	ϵ	α	σ	λ	μ	ν	ξ	o	ι	κ	τ	υ	ϕ	ψ	θ	ρ
γ	γ	δ	ζ	η	ϵ	α	β	τ	μ	ν	ξ	o	ι	κ	λ	υ	ϕ	ψ	θ	ρ	σ
δ	δ	ζ	η	ϵ	α	β	γ	υ	ν	ξ	o	ι	κ	λ	μ	ϕ	ψ	θ	ρ	σ	τ
ζ	ζ	η	ϵ	α	β	γ	δ	ϕ	ξ	o	ι	κ	λ	μ	ν	ψ	θ	ρ	σ	τ	υ
η	η	ϵ	α	β	γ	δ	ζ	ψ	o	ι	κ	λ	μ	ν	ξ	θ	ρ	σ	τ	υ	ϕ
θ	θ	υ	ρ	ϕ	σ	ψ	τ	ι	ϵ	δ	α	ζ	β	η	γ	ν	κ	ξ	λ	o	μ
ι	ι	λ	ν	o	κ	μ	ξ	ϵ	θ	σ	υ	ψ	ρ	τ	ϕ	β	δ	η	α	γ	ζ
κ	κ	μ	ξ	ι	λ	ν	o	α	ρ	τ	ϕ	θ	σ	υ	ψ	γ	ζ	ϵ	β	δ	η
λ	λ	ν	o	κ	μ	ξ	ι	β	σ	υ	ψ	ρ	τ	ϕ	θ	δ	η	α	γ	ζ	ϵ
μ	μ	ξ	ι	λ	ν	o	κ	γ	τ	ϕ	θ	σ	υ	ψ	ρ	ζ	ϵ	β	δ	η	α
ν	ν	o	κ	μ	ξ	ι	λ	δ	υ	ψ	ρ	τ	ϕ	θ	σ	η	α	γ	ζ	ϵ	β
ξ	ξ	ι	λ	ν	o	κ	μ	ζ	ϕ	θ	σ	υ	ψ	ρ	τ	ϵ	β	δ	η	α	γ
o	o	κ	μ	ξ	ι	λ	ν	η	ψ	ρ	τ	ϕ	θ	σ	υ	α	γ	ζ	ϵ	β	δ
ρ	ρ	ϕ	σ	ψ	τ	θ	υ	κ	α	ζ	β	η	γ	ϵ	δ	ξ	λ	o	μ	ι	ν
σ	σ	ψ	τ	θ	υ	ρ	ϕ	λ	β	η	γ	ϵ	δ	α	ζ	o	μ	ι	ν	κ	ξ
τ	τ	θ	υ	ρ	ϕ	σ	ψ	μ	γ	ϵ	δ	α	ζ	β	η	ι	ν	κ	ξ	λ	o
υ	υ	ρ	ϕ	σ	ψ	τ	θ	ν	δ	α	ζ	β	η	γ	ϵ	κ	ξ	λ	o	μ	ι
ϕ	ϕ	σ	ψ	τ	θ	υ	ρ	ξ	ζ	β	η	γ	ϵ	δ	α	λ	o	μ	ι	ν	κ
ψ	ψ	τ	θ	υ	ρ	ϕ	σ	o	η	γ	ϵ	δ	α	ζ	β	μ	ι	ν	κ	ξ	λ

Figure 21: The group table of $\mathrm{Aut}_+(\mathbb{O})$.

$$
\begin{array}{lllllll}
\alpha=\alpha, & \beta=\alpha^2, & \gamma=\alpha^3, & \delta=\alpha^4, & \zeta=\alpha^5, & \eta=\alpha^6, & \theta=\theta, \\
\iota=\theta^2, & \kappa=\alpha\theta^2 & \lambda=\alpha^2\theta^2, & \mu=\alpha^3\theta^2, & \nu=\alpha^4\theta^2, & \xi=\alpha^5\theta^2, & o=\alpha^6\theta^2, \\
\rho=\alpha\theta, & \sigma=\alpha^2\theta, & \tau=\alpha^3\theta, & \upsilon=\alpha^4\theta, & \phi=\alpha^5\theta, & \psi=\alpha^6\theta, & \epsilon=\alpha^7=\theta^3
\end{array}
$$

Figure 22: The $\alpha^p\theta^q$ form of all the automorphisms in $\langle\alpha,\theta\rangle$ where $p,q\in\mathbb{Z}$.

