Lecture notes

# Introductory fluid mechanics

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## 1 Introduction

The derivation of the equations of motion for an *ideal fluid* by Euler in 1755, and then for a *viscous fluid* by Navier (1822) and Stokes (1845) were a tour-de-force of 18th and 19th century mathematics. These equations have been used to describe and explain so many physical phenomena around us in nature, that currently billions of dollars of research grants in mathematics, science and engineering now revolve around them. They can be used to model the coupled atmospheric and ocean flow used by the meteorological office for weather prediction down to any application in chemical engineering you can think of, say to development of the thrusters on NASA's Apollo programme rockets. The incompressible *Navier–Stokes equations* are given by

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nu \, \nabla^2 \boldsymbol{u} - \frac{1}{\rho} \nabla p + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is a three dimensional fluid velocity, $p = p(\boldsymbol{x}, t)$ is the pressure and $\boldsymbol{f}$ is an external force field. The constants $\rho$ and $\nu$ are the mass density and kinematic viscosity, respectively. The frictional force due to stickiness of a fluid is represented by the term $\nu \, \nabla^2 \boldsymbol{u}$. An ideal fluid corresponds to the case $\nu = 0$, when the equations above are known as the *Euler equations* for a homogeneous incompressible ideal fluid. We will derive the Navier–Stokes equations and in the process learn about the subtleties of fluid mechanics and along the way see lots of interesting applications.

## 2 Fluid flow, the Continuum Hypothesis and conservation principles

A material exhibits *flow* if shear forces, however small, lead to a deformation which is unbounded—we could use this as definition of a *fluid*. A *solid* has a fixed shape, or at least a strong limitation on its deformation when force is applied to it. With the category of "fluids", we include liquids and gases. The main distinguishing feature between these two fluids is the notion of compressibility. Gases are usually compressible—as we know from everyday aerosols and air canisters. Liquids are generally incompressible—a feature essential to all modern car braking mechanisms.

Fluids can be further subcatergorized. There are *ideal* or *inviscid* fluids. In such fluids, the *only* internal force present is pressure which acts so that fluid flows from a region of high pressure to one of low pressure. The equations for an ideal fluid have been applied to wing and aircraft design (as a limit of high Reynolds number flow). However fluids can exhibit internal frictional forces which model a "stickiness" property of the fluid which involves energy loss—these are known as *viscous* fluids. Some fluids/material known as "non-Newtonian or complex fluids" exhibit even stranger behaviour, their reaction to deformation may depend on: (i) past history (earlier deformations), for example some paints; (ii) temperature, for example some polymers or glass; (iii) the size of the deformation, for example some plastics or silly putty.

For any *real* fluid there are three natural length scales:

1. $L_{\text{molecular}}$, the molecular scale characterized by the mean free path distance of molecules between collisions;
2. $L_{\text{fluid}}$, the medium scale of a fluid parcel, the fluid droplet in the pipe or ocean flow;

3. $L_{\text{macro}}$, the macro-scale which is the scale of the fluid geometry, the scale of the container the fluid is in, whether a beaker or an ocean.

And, of course we have the asymptotic inequalities:

$$L_{\text{molecular}} \ll L_{\text{fluid}} \ll L_{\text{macro}}.$$

**Continuum Hypothesis** We will assume that the properties of an elementary volume/parcel of fluid, however small, are the same as for the fluid as a whole—i.e. we suppose that the properties of the fluid at scale $L_{\text{fluid}}$ propagate all the way down and through the molecular scale $L_{\text{molecular}}$. This is the *continuum assumption*. For everyday fluid mechanics engineering, this assumption is extremely accurate (Chorin and Marsden [3, p. 2]).

Our derivation of the basic equations underlying the dynamics of fluids is based on three basic conservation principles:

1. *Conservation of mass*, mass is neither created or destroyed;
2. *Newton's 2nd law/balance of momentum*, for a parcel of fluid the rate of change of momentum equals the force applied to it;
3. *Conservation of energy*, energy is neither created nor destroyed.

In turn these principles generate the:

1. *Continuity equation* which governs how the density of the fluid evolves locally and thus indicates compressibility properties of the fluid;
2. *Navier–Stokes equations* of motion for a fluid which indicates how the fluid moves around from regions of high pressure to those of low pressure and under the effects of viscosity;
3. *Equation of state* which indicates the mechanism of energy exchange within the fluid.

