# Fluid Mechanics

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## Contents

We consider velocities $\mathbf{u} = (u, v, w)$. The Lagrangian description tracks particular particles; whereas the Eulerian description looks at a window in space.

# 1 Proofs

## 1.1 Prove $\{\nabla(\phi\mathbf{a}) = \phi(\nabla \cdot \mathbf{a}) + (\mathbf{a} \cdot \nabla)\phi\}$

Starting with:

$$\nabla(\phi\mathbf{a}) = \phi(\nabla \cdot \mathbf{a}) + (\mathbf{a} \cdot \nabla)\phi \tag{1}$$

Now, in suffix notation, using the chain rule for differentiation:

$$\frac{\partial}{\partial x_i}(\phi a_i) = \phi\frac{\partial a_i}{\partial x_i} + a_i\frac{\partial \phi}{\partial x_i} \tag{2}$$

Which, putting back into vector notation, gives:

$$\nabla(\phi\mathbf{a}) = \phi(\nabla \cdot \mathbf{a}) + (\mathbf{a} \cdot \nabla)\phi \tag{3}$$

## 1.2 Use the Divergence Theorem to Prove That $\{\int_V \nabla\phi \, dV = \int_S \phi\mathbf{n} \, dS\}$

Now, the divergence theorem is:

$$\int_V \nabla \cdot \mathbf{a} \, dV = \int_S \mathbf{a} \cdot \mathbf{n} \, dS \tag{4}$$

Now, let $\mathbf{a} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector. Thus:

$$\nabla \cdot \mathbf{a} = \nabla \cdot (\phi\mathbf{c}) = \phi(\nabla \cdot \mathbf{c}) + (\mathbf{c} \cdot \nabla)\phi \tag{5}$$
$$= (\mathbf{c} \cdot \nabla)\phi \tag{6}$$

Thus, (4) becomes:

$$\int_V (\mathbf{c} \cdot \nabla)\phi \, dV = \int_S \phi\mathbf{c} \cdot \mathbf{n} \, dS \tag{7}$$
$$\Rightarrow \mathbf{c} \cdot \int_V \nabla\phi \, dV = \mathbf{c} \cdot \int_S \phi\mathbf{n} \, dS \tag{8}$$
$$\Rightarrow \int_V \nabla\phi \, dV = \int_S \phi\mathbf{n} \, dS \tag{9}$$

Thus proven.

## 1.3 Prove $\{\nabla \times (\phi\mathbf{a}) = \phi(\nabla \times \mathbf{a}) + (\nabla\phi) \times \mathbf{a}\}$

Now, looking at the LHS of:

$$\nabla \times (\phi\mathbf{a}) = \phi(\nabla \times \mathbf{a}) + (\nabla\phi) \times \mathbf{a} \tag{10}$$

We have that the $i$-component is:

$$\nabla \times (\phi \mathbf{a})|_i = \frac{\partial}{\partial y}(\phi a_3) - \frac{\partial}{\partial z}(\phi a_2) \tag{11}$$

Which can be expanded by the chain rule:

$$\begin{aligned} \nabla \times (\phi \mathbf{a})|_i &= \phi\frac{\partial a_3}{\partial y} + a_3\frac{\partial \phi}{\partial y} - \phi\frac{\partial a_2}{\partial z} - a_2\frac{\partial \phi}{\partial z} & (12) \\ &= \phi\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + \left(a_3\frac{\partial \phi}{\partial y} - a_2\frac{\partial \phi}{\partial z}\right) & (13) \\ &= \phi(\nabla \times \mathbf{a})|_i + (\nabla\phi) \times \mathbf{a}|_i & (14) \end{aligned}$$

All other components will be similar. Thus, we have proven:

$$\nabla \times (\phi \mathbf{a}) = \phi(\nabla \times \mathbf{a}) + (\nabla\phi) \times \mathbf{a} \tag{15}$$

