Inversion Invariant Bilipschitz Homogeneity

David Freeman

Department of Math, Physics, and Computer Science
University of Cincinnati - Blue Ash

Eastern Sectional Meeting of the AMS
Syracuse University, Syracuse, NY
October 2, 2010

An embedding $f : X \to Y$ is L-bilipschitz provided that for all $x_1, x_2 \in X$ we have

$$L^{-1} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq L\, d_X(x_1, x_2).$$

An embedding $f : X \to Y$ is L-bilipschitz provided that for all $x_1, x_2 \in X$ we have

$$L^{-1} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq L\, d_X(x_1, x_2).$$

A space X is bilipschitz homogeneous (BLH) provided that for every pair $x_1, x_2 \in X$ there exists a bilipschitz homeomorphism $f : X \to X$ such that $f(x_1) = x_2$.

An embedding $f : X \to Y$ is L-bilipschitz provided that for all $x_1, x_2 \in X$ we have

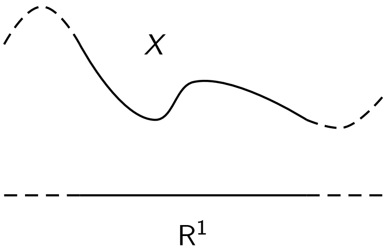
$$L^{-1} d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le L\, d_X(x_1, x_2).$$

A space X is bilipschitz homogeneous (BLH) provided that for every pair $x_1, x_2 \in X$ there exists a bilipschitz homeomorphism $f : X \to X$ such that $f(x_1) = x_2$.

If X is BLH with respect to L-bilipschitz maps, we say that X is uniformly BLH, and in particular, L-BLH.

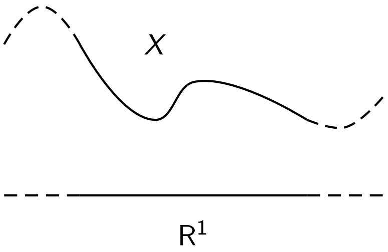
A Jordan curve is a metric space $X \approx \mathbb{S}$. A Jordan line is a (proper) metric space $X \approx \mathbb{R}$.

A Jordan curve is a metric space $X \approx \mathbb{S}$. A Jordan line is a (proper) metric space $X \approx \mathbb{R}$.



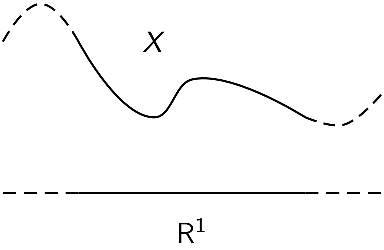
A Jordan curve is a metric space $X \approx \mathbb{S}$. A Jordan line is a (proper) metric space $X \approx \mathbb{R}$.

- *BLH* Jordan curves and lines have been extensively studied over the past decade (Bishop, Ghamsari, Herron, Mayer, Rohde, F).



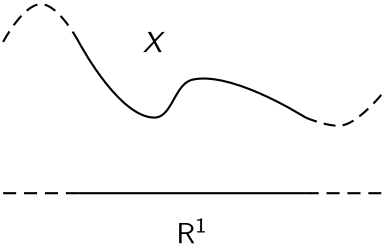
A Jordan curve is a metric space $X \approx \mathbb{S}$. A Jordan line is a (proper) metric space $X \approx \mathbb{R}$.

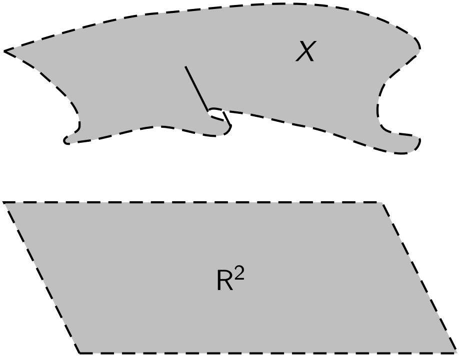
- *BLH* Jordan curves and lines have been extensively studied over the past decade (Bishop, Ghamsari, Herron, Mayer, Rohde, F).
- What about higher dimensional *BLH* spaces?