5 Geometric Interpretations of $\mathbf{Aut}_+(\mathbb{O})$

The results we have presented herein can be interpreted geometrically. The automorphism group we have presented is closely related to both the 7-sphere in eight dimensions and to the 7-dimensional cross product. The fact that the cross product is well defined in seven dimensions is closely related to the multiplicative properties of the octonions. For a more detailed explanation of the 7-dimensional cross product, see [6].

5.1 The 7-sphere in 8-D

Consider the set of points $\{(1,0),(-1,0),(0,1),(0,-1)\}$ on the unit circle in two dimensions, shown in Figure 23. These points correspond to the numbers $\{1,-1,i,-i\}$ in $\mathbb{C}$. Taking the usual definition of multiplication for complex numbers, the product of any two of these numbers is another number in the set. Hence, what we have is a set of points on the 1-sphere in two dimensions that is multiplicatively closed.

As a direct analog to this, we take the unit 7-sphere in eight dimensions, again considering the set of points where the sphere intersects the coordinate axes. Since any octonion can be represented as a point in eight dimensions, this set of points corresponds to the set of octonions

$$\mathbb{O}_{1\mathbb{Z}} = \{1, i, j, k, k\ell, j\ell, i\ell, \ell, -1, -i, -j, -k, -k\ell, -j\ell, -i\ell, -\ell\}. \qquad (37)$$

The fact that this set is closed under octonionic multiplication is clear from the mnemonic in Figure 5. Since the automorphisms we are considering act non-trivially on $i, j, k, k\ell, j\ell, i\ell$ and ℓ, one geometric interpretation of our results is as the 90 degree rotations in eight dimensions that preserve the multiplicative closure of $\mathbb{O}_{1\mathbb{Z}}$. The fact that these automorphisms correspond to 90 degree rotations is due to the fact that every coordinate axis is mapped to another coordinate axis.

5.2 The 7-Dimensional Cross Product

That the cross product exists in three dimensions is well known. Moreover, as we have noted, when considering only the imaginary part of quaternions, there is an isomorphism between quaternionic multiplication and the 3-dimensional vector cross product. Since the 3-dimensional cross product does not change under any of the rotation in $SO(3)$, the 3-dimensional rotation group, we can see that $SO(3)$ is also the automorphism group of the quaternions [6].

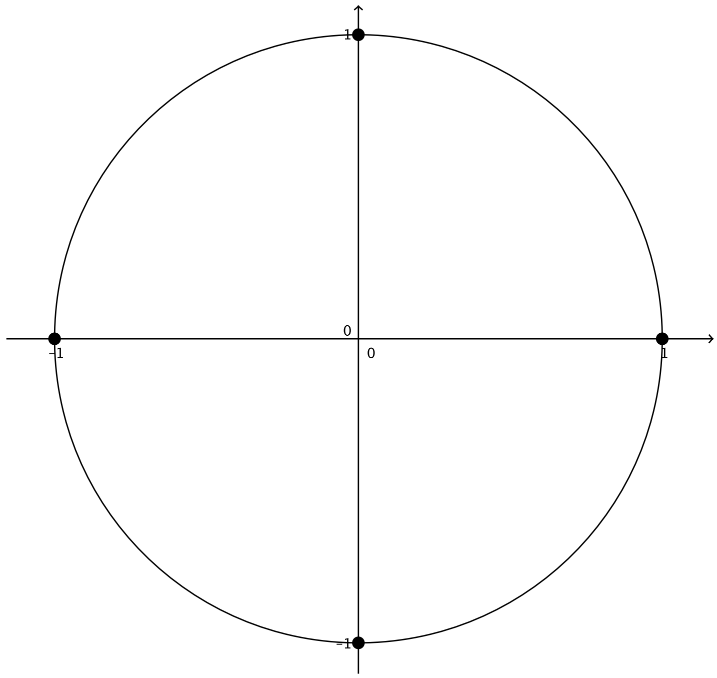

Figure 23: The unit cirlce in 2-D with the points $\{(1,0),(-1,0),(0,1),(0,-1)\}$ corresponding to the complex numbers $\{1,-1,i,-i\}$ respectively.

