

530.327 - Introduction to Fluid Mechanics - Su  
 Statics: Derivation of the basic equation.

Reading: Text, §3.1

Having defined a “fluid” and covered the fundamental concepts of stress, viscosity, etc., we’re prepared to look at the first major category of problems in fluid mechanics, namely *fluid statics*, which is concerned with fluids that are stationary. (The other category of fluid mechanics problems is fluid dynamics, which naturally involves fluids in motion.)

As is familiar in physical problems, our goal is to write the differential equation that is relevant to fluid statics. The approach we will take is to start with an infinitesimal fluid particle, integrate the forces on that particle, and take advantage of the infinitesimal-ness of the particle to end up with a differential equation. This approach is ubiquitous in fluid mechanics – we’ll use it again later in the semester – and is also significant in numerous other areas of science and engineering, such as structural mechanics.

The fluid particle we’re interested in is shown in Fig. 1. The particle is centered at the origin of

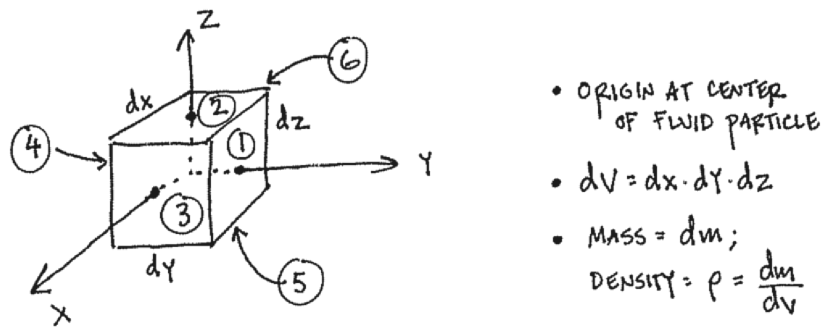


Figure 1: A fluid particle with volume  $dV$  and density  $\rho$ , centered at the origin of  $x - y - z$  space.

Cartesian coordinate space, and has side lengths  $dx$ ,  $dy$  and  $dz$ , volume  $dV = dx dy dz$ , and density  $\rho$ . The positive  $z$ -axis points vertically upward. Assuming the particle is stationary, what forces act on the particle? Reviewing the possibilities as we know them:

- Gravity (a body force): Yes, gravity acts on the particle. Gravity doesn’t care whether the fluid particle is moving or not.
- Pressure (a surface force): Yes, pressure forces act on each of the particle’s faces. Pressure is present in fluids also regardless of whether or not they’re moving.
- Shear stress (a surface force): There are **no** shear stresses. To understand this, recall our definition that fluids move under the action of *any* shear stress. Since the fluid particle is stationary, there can be no shear stresses.

So, we’re interested in gravity and pressure. Let’s now quantify these terms mathematically.

**Gravity.** We know the force of gravity on a given body as being the product of the gravitational acceleration and the mass of that body. For the fluid particle in question here, we’ll write the gravitational force,  $d\mathbf{F}_B$ , as

$$d\mathbf{F} = \mathbf{g} dm = \mathbf{g}\rho dV = g\rho dx dy dz(-\hat{k}). \quad (1)$$

The  $\hat{k}$  is the unit vector (vector of length 1) pointing in the  $z$ -direction (similarly, following the text's notation,  $\hat{i}$  and  $\hat{j}$  are the unit vectors in the  $x$ - and  $y$ -directions, respectively – why we just can't use  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$ , I don't know); the gravitational force points in the minus- $z$  direction since the  $z$ -direction is positive upward.

**Pressure.** Now we want to compile the pressure forces acting on the fluid particle. The pressure field can be written in general form as

$$p = p(x, y, z). \quad (2)$$

For our fluid particle, we will begin by writing the pressure at the origin as  $p(0, 0, 0) = p_0$ , and then we'll fill out the pressure field by using Taylor series expansions. To refresh our memories, let's take a detour back into calculus (with substantial simplifications). Suppose we have a function  $f$  whose value, and the values of whose derivatives, are known at a position  $x$ . Then the value of the function at some location  $x + \Delta x$  is given, as a Taylor series, by

$$f(x + \Delta x) = f(x) + \frac{df}{dx} \Delta x + O(\Delta x^2) \quad (3)$$

where we are free to ignore the terms of order  $\Delta x^2$  and higher if  $\Delta x$  is small. This condition will be easily satisfied here since we're considering the fluid particle as infinitesimal.