A Jordan curve is a metric space $X \approx S$. A Jordan line is a (proper) metric space $X \approx R$.

- *BLH* Jordan curves and lines have been extensively studied over the past decade (Bishop, Ghamsari, Herron, Mayer, Rohde, F).
- What about higher dimensional *BLH* spaces?





Theorem (Bishop, 01; F, 09)

Suppose X is a bilipschitz homogeneous Jordan curve or line. If $X \subset \mathbb{R}^2$, then X is bounded turning.

Theorem (Bishop, 01; F, 09)

Suppose X is a bilipschitz homogeneous Jordan curve or line. If $X \subset \mathbb{R}^2$, then X is bounded turning.

A space X is bounded turning provided that any two points $x, y \in X$ can be joined by a continuum E such that

$$\operatorname{diam}(E) \leq B|x - y|,$$

where B is independent of x, y.

Theorem (Bishop, 01; F, 09)

Suppose X is a bilipschitz homogeneous Jordan curve or line. If $X \subset \mathbb{R}^2$, then X is bounded turning.

A space X is bounded turning provided that any two points $x, y \in X$ can be joined by a continuum E such that

$$\operatorname{diam}(E) \leq B|x - y|,$$

where B is independent of x, y.

Note: There exist bilipschitz homogeneous curves in $\mathbb{R}^3$ that are not bounded turning ([Bishop, 01; Herron, Mayer, 99]).

Theorem (F, 09)

Suppose X is a bounded turning Jordan line in $\mathbb{R}^n$ containing 0. If both X and the inversion of X at 0 are uniformly bilipschitz homogeneous, then X is Ahlfors Q-regular.

Theorem (F, 09)

Suppose X is a bounded turning Jordan line in $\mathbb{R}^n$ containing 0. If both X and the inversion of X at 0 are uniformly bilipschitz homogeneous, then X is Ahlfors Q-regular.

A space X is Ahlfors Q-regular provided that for every $x \in X$ and $0 < r < \mathrm{diam}(X)$ we have

$$A^{-1} r^Q \leq \mathcal{H}^Q(B(x;r)) \leq A\, r^Q,$$

where A is independent of x and r.

Theorem (F, 09)

Suppose X is a bounded turning Jordan line in $\mathbb{R}^n$ containing 0. If both X and the inversion of X at 0 are uniformly bilipschitz homogeneous, then X is Ahlfors Q-regular.

A space X is Ahlfors Q-regular provided that for every $x \in X$ and $0 < r < \operatorname{diam}(X)$ we have

$$A^{-1} r^Q \le \mathcal{H}^Q(B(x;r)) \le A\, r^Q,$$

where A is independent of x and r.

Can the above theorem be strengthened? generalized?

Herron, Buckley, and Xie usefully generalized inversions to the metric space setting (2008).

Herron, Buckley, and Xie usefully generalized inversions to the metric space setting (2008).

Choose $p \in X$. Then for $x, y \in X_p := X \setminus \{p\}$,

Herron, Buckley, and Xie usefully generalized inversions to the metric space setting (2008).

Choose $p \in X$. Then for $x, y \in X_p := X \setminus \{p\}$,

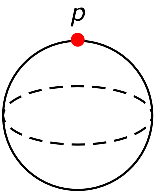
$$d(x,y) \longmapsto d_p(x,y) \simeq \frac{d(x,y)}{d(x,p)d(y,p)}$$

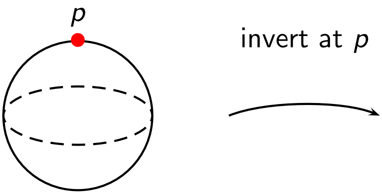
Herron, Buckley, and Xie usefully generalized inversions to the metric space setting (2008).