## 3 Trajectories and streamlines

Suppose that our fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ where $d = 2$ or 3, and $\boldsymbol{x} = (x, y, z)^{\mathrm{T}} \in \mathcal{D}$ is a position/point in $\mathcal{D}$. Imagine a small fluid particle or a speck of dust moving in a fluid flow field prescribed by the *velocity field* $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)^{\mathrm{T}}$. Suppose the position of the particle at time $t$ is recorded by the variables $\big(x(t), y(t), z(t)\big)^{\mathrm{T}}$. The velocity of the particle at time $t$ at position $\big(x(t), y(t), z(t)\big)^{\mathrm{T}}$ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= u\big(x(t), y(t), z(t), t\big), \\
\frac{\mathrm{d}}{\mathrm{d}t}y(t) &= v\big(x(t), y(t), z(t), t\big), \\
\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= w\big(x(t), y(t), z(t), t\big).
\end{aligned}$$

**Definition 1 (Particle path or trajectory)** The *particle path* or *trajectory* of a fluid particle is the curve traced out by the particle as time progresses. If the particle starts at position $(x_0, y_0, z_0)^{\mathrm{T}}$ then its particle path is the solution to the system of differential equations (the same as those above but here in shorter vector notation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t),$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$.

**Definition 2 (Streamline)** Suppose for a given fluid flow $\boldsymbol{u}(\boldsymbol{x},t)$ we fix time $t$. A *streamline* is an integral curve of $\boldsymbol{u}(\boldsymbol{x},t)$ for $t$ fixed, i.e. it is a curve $\boldsymbol{x}=\boldsymbol{x}(s)$ parameterized by the variable $s$, that satisfies the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s)=\boldsymbol{u}(\boldsymbol{x}(s),t),$$

with $t$ held constant.

*Remark 1* If the velocity field $\boldsymbol{u}$ is time-independent, i.e. $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x})$ only, or equivalently $\partial\boldsymbol{u}/\partial t=\mathbf{0}$, then trajectories and streamlines coincide. Flows for which $\partial\boldsymbol{u}/\partial t=\mathbf{0}$ are said to be *stationary.*

**Example.** Suppose a velocity field $\boldsymbol{u}(\boldsymbol{x},t)=(u,v,w)^{\mathrm{T}}$ is given by

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}-\Omega y\\\Omega x\\0\end{pmatrix}$$

for some constant $\Omega>0$. Then the particle path for a particle that starts at $(x_0,y_0,z_0)^{\mathrm{T}}$ is the integral curve of the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\Omega y,\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=\Omega x\qquad\text{and}\qquad\frac{\mathrm{d}z}{\mathrm{d}t}=0.$$

This is a coupled pair of differential equations as the solution to the last equation is $z(t)=z_0$ for all $t\geqslant0$. There are several methods for solving the pair of equations, one method is as follows. Differentiating the first equation with respect to $t$ we find

$$\begin{aligned}&&\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=-\Omega\frac{\mathrm{d}y}{\mathrm{d}t}\\ \Leftrightarrow&&\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=-\Omega^2x.\end{aligned}$$

In other words we are required to solve the linear second order differential equation for $x=x(t)$ shown. The general solution is

$$x(t)=A\cos\Omega t+B\sin\Omega t,$$

where $A$ and $B$ are arbitrary constants. We can now find $y=y(t)$ by substituting this solution for $x=x(t)$ into the first of the pair of differential equations as follows:

$$\begin{aligned}y(t)&=-\frac{1}{\Omega}\frac{\mathrm{d}x}{\mathrm{d}t}\\&=-\frac{1}{\Omega}(-A\Omega\sin\Omega t+B\Omega\cos\Omega t)\\&=A\sin\Omega t-B\cos\Omega t.\end{aligned}$$

Using that $x(0)=x_0$ and $y(0)=y_0$ we find that $A=x_0$ and $B=-y_0$ so the particle path of the particle that is initially at $(x_0,y_0,z_0)^{\mathrm{T}}$ is given by

$$x(t)=x_0\cos\Omega t-y_0\sin\Omega t,\quad y(t)=x_0\sin\Omega t+y_0\cos\Omega t\quad\text{and}\quad z(t)=z_0.$$

This particle thus traces out a horizontal circular particle path at height $z = z_0$ of radius $\sqrt{x_0^2 + y_0^2}$. Since this flow is stationary, streamlines coincide with particle paths for this flow.

**Example.** Consider the two-dimensional flow

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \cos(kx - \alpha t) \end{pmatrix},$$

where $u_0$, $v_0$, $k$ and $\alpha$ are constants. Let us find the particle path and streamline for the particle at $(x_0, y_0)^{\mathrm{T}} = (0, 0)^{\mathrm{T}}$ at $t = 0$. Starting with the *particle path*, we are required to solve the coupled pair of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_0 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v_0 \cos(kx - \alpha t).$$

We can solve the first differential equation which tells us

$$x(t) = u_0 t,$$

where we used that $x(0) = 0$. We now substitute this expression for $x = x(t)$ into the second differential equation and integrate with respect to time (using $y(0) = 0$) so

$$\begin{aligned} & & \frac{\mathrm{d}y}{\mathrm{d}t} &= v_0 \cos\big((ku_0 - \alpha)t\big) \\ \Leftrightarrow & & y(t) &= 0 + \int_0^t v_0 \cos\big((ku_0 - \alpha)\tau\big)\,\mathrm{d}\tau \\ \Leftrightarrow & & y(t) &= \frac{v_0}{ku_0 - \alpha} \sin\big((ku_0 - \alpha)t\big). \end{aligned}$$