### 1.4 Prove $\{\mathbf{u} \times \nabla \times \mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}\}$

Now, looking at the $i$-component of $\mathbf{u} \times \nabla \times \mathbf{u}$, we have:

$$\begin{aligned} \mathbf{u} \times \nabla \times \mathbf{u}|_i &= v\left(\frac{\partial w}{\partial z} - \frac{\partial v}{\partial z}\right) + w\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & (16) \\ &= \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) & (17) \\ &= \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}w^2\right) - (\mathbf{u}\cdot\nabla)u & (18) \end{aligned}$$

Thus, all other components will work similarly to prove:

$$\mathbf{u} \times \nabla \times \mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{19}$$

## 2 Derivations

### 2.1 Streamlines

Consider the section of a streamline. We have one coordinate at $\mathbf{x}$, another at $\mathbf{x}+\delta\mathbf{x}$; with velocites at each given by $\mathbf{u}$ and $\mathbf{u}+\delta\mathbf{u}$. Thus, as $\delta\mathbf{x} \to 0$, $\delta\mathbf{x}$ and $\mathbf{u}$ become more and more parallel. Thus, a streamline will have $\mathbf{u}$ as a tangent vector:

$$\begin{aligned} \delta\mathbf{x} &\propto \mathbf{u} & (20) \\ \Rightarrow d\mathbf{x} &= k\mathbf{u} & (21) \\ \Rightarrow (dx, dy, dz) &= k(u, v, w) & (22) \end{aligned}$$

Thus, we have the equation for a streamline:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{23}$$

When $u = v = w = 0$ we have stagnation points; and are the only places where streamlines cross.

## 2.2 Particle Path

The Lagrangian description yields:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \tag{24}$$

$$\Rightarrow \tag{25}$$

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w \tag{26}$$

If the velocity flow is not a function of time (i.e. is a steady flow), then the streamlines are the same as the particle paths.

### 2.2.1 Proof of Streamline and Particle Path Coincidence for Steady Flows

Now:

$$u = \frac{dx}{dt} \tag{27}$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{u} = \int_{t_0}^{t} dt \tag{28}$$

Which can only be done is $u$ is not a function of time.
Now:

$$\int_{t_0}^{t} dt = t - t_0 \tag{29}$$

It follows that:

$$\int_{x_0}^{x} \frac{dx}{u} = \int_{y_0}^{y} \frac{dy}{v} = \int_{z_0}^{z} \frac{dz}{w} = t - t_0 \tag{30}$$

Now, differentiating w.r.t $x$, say:

$$\frac{d}{dx}\int_{x_0}^{x} \frac{dx}{u} = \frac{1}{u} = \frac{d}{dx}\int_{y_0}^{y} \frac{dy}{v} \tag{31}$$

$$= \frac{dy}{dx}\frac{d}{dy}\int_{y_0}^{y} \frac{dy}{v} \tag{32}$$

$$= \frac{1}{v}\frac{dy}{dx} \tag{33}$$

$$= \frac{1}{w}\frac{dz}{dx} \tag{34}$$

That is:

$$\frac{1}{u} = \frac{1}{v}\frac{dy}{dx} = \frac{1}{w}\frac{dz}{dx} \tag{35}$$

Which is back to the streamline equation:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{36}$$

## 2.3 The Material Derivative $\frac{D}{Dt}$

Now, if we have the density as

$$\rho = \rho(x, y, x, t) \tag{37}$$

Then, for small changes we have:

$$\delta\rho = \frac{\partial\rho}{\partial t}\delta t + \frac{\partial\rho}{\partial x_i}\delta x_i \tag{38}$$

And, dividing by $\delta t$, and taking the limit $\delta t \to 0$:

$$\lim_{\delta t\to 0}\left(\frac{\delta\rho}{\delta t}\right) = \frac{\partial\rho}{\partial t} + \lim_{\delta t\to 0}\left(\frac{\delta x_i}{\delta t}\right) \tag{39}$$

And, we have that:

$$\lim_{\delta t\to 0}\left(\frac{\delta x_i}{\delta t}\right) = \frac{\partial x_i}{\partial t} \tag{40}$$

$$= u_i \tag{41}$$

And, we define the LHS of (39) as the material derivative:

$$\frac{D\rho}{Dt} \equiv \lim_{\delta t\to 0}\left(\frac{\delta\rho}{\delta t}\right) \tag{42}$$

Hence, putting all this together, (39) can be written as:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u_i\frac{\partial\rho}{\partial x_i} \tag{43}$$

Which is equivalent to:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho \tag{44}$$

So, the material derivative itself is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla) \tag{45}$$

Notice that it is an operator.