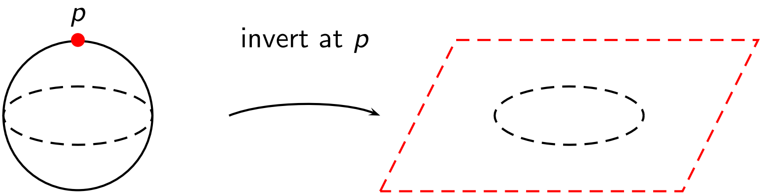
Choose $p \in X$. Then for $x, y \in X_p := X \setminus \{p\}$,

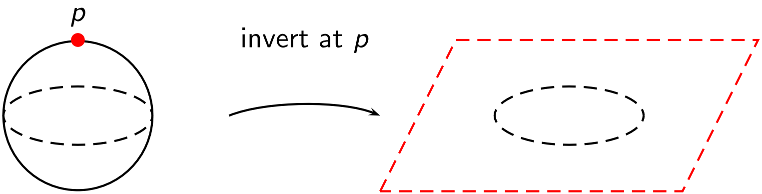
$$d(x,y) \longmapsto d_p(x,y) \simeq \frac{d(x,y)}{d(x,p)d(y,p)}$$

Given $p \in X$, we write $\mathrm{Inv}_p(X) := (\hat{X}_p, d_p)$.


p


p
invert at p


p
invert at p



Inversion invariant bilipschitz homogeneity (the IIBLH property):

Both X and $\mathrm{Inv}_p(X)$ are uniformly bilipschitz homogeneous

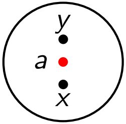
The bounded turning condition is equivalent to the LLC_1 condition:

The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.

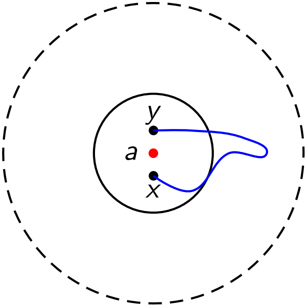
The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.



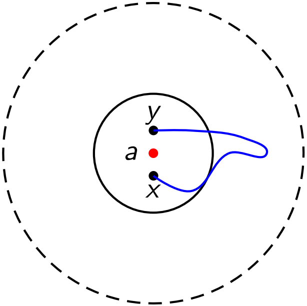
The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.



The bounded turning condition is equivalent to the LLC_1 condition:

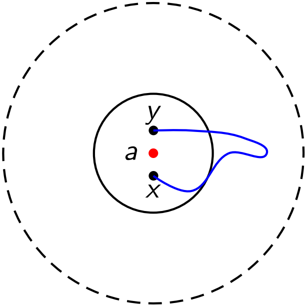
- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.



The LLC_2 condition is 'dual' to the LLC_1 condition:

The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \mathrm{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.

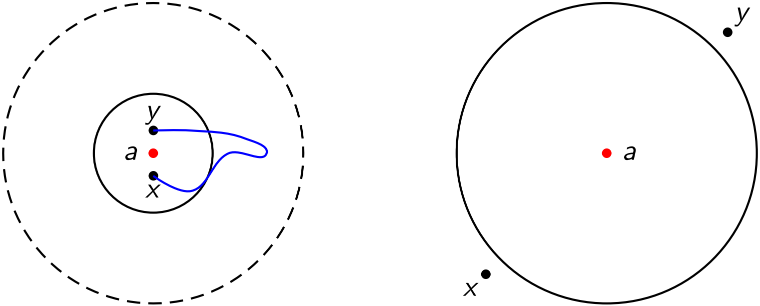


The LLC_2 condition is 'dual' to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \mathrm{diam}(X)$ and any pair $\{x, y\} \subset X \setminus B(a; r)$, there is a continuum $E \subset X \setminus B(a; r/\lambda)$ joining $\{x, y\}$.

The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.

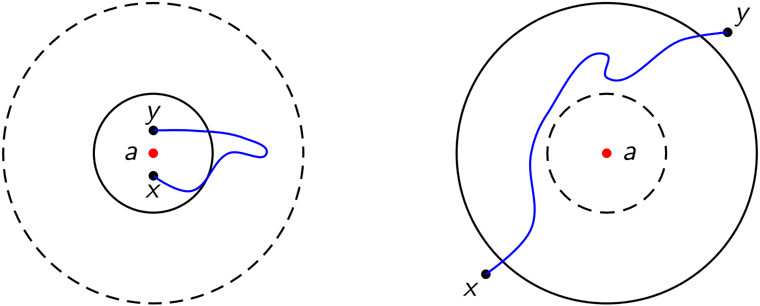


The LLC_2 condition is 'dual' to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \text{diam}(X)$ and any pair $\{x, y\} \subset X \setminus B(a; r)$, there is a continuum $E \subset X \setminus B(a; r/\lambda)$ joining $\{x, y\}$.