If we eliminate time $t$ between the formulae for $x = x(t)$ and $y = y(t)$ we find that the trajectory through $(0, 0)^{\mathrm{T}}$ is

$$y = \frac{v_0}{ku_0 - \alpha} \sin\bigg(\Big(k - \frac{\alpha}{u_0}\Big)x\bigg).$$

To find the *streamline* through $(0, 0)^{\mathrm{T}}$, we fix $t$, and solve the pair of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u_0 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = v_0 \cos(kx - \alpha t).$$

As above we can solve the first equation so that $x(s) = u_0 s$ using that $x(0) = 0$. We can substitute this into the second equation and integrate with respect to $s$—remembering that $t$ is constant—to get

$$\begin{aligned} & & \frac{\mathrm{d}y}{\mathrm{d}s} &= v_0 \cos(ku_0 s - \alpha t) \\ \Leftrightarrow & & y(s) &= 0 + \int_0^s v_0 \cos(ku_0 r - \alpha t)\,\mathrm{d}r \\ \Leftrightarrow & & y(s) &= \frac{v_0}{ku_0}\big(\sin(ku_0 s - \alpha t) - \sin(-\alpha t)\big). \end{aligned}$$

If we eliminate the parameter $s$ between $x = x(s)$ and $y = y(s)$ above, we find the equation for the streamline is

$$y = \frac{v_0}{ku_0}\big(\sin(kx - \alpha t) + \sin(\alpha t)\big).$$

The equation of the streamline through $(0,0)^{\mathrm{T}}$ at time $t=0$ is thus given by

$$y = \frac{v_0}{ku_0}\sin(kx).$$

As the underlying flow is *not* stationary, as expected, the particle path and streamline through $(0,0)^{\mathrm{T}}$ at time $t=0$ are distinguished. Finally let us examine two special limits for this flow. As $\alpha \to 0$ the flow becomes stationary and correspondingly the particle path and streamline coincide. As $k \to 0$ the flow is not stationary. In this limit the particle path through $(0,0)^{\mathrm{T}}$ is $y = (v_0/\alpha)\sin(\alpha x/u_0)$, i.e. it is sinusoidal, whereas the streamline is given by $x = u_0 s$ and $y = 0$, which is a horizontal straight line through the origin.

*Remark 2* A *streakline* is the locus of all the fluid elements which at some time have past through a particular point, say $(x_0, y_0, z_0)^{\mathrm{T}}$. We can obtain the equation for a streakline through $(x_0, y_0, z_0)^{\mathrm{T}}$ by solving the equations $(\mathrm{d}/\mathrm{d}t)\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t)$ assuming that at $t = t_0$ we have $\big(x(t_0), y(t_0), z(t_0)\big)^{\mathrm{T}} = (x_0, y_0, z_0)^{\mathrm{T}}$. Eliminating $t_0$ between the equations generates the streakline corresponding to $(x_0, y_0, z_0)^{\mathrm{T}}$. For example, ink dye injected at the point $(x_0, y_0, z_0)^{\mathrm{T}}$ in the flow will trace out a streakline.

## 4 Continuity equation

Recall, we suppose our fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ (here we will assume $d = 3$, but everything we say is true for the collapsed two dimensional case $d = 2$). Hence $\boldsymbol{x} = (x, y, z)^{\mathrm{T}} \in \mathcal{D}$ is a position/point in $\mathcal{D}$. At each time $t$ we will suppose that the fluid has a well defined *mass density* $\rho(\boldsymbol{x}, t)$ at the point $\boldsymbol{x}$. Further, each fluid particle traces out a well defined path in the fluid, and its motion along that path is governed by the *velocity field* $\mathbf{u}(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ at time $t$. Consider an arbitrary subregion $\Omega \subseteq \mathcal{D}$. The total mass of fluid contained inside the region $\Omega$ at time $t$ is

$$\int_{\Omega} \rho(\boldsymbol{x}, t)\,\mathrm{d}V.$$

where $\mathrm{d}V$ is the volume element in $\mathbb{R}^d$. Let us now consider the rate of change of mass inside $\Omega$. By the principle of conservation of mass, the rate of increase of the mass in $\Omega$ is given by the mass of fluid entering/leaving the boundary $\partial\Omega$ of $\Omega$ per unit time.