## 2.4 Integral Form of the Continuity Equation

The mass in a volume $V$ is:

$$\int_V \rho\, dV \tag{46}$$

Hence, the rate of change of mass in $V$ is:

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial\rho}{\partial t}\, dV \tag{47}$$

The rate of mass flow out of $V$, through a bounding closed surface $S$ is;

$$\int_S \rho\mathbf{u}\cdot\mathbf{n}\,dS \tag{48}$$

Hence, if mass is conserved,(47) and (48) must balance:

$$\int_V \frac{\partial\rho}{\partial t}\,dV = -\int_S \rho\mathbf{u}\cdot\mathbf{n}\,dS \tag{49}$$

Which is the integral form of the continuity equation

## 2.5 Derivative Form of the Continuity Equation

Starting with the divergence theorem, and the integral form of the continuity equation:

$$\int_V (\nabla\cdot\mathbf{a})\,dV = \int_S \mathbf{a}\cdot\mathbf{n}\,dS \tag{50}$$

$$\int_V \frac{\partial\rho}{\partial t}\,dV = -\int_S \rho\mathbf{u}\cdot\mathbf{n}\,dS \tag{51}$$

Now, the RHS of (51), using (50) becomes:

$$\int_S \rho\mathbf{u}\cdot\mathbf{n}\,dS = \int_V \nabla\cdot(\rho\mathbf{u})\,dV \tag{52}$$

Now, expanding out the divergence in the RHS of (52), by vector calculus:

$$\nabla\cdot(\rho\mathbf{u}) = \rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho \tag{53}$$

Hence, (52) becomes:

$$\int_S \rho\mathbf{u}\cdot\mathbf{n}\,dS = \int_V \{\rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\}\,dV \tag{54}$$

Now, putting (54) into (51); bringing everything over to the other side & putting under a single integral:

$$\int_V \frac{\partial\rho}{\partial t} + \rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\,dV = 0 \tag{55}$$

Now, the volume $V$ can be shrunk down to a point, hence the integrand is zero:

$$\frac{\partial\rho}{\partial t} + \rho(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho = 0 \tag{56}$$

We notice elements of the material derivative:

$$\frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho \equiv \frac{D\rho}{Dt} \tag{57}$$

Hence, we have:

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{u}) = 0 \tag{58}$$

Which is the pointwise (derivative) form of the continuity equation.

### 2.6 Hydrostatic Equilibrium

Now, suppose there is a body force $\mathbf{F}(\mathbf{x}, t)$ per unit mass. Thus, the total body force is:

$$\int_V \rho \mathbf{F}\, dV \tag{59}$$

The internal pressure force is:

$$-\int_S p\mathbf{n}\, dS \tag{60}$$

If the system is in equilibrium, (59) and (60) must balance. Hence:

$$\int_V \rho \mathbf{F}\, dV = \int_S p\mathbf{n}\, dS \tag{61}$$

Now, if we use the divergence theorem in the form:

$$\int_V \nabla\phi\, dV = \int_S \phi\mathbf{n}\, dS \tag{62}$$

the RHS of (61) becomes:

$$\int_S p\mathbf{n}\, dS = \int_V \nabla p\, dV \tag{63}$$

Hence, (61) becomes:

$$\int_V \rho\mathbf{F} - \nabla p\, dV = 0 \tag{64}$$

Again, we can shrink the volume down to a point, giving the equation for hydrostatic equlibrium:

$$\rho\mathbf{F} = \nabla p \tag{65}$$

### 2.7 Euler's Equation

Now, if we start from Newtons $2^{nd}$ law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{u}) \tag{66}$$