The bounded turning condition is equivalent to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \mathrm{diam}(X)$ and any pair $\{x, y\} \subset B(a; r)$, there is a continuum $E \subset B(a; \lambda r)$ joining $\{x, y\}$.



The LLC_2 condition is 'dual' to the LLC_1 condition:

- There exists $1 \leq \lambda < +\infty$ s.t. for any $r < \mathrm{diam}(X)$ and any pair $\{x, y\} \subset X \setminus B(a; r)$, there is a continuum $E \subset X \setminus B(a; r/\lambda)$ joining $\{x, y\}$.

Theorem

Suppose X is proper, connected, locally connected, and doubling. Then the IIBLH property implies the LLC_1 condition. If, in addition, X contains no cut points, it also implies the LLC_2 condition.

Theorem

Suppose X is proper, connected, locally connected, and doubling. Then the IIBLH property implies the LLC_1 condition. If, in addition, X contains no cut points, it also implies the LLC_2 condition.

Corollary

Suppose $X \subset \mathbb{R}^n$ is a Jordan curve or line. Then X has the IIBLH property if and only if X is bounded turning and Ahlfors Q-regular.

Theorem

Suppose X is a proper, connected, D-doubling metric space. Then the L-IIBLH property implies Ahlfors Q-regularity, with regularity constant depending only on D, L.

Theorem

Suppose X is a proper, connected, D-doubling metric space. Then the L-IIBLH property implies Ahlfors Q-regularity, with regularity constant depending only on D, L.

Quantitative!

Theorem

Suppose X is a proper, connected, D-doubling metric space. Then the L-IIBLH property implies Ahlfors Q-regularity, with regularity constant depending only on D, L.

Quantitative!

The previous two theorems yield...

Theorem

Suppose X is a proper, connected, D-doubling metric space. Then the L-IIBLH property implies Ahlfors Q-regularity, with regularity constant depending only on D, L.

Quantitative!

The previous two theorems yield...

Corollary

When $X \approx \mathbb{S}^2$ and has Hausdorff dimension 2, the IIBLH property implies the existence of a quasisymmetric homeomorphism $f : \mathbb{S}^2 \to X$.

To verify the LLC condition:

To verify the LLC condition:

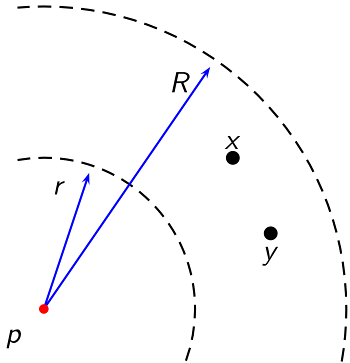
- We first use bilipschitz homogeneity to show that X must be LLC at fixed scales.

To verify the *LLC* condition:

- We first use bilipschitz homogeneity to show that X must be *LLC* at fixed scales.
- Then we use inversion invariance to show that a single *LLC* constant holds at all scales.

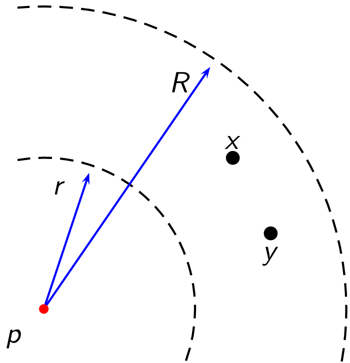
To verify the *LLC* condition:

- We first use bilipschitz homogeneity to show that X must be *LLC* at fixed scales.
- Then we use inversion invariance to show that a single *LLC* constant holds at all scales.