To compute the total mass of fluid entering/leaving the boundary $\partial\Omega$ per unit time, we consider a small area patch $\mathrm{d}S$ on the boundary of $\partial\Omega$, which has unit outward normal $\boldsymbol{n}$. The total mass of fluid flowing out of $\Omega$ through the area patch $\mathrm{d}S$ per unit time is

$$\text{mass density} \times \text{fluid volume leaving per unit time} = \rho(\boldsymbol{x}, t)\,\boldsymbol{u}(\boldsymbol{x}, t)\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S,$$

where $\boldsymbol{x}$ is at the centre of the area patch $\mathrm{d}S$ on $\partial\Omega$. Note that to estimate the fluid volume leaving per unit time we have decomposed the fluid velocity at $\boldsymbol{x} \in \partial\Omega$, time $t$, into velocity components normal $(\boldsymbol{u}\cdot\boldsymbol{n})$ and tangent to the surface $\partial\Omega$ at that point. The velocity component tangent to the surface pushes fluid across the surface—no fluid enters or leaves $\Omega$ via this component. Hence we only retain the normal component—see Fig. 2.

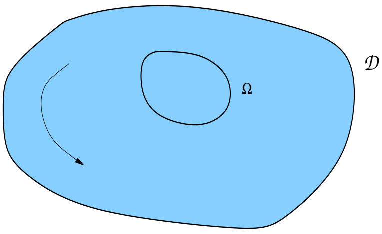


**Fig. 1** The fluid of mass density $\rho(\boldsymbol{x},t)$ swirls around inside the container $\mathcal{D}$, while $\Omega$ is an imaginary subregion.

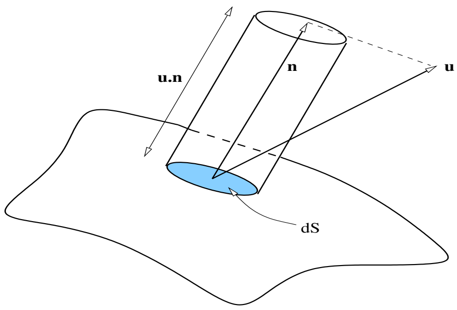


**Fig. 2** The total mass of fluid moving through the patch $\mathrm{d}S$ on the surface $\partial\Omega$ per unit time, is given by the mass density $\rho(\boldsymbol{x},t)$ times the volume of the cylinder shown which is $\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S$.

Returning to the principle of conservation of mass, this is now equivalent to the *integral form of the law of conservation of mass*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho(\boldsymbol{x},t)\,\mathrm{d}V=-\int_{\partial\Omega}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

The divergence theorem and that the rate of change of the total mass inside $\Omega$ equals the total rate of change of mass density inside $\Omega$ imply, respectively,

$$\int_{\Omega}\nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V=\int_{\partial\Omega}(\rho\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}S\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\,\mathrm{d}V=\int_{\Omega}\frac{\partial\rho}{\partial t}\,\mathrm{d}V.$$

Using these two relations, the law of conservation of mass is equivalent to

$$\int_{\Omega}\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V=0.$$

Now we use that $\Omega$ is arbitrary to deduce the *differential form of the law of conservation of mass* or *continuity equation* that applies pointwise:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0.$$

This is the first of our three conservation laws.

## 5 Transport Theorem

Recall our image of a small fluid particle moving in a fluid flow field prescribed by the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$. The velocity of the particle at time $t$ at position $\boldsymbol{x}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=\boldsymbol{u}(\boldsymbol{x}(t),t).$$

As the particle moves in the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, say from position $\boldsymbol{x}(t)$ to a nearby position an instant in time later, two dynamical contributions change: (i) a small instant in time has elapsed and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on time, will have changed a little; (ii) the position of the particle has changed in that short time as it moved slightly, and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on position, will be slightly different at the new position.

Let us compute the *acceleration* of the particle to explicitly observe these two contributions. By using the chain rule we see that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}\big(\boldsymbol{x}(t),t\big)\\&=\frac{\partial\boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial\boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial\boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\frac{\partial\boldsymbol{u}}{\partial t}\\&=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}+\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y}+\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}\right)\boldsymbol{u}+\frac{\partial\boldsymbol{u}}{\partial t}\\&=\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{\partial\boldsymbol{u}}{\partial t}.\end{aligned}$$

Indeed for any function $F(x,y,z,t)$, scalar or vector valued, the chain rule implies

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t),y(t),z(t),t\big)=\frac{\partial F}{\partial t}+\boldsymbol{u}\cdot\nabla F.$$

**Definition 3 (Material derivative)** If the velocity field components are

$$\boldsymbol{u}=(u,v,w)^{\mathrm{T}}\qquad\text{and}\qquad\boldsymbol{u}\cdot\nabla\equiv u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z},$$

then we define the *material derivative* following the fluid to be

$$\frac{\mathrm{D}}{\mathrm{D}t}:=\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla.$$

Suppose that the region within which the fluid is moving is $\mathcal{D}$. Suppose $\Omega$ is a subregion of $\mathcal{D}$ identified at time $t=0$. As the fluid flow evolves the fluid particles that originally made up $\Omega$ will subsequently fill out a volume $\Omega_t$ at time $t$. We think of $\Omega_t$ as the volume *moving with the fluid*.