The fluid analogue is:

$$\frac{d}{dt}\left(\int_V \mathbf{u}\rho\, dV\right) = -\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, dS + \int_V \rho\mathbf{F} - \nabla p\, dV \tag{67}$$

That is, the rate of change of momentum is mass flux in, plus the resultant force.
Now, if we look at the middle integral in (67):

$$\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, dS = \left(\int_S \rho u(\mathbf{u}\cdot\mathbf{n})\, dS, \int_S \rho v(\mathbf{u}\cdot\mathbf{n})\, dS, \int_S \rho w(\mathbf{u}\cdot\mathbf{n})\, dS\right) \tag{68}$$

Using the divergence theorem (50) on one of the components of (68):

$$\int_S \rho u(\mathbf{u}\cdot\mathbf{n})\,dS = \int_V \nabla\cdot(\rho u\mathbf{u})\,dV \tag{69}$$

Which, upon expansion of the RHS of (69) gives:

$$\int_V \nabla\cdot(\rho u\mathbf{u})\,dV = \int_V \{u\nabla\cdot(\rho\mathbf{u}) + \rho(\mathbf{u}\cdot\nabla)u\}\,dV \tag{70}$$

And, doing this for all components of (68), the middle integral in (67) becomes:

$$\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,dS = \int_V \{\mathbf{u}(\nabla\cdot(\rho\mathbf{u})) + \rho(\mathbf{u}\cdot\nabla)\mathbf{u})\}\,dV \tag{71}$$

Hence, (67) becomes:

$$\int_V \left(\rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t} + \mathbf{u}(\nabla\cdot\rho\mathbf{u}) + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho\mathbf{F} + \nabla p\right)\,dV = 0 \tag{72}$$

Now, notice that some components can be simplified:

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \rho\frac{D\mathbf{u}}{Dt} \tag{73}$$

$$\mathbf{u}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u}\right) = 0 \tag{74}$$

(74) by the continuity equation (58). After doing these simplifications, and shrinking the volume $V$ down to a point, we have that (72) becomes:

$$\rho\frac{D\mathbf{u}}{Dt} - \rho\mathbf{F} + \nabla p = 0 \tag{75}$$

Or, rearanging:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{76}$$

Which is known as Euler's equation.

## 2.8 Bernoulli's Equation

Starting with Euler's equation:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{77}$$

Now, under the assumption that $\rho$ is barotropic $\Rightarrow \rho = \rho(p)$, we have:

$$\frac{1}{\rho}\nabla p = \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{78}$$

$$= \frac{d}{dp}\left(\int\frac{dp}{\rho}\right)\frac{\partial p}{\partial x_i} \tag{79}$$

$$= \frac{\partial}{\partial x_i}\int\frac{dp}{\rho} \tag{80}$$

$$= \nabla\left(\int\frac{dp}{\rho}\right) \tag{81}$$

$$\Rightarrow \frac{1}{\rho}\nabla p = \nabla\left(\int\frac{dp}{\rho}\right) \tag{82}$$

Under the assumption that $\mathbf{F}$ is a conservative force, we can write an associated scalar potential:

$$\mathbf{F} = -\nabla\Omega \tag{83}$$

Under the assumption that $\mathbf{u}$ is a steady flow ($\Rightarrow \frac{\partial \mathbf{u}}{\partial t} = 0$), the material derivative compresses to:

$$\frac{D\mathbf{u}}{Dt} = (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{84}$$

Now, there is a vector identity:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\nabla\times\mathbf{u} \tag{85}$$

We shall define the vorticity $\omega \equiv \nabla\times\mathbf{u}$.
Hence, putting (85) on the LHS of (77); and using (82) and (83) on the RHS of (77), we have:

$$\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \int\frac{dp}{\rho} + \Omega\right) = \mathbf{u}\times\omega \tag{86}$$

Now, if the flow is irrotational, i.e. $\omega = 0$, then the object in the grad is a constant. Hence:

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \int\frac{dp}{\rho} + \Omega = const \tag{87}$$

Which is known as Bernoulli's equation.