To verify the *LLC* condition:

- We first use bilipschitz homogeneity to show that X must be *LLC* at fixed scales.
- Then we use inversion invariance to show that a single *LLC* constant holds at all scales.



$$\frac{d(x,y)}{4R^2} \leq d_p(x,y) \leq \frac{d(x,y)}{r^2}$$

To verify the Ahlfors Q-regularity:

To verify the Ahlfors Q-regularity:

- First we show that bilipschitz homogeneity implies generalized Ahlfors regularity for some dimension gauge $\delta : (0, +\infty) \to (0, +\infty)$

$$\mathcal{G}^{\delta}(B(x; r)) \simeq \delta(r).$$

To verify the Ahlfors Q-regularity:

- First we show that bilipschitz homogeneity implies generalized Ahlfors regularity for some dimension gauge $\delta : (0, +\infty) \to (0, +\infty)$

$$\mathcal{G}^{\delta}(B(x; r)) \simeq \delta(r).$$

- Then we show that $\delta(t) \simeq t^{Q}$.

To verify the Ahlfors Q-regularity:

- First we show that bilipschitz homogeneity implies generalized Ahlfors regularity for some dimension gauge $\delta : (0, +\infty) \to (0, +\infty)$

$$\mathcal{G}^{\delta}(B(x; r)) \simeq \delta(r).$$

- Then we show that $\delta(t) \simeq t^{Q}$.

When X is bounded,

$$\delta(r) := N(r; X)$$

To verify the Ahlfors Q-regularity:

- First we show that bilipschitz homogeneity implies generalized Ahlfors regularity for some dimension gauge $\delta : (0, +\infty) \to (0, +\infty)$

$$\mathcal{G}^{\delta}(B(x; r)) \simeq \delta(r).$$

- Then we show that $\delta(t) \simeq t^Q$.

When X is bounded,

$$\delta(r) := N(r; X)$$

When X is unbounded,

$$\delta(r) := \begin{cases} N(r; B(x; 1)) & \text{if } r \leq 1 \\ N(1; B(x; r)) & \text{if } r \geq 1 \end{cases}$$

Let Y denote a non-bounded turning bilipschitz homogeneous curve in $\mathbb{R}^3$. Then $Y \times \mathbb{R}$ is a non-LLC bilipschitz homogeneous surface in $\mathbb{R}^4$.

Let Y denote a non-bounded turning bilipschitz homogeneous curve in $\mathbb{R}^3$. Then $Y \times \mathbb{R}$ is a non-LLC bilipschitz homogeneous surface in $\mathbb{R}^4$.

- **Question:** Does bilipschitz homogeneity imply that a space X is LLC when $X \subset \mathbb{R}^n$ and $X \approx \mathbb{R}^{n-1}$?

Let Y denote a non-bounded turning bilipschitz homogeneous curve in $\mathbb{R}^3$. Then $Y \times \mathbb{R}$ is a non-LLC bilipschitz homogeneous surface in $\mathbb{R}^4$.

- **Question:** Does bilipschitz homogeneity imply that a space X is LLC when $X \subset \mathbb{R}^n$ and $X \approx \mathbb{R}^{n-1}$?

Let Y denote a planar bilipschitz homogeneous Jordan curve that is not Ahlfors Q-regular for any Q. Then $Y \times \mathbb{R}$ is an LLC bilipschitz homogeneous surface in $\mathbb{R}^3$ that is not Ahlfors Q-regular for any Q.

Let Y denote a non-bounded turning bilipschitz homogeneous curve in $\mathbb{R}^3$. Then $Y \times \mathbb{R}$ is a non-LLC bilipschitz homogeneous surface in $\mathbb{R}^4$.

- **Question:** Does bilipschitz homogeneity imply that a space X is LLC when $X \subset \mathbb{R}^n$ and $X \approx \mathbb{R}^{n-1}$?

Let Y denote a planar bilipschitz homogeneous Jordan curve that is not Ahlfors Q-regular for any Q. Then $Y \times \mathbb{R}$ is an LLC bilipschitz homogeneous surface in $\mathbb{R}^3$ that is not Ahlfors Q-regular for any Q.

- **Question:** Does there exist a property that - when coupled with bilipschitz homogeneity - will imply that a space X is LLC but *not* Ahlfors Q-regular?

THANKS!