**Theorem 1 (Transport Theorem)** *For any function $F$ and density function $\rho$ satisfying the continuity equation, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho\,F\,\mathrm{d}V=\int_{\Omega_t}\rho\,\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}V.$$

We will use the Transport Theorem to deduce both the Euler and Cauchy equations of motion from the *primitive* integral form of the balance of momentum; see Sections 10 and 15. We now carefully elucidate the steps required for the proof of the Transport Theorem—see Chorin and Marsden [3, pp. 6–11]. Importantly the concepts of the *flow map* in Step 1 and the evolution of its Jacobian in Step 3 will have important ramifications in coming sections. The four steps are as follows.

*Step 1: Fluid flow map.* For a fixed position $\boldsymbol{x} \in \mathcal{D}$ we denote by $\boldsymbol{\xi}(\boldsymbol{x},t) = (\xi,\eta,\zeta)^{\mathrm{T}}$ the position of the particle at time $t$, which at time $t=0$ was at $\boldsymbol{x}$. We use $\varphi_t$ to denote the map $\boldsymbol{x} \mapsto \boldsymbol{\xi}(\boldsymbol{x},t)$, i.e. $\varphi_t$ is the map that advances each particle at position $\boldsymbol{x}$ at time $t=0$ to its position at time $t$ later; it is the *fluid flow-map*. Hence, for example $\varphi_t(\Omega) = \Omega_t$. We assume $\varphi_t$ is sufficiently smooth and invertible for all our subsequent manipulations.

*Step 2: Change of variables.* For any two functions $\rho$ and $F$ we can perform the change of variables from $(\boldsymbol{\xi},t)$ to $(\boldsymbol{x},t)$—with $J(\boldsymbol{x},t)$ the Jacobian for this transformation given by definition as $J(\boldsymbol{x},t) := \det\bigl(\nabla\boldsymbol{\xi}(\boldsymbol{x},t)\bigr)$. Here the gradient operator is with respect to the $\boldsymbol{x}$ coordinates, i.e. $\nabla = \nabla_{\boldsymbol{x}}$. Note for $\Omega_t$ we integrate over volume elements $\mathrm{d}V = \mathrm{d}V(\boldsymbol{\xi})$, i.e. with respect to the $\boldsymbol{\xi}$ coordinates, whereas for $\Omega$ we integrate over volume elements $\mathrm{d}V = \mathrm{d}V(\boldsymbol{x})$, i.e. with respect to the *fixed* coordinates $\boldsymbol{x}$. Hence by direct computation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\, F\,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} (\rho\, F)(\boldsymbol{\xi},t)\,\mathrm{d}V(\boldsymbol{\xi}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} (\rho\, F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\, J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}) \\
&= \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}t}\Bigl((\rho\, F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\, J(\boldsymbol{x},t)\Bigr)\,\mathrm{d}V \\
&= \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}t}(\rho\, F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\, J(\boldsymbol{x},t) + (\rho\, F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t)\,\mathrm{d}V \\
&= \int_{\Omega} \left(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\, F)\right)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\, J(\boldsymbol{x},t) + (\rho\, F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t)\,\mathrm{d}V.
\end{aligned}
$$

*Step 3: Evolution of the Jacobian.* We establish the following result for the Jacobian:

$$
\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t) = \bigl(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi}(\boldsymbol{x},t),t)\bigr)\, J(\boldsymbol{x},t).
$$

We know that a particle at position $\boldsymbol{\xi}(\boldsymbol{x},t) = \bigl(\xi(\boldsymbol{x},t),\eta(\boldsymbol{x},t),\zeta(\boldsymbol{x},t)\bigr)^{\mathrm{T}}$, which started at $\boldsymbol{x}$ at time $t=0$, evolves according to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(\boldsymbol{x},t) = \boldsymbol{u}\bigl(\boldsymbol{\xi}(\boldsymbol{x},t),t\bigr).
$$

Taking the gradient with respect to $\boldsymbol{x}$ of this relation, and swapping over the gradient and $\mathrm{d}/\mathrm{d}t$ operations on the left, we see that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\nabla\boldsymbol{\xi}(\boldsymbol{x},t) = \nabla\boldsymbol{u}\bigl(\boldsymbol{\xi}(\boldsymbol{x},t),t\bigr).
$$

Using the chain rule we have

$$
\nabla_{\boldsymbol{x}}\,\boldsymbol{u}\bigl(\boldsymbol{\xi}(\boldsymbol{x},t),t\bigr) = \Bigl(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u}\bigl(\boldsymbol{\xi}(\boldsymbol{x},t),t\bigr)\Bigr)\cdot\bigl(\nabla_{\boldsymbol{x}}\,\boldsymbol{\xi}(\boldsymbol{x},t)\bigr).
$$

Combining the last two relations we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla\boldsymbol{\xi} = (\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u})\,\nabla\boldsymbol{\xi}.$$

Abel's Theorem then tells us that $J = \det\nabla\boldsymbol{\xi}$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\nabla\boldsymbol{\xi} = \big(\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u})\big)\ \det\nabla\boldsymbol{\xi},$$

where Tr denotes the *trace* operator on matrices—the trace of a matrix is the sum of its diagonal elements. Since $\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u}) \equiv \nabla\cdot\boldsymbol{u}$ we have established the required result.