The cross product also exists in seven dimensions. Moreover, this is the only other space in which the cross product exists [6]. We make the important distinction here that, unlike the imaginary part of a complex number, which is the real coefficient attached to the i term, by the imaginary part of an octonion, we mean the part of the number that is not real. Hence the imaginary part of i is i itself! Now, as we have previously stated, there is a close relationship between octonionic multiplication and the 7-dimensional cross product. In fact, just as we did for the quaternions, if we consider only the imaginary part of the product, the relationship is an isomorphism. Considering only the imaginary part of any of the products defined on $\mathbb{O}_{1\mathbb{Z}}$ accounts for the fact that the square of any of $i, j, k, k\ell, j\ell, i\ell, \ell$ is -1 and the cross product of any vector with itself is 0.

Since the 7-dimensional cross product is isomorphic to the imaginary part of octonionic multiplication, we can consider $\mathbb{R}^7$ with basis given by $i, j, k, k\ell, j\ell, i\ell$, and ℓ and use the mnemonic for octonionic multiplication in Figure 5 to determine the cross product in 7-dimensions, taking only the imaginary part of octonionic products. Moreover, given the isomorphism between the imaginary part of octonionic multiplication and the 7-dimensional cross product, we see that the automorphism group of the 7-dimensional cross product is also G_2, of which $\mathrm{Aut}_+(\mathbb{O})$ is a subgroup, as we have already noted. Hence all elements of $\mathrm{Aut}_+(\mathbb{O})$ preserve the cross product in 7 dimensions with basis given by $i, j, k, k\ell, j\ell, i\ell$, and ℓ.

6 Conclusion

In conclusion, we have shown that there are 28 nonequivalent ways to label the Fano plane as a mnemonic for the multiplication table of the octonions when not permitting the use of minus signs. This is likely tied to the fact that the dimension of the automorphism group of the octonions is 14 [4]. The exact nature of this relationship is unclear at present. However, this doubling could possibly be tied to there being a positive and negative direction to each dimension.

We have also shown that there are 21 automorphisms on the octonions that only permute the imaginary basis units. It is surprising that there are 28 nonequivalent ways to label the Fano plane as a mnemonic for the octonionic multiplication table but only 21 automorphisms in $\mathrm{Aut}_+(\mathbb{O})$. Since counting automorphisms was motivated by counting nonequivalent ways to label the Fano plane, we reasonably expected the same number of automorphisms as ways to label the Fano plane, especially since we do not allow for minus signs when labeling the Fano plane and require our automorphisms to have the positivity property. The reason for this difference is tied to the fact that all elements of $\mathrm{Aut}_+(\mathbb{O})$ preserve the orientation of the Fano plane when applied to a given labeling. However, when we defined equivalence of labelings, we included reflections, which do not preserve orientation. Therefore, the automorphisms in $\mathrm{Aut}_+(\mathbb{O})$ do not move between all elements of an equivalence class for a labeling of the Fano plane. However, given an orientation of the Fano plane and a labeling of that orientation that works as a mnemonic, we can use the elements of $\mathrm{Aut}_+(\mathbb{O})$ to generate twenty more labelings. We know that, if we do not account for equivalence between labelings, there are 168 ways to label the Fano plane as a mnemonic for octonionic multiplication. Therefore, since our automorphisms preserve the orientation of the Fano plane, we have that there must be $168 \div 21 = 8$ ways to orient the Fano plane such that it is possible to construct a mnemonic for octonionic multiplication.

The natural extension of the work presented in this paper would be to consider labellings of the Fano plane that permit minus signs and automorphisms that do not have the positivity property. From our work, we do know a little about what to expect with these extensions. Specifically, allowing minus signs is directly related to the orientation provided for each line in the Fano plane. In particular, on any one line, we can reverse the orientation and put a minus sign on any one of the units on the line to account for the reversal in orientation. However, the sign change will then affect the other two lines that share the unit with a minus sign. This necessitates two more orientation changes to construct a correct table. Moreover, given a properly labeled Fano plane, it is possible to create another correct table only by reversing three orientations or all seven orientations. If we consider removing the positivity requirement for automorphisms, we know the number of automorphisms we obtain will be a multiple of 21 since $\mathrm{Aut}_+(\mathbb{O})$ is a subgroup of this larger group of automorphisms.

In conclusion, we have shown that there are 28 nonequivalent ways to label the Fano plane a a mnemonic for the multiplication table of the octonions when not permitting the use of minus signs. Moreover, these nonequivalent labellings are related by automorphisms with the property that if ϕ is an automorphism, $\phi(a)$ and a have the same sign. There are 21 such automorphisms. Removing the requirement of positivity in both cases gives rise to a larger structure, the nature of which can be determined using methods similar to those employed in this paper.

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