Using Eq. 3, we can go ahead and write the values of the pressure field on each face of the fluid particle, basing the Taylor expansions on the values of the pressure and its derivatives at the origin. For faces 1 and 4, which are located at  $y = dy/2$  and  $-dy/2$ , respectively, the pressures can be written

$$p_1 = p(dy/2) = p_0 + \frac{\partial p}{\partial y} \left( \frac{dy}{2} \right), \quad \text{and} \quad p_4 = p(-dy/2) = p_0 + \frac{\partial p}{\partial y} \left( -\frac{dy}{2} \right). \quad (4)$$

That is, we assume that the pressure on each face differs from the pressure at the origin solely by virtue of their separation from the origin on the  $y$ -axis. (This in turn assumes that the pressure on each face is constant, which is also justifiable by the assumption that the fluid particle is small.) For faces 2 and 5, located at  $z = dz/2$  and  $-dz/2$ , and faces 3 and 6, located at  $x = dx/2$  and  $-dx/2$ , we can similarly write

$$p_2 = p_0 + \frac{\partial p}{\partial z} \left( \frac{dz}{2} \right), \quad \text{and} \quad p_5 = p_0 + \frac{\partial p}{\partial z} \left( -\frac{dz}{2} \right) \quad (5)$$

$$p_3 = p_0 + \frac{\partial p}{\partial x} \left( \frac{dx}{2} \right), \quad \text{and} \quad p_6 = p_0 + \frac{\partial p}{\partial x} \left( -\frac{dx}{2} \right) \quad (6)$$

Given these pressure values, we can write the pressure *forces* by using the relation

$$(\text{pressure}) = \frac{(\text{force})}{(\text{area})}, \quad (7)$$

and by being careful to define the direction of the pressure forces properly so that they represent forces acting *on* the fluid particle. Thus, we get for the pressure force on face 1

$$\begin{aligned} d\mathbf{F}_1 &= p_1 \overbrace{(dx \, dz)}^{\text{Area of face}} \overbrace{(-\hat{j})}^{\text{Direction of force}} \\ &= \left( p_0 + \frac{\partial p}{\partial y} \left( \frac{dy}{2} \right) \right) dx \, dz (-\hat{j}) \end{aligned} \quad (8)$$

where the force acts in the negative  $y$ -direction because that's the direction in which the surroundings push on the fluid particle on face 1; for face 4, we get

$$d\mathbf{F}_4 = p_4 dx dz \hat{j} = \left( p_0 + \frac{\partial p}{\partial y} \left( -\frac{dy}{2} \right) \right) dx dz \hat{j} \quad (9)$$

which acts in the positive  $y$ -direction. The results are similar for the pressure forces acting on the other faces:

$$d\mathbf{F}_2 = \left( p_0 + \frac{\partial p}{\partial z} \left( \frac{dz}{2} \right) \right) dx dy (-\hat{k}), \quad \text{and} \quad d\mathbf{F}_5 = \left( p_0 + \frac{\partial p}{\partial z} \left( -\frac{dz}{2} \right) \right) dx dy \hat{k} \quad (10)$$

$$d\mathbf{F}_3 = \left( p_0 + \frac{\partial p}{\partial x} \left( \frac{dx}{2} \right) \right) dy dz (-\hat{i}), \quad \text{and} \quad d\mathbf{F}_6 = \left( p_0 + \frac{\partial p}{\partial x} \left( -\frac{dx}{2} \right) \right) dy dz \hat{i}. \quad (11)$$

It is subtle and maybe even somewhat unintuitive to think only of the pressure forces acting *on* the fluid element. In reality, if we place any surface in a fluid, pressure will act on both sides; therefore, on any single face of our fluid particle, not only do the surroundings push in, but the fluid contained in the particle also pushes out. We are concerned only with the forces that are imposed by the surroundings on the fluid particle because we're explicitly interested in the reaction of the fluid particle to these external forces.