*Step 4: Conservation of mass.* We see that we thus have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho\,F\,\mathrm{d}V &= \int_{\Omega}\left(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,F) + (\rho\,F)\big(\nabla\cdot\boldsymbol{u}\big)\right)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\,J(\boldsymbol{x},t)\,\mathrm{d}V\\ &= \int_{\Omega_t}\left(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,F) + \big(\rho\,\nabla\cdot\boldsymbol{u}\big)\,F\right)\mathrm{d}V\\ &= \int_{\Omega_t}\rho\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}V,\end{aligned}$$

where in the last step we have used the conservation of mass equation.

We have thus completed the proof of the Transport Theorem. A straightforward corollary proved in a manner analogous to that of the Transport Theorem is as follows.

**Corollary 1** *For any function $F = F(\boldsymbol{\xi}(t),t)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}F\,\mathrm{d}V = \int_{\Omega_t}\left(\frac{\partial F}{\partial t} + \nabla\cdot(F\boldsymbol{u})\right)\mathrm{d}V.$$

## 6 Incompressible flow

We now characterize a subclass of flows which are *incompressible*. The classic examples are water, and the brake fluid in your car whose incompressibility properties are vital to the effective transmission of pedal pressure to brakepad pressure. Herein we closely follow the presentation given in Chorin and Marsden [3, pp. 10–11].

**Definition 4 (Incompressible flow)** A flow is said to be *incompressible* if for any subregion $\Omega\subseteq\mathcal{D}$, the volume of $\Omega_t$ is constant in time.

**Corollary 2 (Equivalent incompressibility statements)** *The following statements are equivalent:*

1. *Fluid is incompressible;*
2. *Jacobian $J\equiv 1$;*
3. *The velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$ is divergence free, i.e. $\nabla\cdot\boldsymbol{u} = 0$.*

*Proof* Using the result for the Jacobian of the flow map in Step 3 of the proof of the Transport Theorem, for any subregion $\Omega$ of the fluid, we see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\Omega_t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \mathrm{d}V(\boldsymbol{\xi}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}) \\ &= \int_{\Omega} \big(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi}(\boldsymbol{x},t),t)\big)\, J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}) \\ &= \int_{\Omega_t} \big(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi},t)\big)\,\mathrm{d}V(\boldsymbol{\xi}).\end{aligned}$$

Further, noting that by definition $J(\boldsymbol{x},0)=1$, establishes the result. □

The continuity equation and the identity, $\nabla\cdot(\rho\,\boldsymbol{u}) = \nabla\rho\cdot\boldsymbol{u} + \rho\,\nabla\cdot\boldsymbol{u}$, imply

$$\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho\,\nabla\cdot\boldsymbol{u} = 0.$$

Hence since $\rho > 0$, a flow is incompressible if and only if

$$\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = 0,$$

i.e. the fluid density is constant following the fluid.

**Definition 5 (Homogeneous fluid)** A fluid is said to be *homogeneous* if its mass density $\rho$ is constant in space.

If we set $F \equiv 1$ in the Transport Theorem we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\,\mathrm{d}V = 0.$$

This is equivalent to the statement

$$\begin{aligned} & \int_{\Omega_t} \rho(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega} \rho(\boldsymbol{x},0)\,\mathrm{d}V \\ \Leftrightarrow\quad & \int_{\Omega} \rho\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big)\,J(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega} \rho(\boldsymbol{x},0)\,\mathrm{d}V \quad , \end{aligned}$$

where we made a change of variables. Since $\Omega$ is arbitrary, we deduce

$$\rho\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big)\,J(\boldsymbol{x},t) = \rho(\boldsymbol{x},0).$$

From this relation we see that if the flow is incompressible so $J(\boldsymbol{x},t) \equiv 1$ then $\rho\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big) = \rho(\boldsymbol{x},0)$. Thus if an incompressible fluid is homogeneous at time $t=0$ then it remains so. If we combine this with the result that mass density is constant following the fluid, then we conclude that $\rho$ is constant in time.