# 3 Complex Potential

We define vorticity as $\omega \equiv \nabla\times\mathbf{u}$. Now, if a flow is irrotational, then:

$$\omega = \nabla\times\mathbf{u} = 0 \tag{88}$$

Hence, we can introduce some scalar potential:

$$\mathbf{u} = \nabla\phi \qquad \Rightarrow \qquad \nabla\times\nabla\phi = 0 \tag{89}$$

If a flow is incompressible, then:

$$\nabla\cdot\mathbf{u} = 0 \qquad \Rightarrow \qquad \nabla\cdot\nabla\phi = \nabla^2\phi = 0 \tag{90}$$

That is, in a 2D incompressible flow, with $\mathbf{u} = (u, v)$:

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{91}$$

Now, if we let:

$$u = \frac{\partial\psi}{\partial y} \tag{92}$$

$$v = -\frac{\partial\psi}{\partial x} \tag{93}$$

Then:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{94}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \tag{95}$$

$$= 0 \tag{96}$$

Hence, with this condition for the stream function $\psi$, the condition for incompressibility is satisfied. The same is true for the divergence expressed in plane- or spherical-polars.
Now, for a 2D irrotational flow, we have $\mathbf{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$:

$$\omega = \nabla \times \mathbf{u} \tag{97}$$

$$= -\mathbf{k}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \tag{98}$$

$$= 0 \tag{99}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \tag{100}$$

Thus, Laplace's equation is satisfied for both the stream function $\psi$, and scalar potential $\phi$, in an incompressible, irrotational 2D flow.
Notice:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{101}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{102}$$

That is:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{103}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{104}$$

Which are the Cauchy-Riemann equations.
That is, there exists an analytic complex function $w(z) = \phi + i\psi$. This is known as the complex potential.

$$w(z) = \phi + i\psi \tag{105}$$

Lines on which $\Im\{w(z)\} = \psi = \text{const}$ are streamlines.
Also notice:

$$\nabla^2 w = \nabla^2(\phi + i\psi) \tag{106}$$

$$= \nabla^2 \phi + i\nabla^2 \psi \tag{107}$$

$$= 0 \tag{108}$$

$$\Rightarrow \nabla^2 w = 0 \tag{109}$$

Now, we can derive a complex velocity, given that the complex potential is only a function of $z$, and not its conjugate:

$$\frac{dw}{dz} = u - iv \tag{110}$$

## 3.1 Special 2D Flows

### 3.1.1 Uniform Stream

Suppose we have a uniform flow of speed $U$, inclined at an angle $\alpha$ to the $x$-axis. Thus, the velocity vector is:

$$\mathbf{u} = (U\cos\alpha, U\sin\alpha) = (u, v) \tag{111}$$

Now, the complex velocity is given by:

$$\begin{aligned}
\frac{dw}{dz} &= u - iv && (112)\\
&= U\cos\alpha - iU\sin\alpha && (113)\\
&= Ue^{-i\alpha} && (114)\\
\Rightarrow \frac{dw}{dz} &= Ue^{-i\alpha} && (115)\\
\Rightarrow dw &= Ue^{-i\alpha}dz && (116)
\end{aligned}$$

Thus, integrating, we have the complex potential for a uniform flow, of strength $U$, inclined by an angle $\alpha$ to the $x$-axis:

$$w(z) = Uze^{-i\alpha} \tag{117}$$

### 3.1.2 Source

We look at a source of strength $m$, at the origin.
At any radius, mass flux must be the same:

$$2\pi rU = m \tag{118}$$

The velocity field is also purely radial. Hence, we have:

$$\begin{aligned}
\frac{dw}{dz} &= u - iv && (119)\\
&= U\cos\theta - Ui\sin\theta && (120)\\
&= Ue^{-i\theta} && (121)\\
&= \frac{U}{e^{i\theta}} && (122)\\
&= \frac{m}{2\pi re^{i\theta}} && (123)\\
&= \frac{m}{2\pi z} && (124)\\
\Rightarrow dw &= \frac{m}{2\pi z}dz && (125)
\end{aligned}$$

Hence, integrating, we have an expression for the complex potential of a source at the origin:

$$w(z) = \frac{m}{2\pi} \log z \tag{126}$$

Which may be easily generalised to a source of strength $m$, at $z = z_0$:

$$w(z) = \frac{m}{2\pi} \log(z - z_0) \tag{127}$$

If $m < 0$, then we have a sink.