Now we can sum, or integrate, the surface forces over the entire fluid particle, by writing

$$d\mathbf{F}_S = d\mathbf{F}_1 + d\mathbf{F}_4 + d\mathbf{F}_2 + d\mathbf{F}_5 + d\mathbf{F}_3 + d\mathbf{F}_6. \quad (12)$$

An interesting thing happens if we add the  $d\mathbf{F}_i$  in the pairs 1 and 4, 2 and 5, and 3 and 6 as suggested by the form of Eq. 12. Consider the sum of  $d\mathbf{F}_1$  and  $d\mathbf{F}_4$ , using Eqs. 8 and 9:

$$\begin{aligned} d\mathbf{F}_1 + d\mathbf{F}_4 &= \left( p_0 + \frac{\partial p}{\partial y} \left( \frac{dy}{2} \right) \right) dx dz (-\hat{j}) + \left( p_0 + \frac{\partial p}{\partial y} \left( -\frac{dy}{2} \right) \right) dx dz \hat{j} \\ &= -\frac{\partial p}{\partial y} dx dy dz \hat{j} \end{aligned} \quad (13)$$

and the  $p_0$  terms cancel out, leaving only  $\partial p/\partial y$ , the  $y$ -partial derivative of the pressure at the origin. Similarly, for faces 2 & 5 and 3 & 6, we get

$$d\mathbf{F}_2 + d\mathbf{F}_5 = -\frac{\partial p}{\partial z} dx dy dz \hat{k} \quad (14)$$

$$d\mathbf{F}_3 + d\mathbf{F}_6 = -\frac{\partial p}{\partial x} dx dy dz \hat{i}. \quad (15)$$

Here, we remember from vector calculus that the *gradient* of the pressure can be written

$$\nabla p \equiv \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) p, \quad (16)$$

which means that the total, integrated surface pressure force  $d\mathbf{F}_S$  acting on the fluid element can be written

$$d\mathbf{F}_S = -\nabla p dx dy dz = -\nabla p dV \quad (17)$$

which is notable because only changes in pressure, not the actual magnitudes of pressure, are significant.

**Total forces.** Since we now know the total body and pressure forces acting on the fluid particle (through Eqs. 1 and 17), we can write the total forces acting on the particle as

$$d\mathbf{F} = d\mathbf{F}_B + d\mathbf{F}_S = (-\nabla p + \rho \mathbf{g}) dV. \quad (18)$$

Now we take another detour, this time into classical mechanics, and remember Newton's second law, namely that the total force on a particle equals the product of its mass and its acceleration. In our case, the fluid particle has mass  $dm$ , but is stationary, so its acceleration is zero. Therefore the total force acting on it must also be zero –

$$0 = d\mathbf{F} = d\mathbf{F}_B + d\mathbf{F}_S = (-\nabla p + \rho\mathbf{g}) dV. \quad (19)$$

At this point Eq. 19 is still technically an integral equation. The reason there's no integral sign involved is because of our assumption that the fluid particle is infinitesimally small, which allowed us to write sums instead of integrals. Looking at Eq. 19, we note one more thing. The volume of the fluid particle  $dV$  that appears on the right is always non-zero; so for Eq. 19 to hold, we require that

$$-\nabla p + \rho\mathbf{g} = 0 \quad (20)$$

which is now a *differential* equation. We can simplify this some more by looking at the individual vector components (remember that  $\nabla p$  is a vector, as we saw in Eq. 16). There is no gravity component in either the  $x$ - or  $y$ -directions, so

$$x - \text{component:} \quad -\frac{\partial p}{\partial x} = 0 \quad (21)$$

$$y - \text{component:} \quad -\frac{\partial p}{\partial y} = 0, \quad (22)$$

which tells us that pressure in a stationary fluid doesn't vary in the horizontal,  $x$ - or  $y$ -directions. As for the  $z$ -component of Eq. 20, we get

$$\boxed{z - \text{component:} \quad \frac{dp}{dz} = -\rho g} \quad (23)$$

(we are allowed to write the  $z$ -derivative of  $p$  as a total derivative because pressure doesn't vary with  $x$  or  $y$  anymore). Equation 23 tells us how pressure in a static fluid varies with depth, and is the basic equation in fluid statics.