## 7 Stream functions

A *stream function* exists for a given flow $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$ if the velocity field $\boldsymbol{u}$ is *solenoidal*, i.e. $\nabla \cdot \boldsymbol{u} = 0$, *and* we have an additional symmetry that allows us to eliminate one coordinate. For example, a two dimensional incompressible fluid flow $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ is solenoidal since $\nabla \cdot \boldsymbol{u} = 0$, and has the symmetry that it is uniform with respect to $z$. For such a flow we see that

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This equation is satisfied if and only if there exists a function $\psi(x, y, t)$ such that

$$\frac{\partial \psi}{\partial y} = u(x, y, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = v(x, y, t).$$

The function $\psi$ is called *Lagrange's stream function.* A stream function is always only defined up to any arbitrary additive constant. Further note that for $t$ fixed, *streamlines* are given by constant contour lines of $\psi$ (note $\nabla \psi \cdot \boldsymbol{u} = 0$ everywhere).

Note that if we use plane polar coordinates so $\boldsymbol{u} = \boldsymbol{u}(r, \theta, t)$ and the velocity components are $\boldsymbol{u} = (u_r, u_\theta)$ then

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\, u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0.$$

This is satisfied if and only if there exists a function $\psi(r, \theta, t)$ such that

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r(r, \theta, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial r} = u_\theta(r, \theta, t).$$

**Example** Suppose that in Cartesian coordinates we have the two dimensional flow $\boldsymbol{u} = (u, v)^{\mathrm{T}}$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k\,x \\ -k\,y \end{pmatrix},$$

for some constant $k$. Note that $\nabla \cdot \boldsymbol{u} = 0$ so there exists a stream function satisfying

$$\frac{\partial \psi}{\partial y} = k\,x \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = -k\,y.$$

Consider the first partial differential equation. Integrating with respect to $y$ we get

$$\psi = k\,xy + C(x)$$

where $C(x)$ is an arbitrary function of $x$. However we know that $\psi$ must simultaneously satisfy the second partial differential equation above. Hence we substitute this last relation into the second partial differential equation above to get

$$-\frac{\partial \psi}{\partial x} = -k\,y \qquad \Leftrightarrow \qquad -k\,y + C'(x) = -k\,y.$$

We deduce $C'(x) = 0$ and therefore $C$ is an arbitrary constant. Since a stream function is only defined up to an arbitrary constant we take $C = 0$ for simplicity and the stream function is given by

$$\psi = k\,xy.$$

Now suppose we used plane polar coordinates instead. The corresponding flow $\boldsymbol{u} = (u_r, u_\theta)^{\mathrm{T}}$ is given by

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} k\,r\cos 2\theta \\ -k\,r\sin 2\theta \end{pmatrix}.$$

First note that $\nabla\cdot\boldsymbol{u} = 0$ using the polar coordinate form for $\nabla\cdot\boldsymbol{u}$ indicated above. Hence there exists a stream function $\psi = \psi(r,\theta)$ satisfying

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = k\,r\cos 2\theta \qquad \text{and} \qquad -\frac{\partial\psi}{\partial r} = -k\,r\sin 2\theta.$$

As above, consider the first partial differential equation shown, and integrate with respect to $\theta$ to get

$$\psi = \tfrac{1}{2}\,k\,r^2\sin 2\theta + C(r).$$

Substituting this into the second equation above reveals that $C'(r) = 0$ so that $C$ is a constant. We can for convenience set $C = 0$ so that

$$\psi = \tfrac{1}{2}\,k\,r^2\sin 2\theta.$$

Comparing this form with its Cartesian equivalent above, reveals they are the same.

## 8 Rate of strain tensor

Consider a fluid flow in a region $\mathcal{D} \subseteq \mathbb{R}^3$. Suppose $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{h}$ are two nearby points in the interior of $\mathcal{D}$. How is the flow, or more precisely the velocity field, at $\boldsymbol{x}$ related to that at $\boldsymbol{x}+\boldsymbol{h}$? From a mathematical perspective, by Taylor expansion we have

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + \big(\nabla\boldsymbol{u}(\boldsymbol{x})\big)\cdot\boldsymbol{h} + \mathcal{O}(h^2),$$

where $(\nabla\boldsymbol{u})\cdot\boldsymbol{h}$ is simply matrix multiplication of the $3\times 3$ matrix $\nabla\boldsymbol{u}$ by the column vector $\boldsymbol{h}$. Recall that $\nabla\boldsymbol{u}$ is given by

$$\nabla\boldsymbol{u} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix}.$$

In the context of fluid flow it is known as the *rate of strain tensor*. This is because, locally, it measures that rate at which neighbouring fluid particles are being pulled apart (it helps to recall that the velocity field $\boldsymbol{u}$ records the rate of change of particle position with respect to time).