### 3.1.3 Vortex

Here, the velocity field is purely tangential. Again, we have that the mass flux through any radius is constant $2\pi r V = k$. So:

$$\begin{aligned} \mathbf{u} &= V\hat{\theta} && (128)\\ &= (-V\sin\theta, V\cos\theta) && (129)\\ \Rightarrow \frac{dw}{dz} &= -iV(\cos\theta - i\sin\theta) && (130)\\ &= -iVe^{-i\theta} && (131) \end{aligned}$$

Thus, along a very similar argument for the source: for a vortex, at $z = z_0$, we have the complex potential:

$$w(z) = -\frac{ik}{2\pi} \log(z - z_0) \tag{132}$$

Convention: if $k > 0$, then vortex is anti-clockwise; $k < 0$ for clockwise.

### 3.1.4 Dipole Flow

Suppose we put a source and a sink very close together, with the source at $z = \delta e^{i\alpha}$:

$$\begin{aligned} w(z) &= \frac{m}{2\pi}\log(z - \delta e^{i\alpha}) - \frac{m}{2\pi}\log z && (133)\\ &= \frac{m}{2\pi}\log\left(1 - \frac{\delta e^{i\alpha}}{z}\right) && (134) \end{aligned}$$

Now, bring the two very close together: $\delta \to 0$, and $m\delta \equiv \mu$ be constant. Using the following Taylor expansion:

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \tag{135}$$

We have:

$$\begin{aligned} w(z) &= \frac{m}{2\pi}\left(-\frac{\delta e^{i\alpha}}{z} - \frac{\delta^2 e^{2i\alpha}}{2z^2} - \dots\right) && (136)\\ &= -\frac{\mu}{2\pi}\left(\frac{e^{i\alpha}}{z} + \frac{\delta e^{2i\alpha}}{2z^2} + \dots\right) && (137) \end{aligned}$$

Which, as $\delta \to 0$, tends to the complex potential for a dipole at $z = z_0$, inclined at an angle $\alpha$ to the $x$-axis:

$$w(z) = -\frac{\mu e^{i\alpha}}{2\pi(z - z_0)} \tag{138}$$

#### 3.1.5 Flow in a Corner

For a corner of angle $m$, we purely have:

$$w(z) = Az^m = Are^{im\theta} \tag{139}$$

### 3.2 Circle Theorem

If $w = f(z)$ is a given flow, and if a circle is placed in the flow at the origin, with a radius $|z| = a$, then, under the assumption that $f(z)$ is analytic inside and on the circle, the new potential is:

$$w(z) = f(z) + \overline{f}\left(\frac{a^2}{z}\right) \tag{140}$$

For example:
Uniform flow past a circle, with $\alpha = 0$:

$$\begin{aligned} f(z) &= Uz && (141)\\ \Rightarrow \overline{f}(z) &= Uz && (142)\\ \overline{f}\left(\frac{a^2}{z}\right) &= \frac{Ua^2}{z} && (143) \end{aligned}$$

Thus:

$$\begin{aligned} w(z) &= f(z) + \overline{f}\left(\frac{a^2}{z}\right) && (144)\\ &= Uz + \frac{Ua^2}{z} && (145)\\ &= U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) && (146) \end{aligned}$$

Thus, the streamlines are on:

$$\begin{aligned} \Im\{w(z)\} = \psi &= U\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right) && (147)\\ &= U\sin\theta\left(r - \frac{a^2}{r}\right) && (148)\\ &= const && (149) \end{aligned}$$

Notice, that for $\psi = 0$, we have $r = a$. So that flow on the surface of the circle is a streamline.