Again from a mathematical perspective, we can decompose $\nabla\boldsymbol{u}$ as follows. We can always write

$$\nabla\boldsymbol{u} = \tfrac{1}{2}\big((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathrm{T}}\big) + \tfrac{1}{2}\big((\nabla\boldsymbol{u}) - (\nabla\boldsymbol{u})^{\mathrm{T}}\big).$$

We set

$$\begin{aligned} D &\coloneqq \tfrac{1}{2}\big((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathrm{T}}\big), \\ R &\coloneqq \tfrac{1}{2}\big((\nabla\boldsymbol{u}) - (\nabla\boldsymbol{u})^{\mathrm{T}}\big). \end{aligned}$$

Note that $D = D(\boldsymbol{x})$ is a $3 \times 3$ symmetric matrix, while $R = R(\boldsymbol{x})$ is the $3 \times 3$ skew-symmetric matrix given by

$$R = \begin{pmatrix} 0 & \partial u/\partial y - \partial v/\partial x & \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y & 0 & \partial v/\partial z - \partial w/\partial y \\ \partial w/\partial x - \partial u/\partial z & \partial w/\partial y - \partial v/\partial z & 0 \end{pmatrix}.$$

Note that if we set

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \text{and} \qquad \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

then $R$ is more simply expressed as

$$R = \tfrac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Further by direct computation we see that

$$R\boldsymbol{h} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h},$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x})$ is the vector with three components $\omega_1$, $\omega_2$ and $\omega_3$. At this point, we have thus established the following.

**Theorem 2** *If $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{h}$ are two nearby points in the interior of $\mathcal{D}$, then*

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x}) \cdot \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h} + \mathcal{O}(h^2).$$

The symmetric matrix $D$ is the *deformation tensor*. Since it is symmetric, there is an orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in which $D$ is diagonal, i.e. if $X = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ then

$$X^{-1}DX = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

By direct computation, the vector field $\boldsymbol{\omega}$ above is equivalently given by $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.

**Definition 6 (Vorticity field)** For any velocity vector field $\boldsymbol{u}$ the associated vector field given by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u},$$

is known as the *vorticity field* of the flow. It encodes the magnitude of, and direction of the axis about which, the fluid rotates, locally.

Now consider the motion of a fluid particle labelled by $\boldsymbol{x}+\boldsymbol{h}$ where $\boldsymbol{x}$ is fixed and $\boldsymbol{h}$ is small (for example suppose that only a short time has elapsed). Then the position of the particle is given by

$$\begin{aligned} & & \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}+\boldsymbol{h}) &= \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) \\ \Leftrightarrow & & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} &= \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) \\ \Leftrightarrow & & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} &\approx \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x}) \cdot \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}. \end{aligned}$$

Let us consider in turn each of the effects on the right shown:

1. The term $\boldsymbol{u}(\boldsymbol{x})$ is simply uniform translational velocity (the particle being pushed by the ambient flow surrounding it).
2. Now consider the second term $D(\boldsymbol{x}) \cdot \boldsymbol{h}$. If we ignore the other terms then, approximately, we have
$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = D(\boldsymbol{x}) \cdot \boldsymbol{h}.$$
Making a local change of coordinates so that $\boldsymbol{h} = X\hat{\boldsymbol{h}}$ we get
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix} = \begin{pmatrix}d_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & d_3\end{pmatrix}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix}.$$
We see that we have pure expansion or contraction (depending on whether $d_i$ is positive or negative, respectively) in each of the characteristic directions $\hat{h}_i$, $i = 1, 2, 3$. Indeed the small linearized volume element $\hat{h}_1\hat{h}_2\hat{h}_3$ satisfies
$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{h}_1\hat{h}_2\hat{h}_3) = (d_1 + d_2 + d_3)(\hat{h}_1\hat{h}_2\hat{h}_3).$$
Note that $d_1 + d_2 + d_3 = \mathrm{Tr}(D) = \nabla \cdot \boldsymbol{u}$.
3. Let us now examine the effect of the third term $\frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}$. Ignoring the other two terms we have
$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}.$$
Direct computation shows that
$$\boldsymbol{h}(t) = \Theta(t, \boldsymbol{\omega}(\boldsymbol{x}))\boldsymbol{h}(0),$$
where $\Theta(t, \boldsymbol{\omega}(\boldsymbol{x}))$ is the matrix that represents the rotation through an angle $t$ about the axis $\boldsymbol{\omega}(\boldsymbol{x})$. Note also that $\nabla \cdot \big(\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}\big) = 0$.

## 9 Internal fluid forces

Let us consider the forces that act on a small parcel of fluid in a fluid flow. There are two types:

1. *external or body forces*, these may be due to gravity or external electromagnetic fields. They exert a force per unit volume on the continuum.
2. *surface or stress forces*, these are forces, molecular in origin, that are applied by the neighbouring fluid across the surface of the fluid parcel.

The surface or stress forces are normal stresses due to pressure differentials, and shear stresses which are the result of molecular diffusion. We explain shear stresses as follows. Imagine two neighbouring parcels of fluid $P$ and $P'$ as shown in Fig. 3, with a mutual contact surface is $S$ as shown. Suppose both parcels of fluid are moving parallel to $S$ and to each other, but the speed of $P$, say $\boldsymbol{u}$, is much faster than that of $P'$, say $\boldsymbol{u}'$. In the kinetic theory of matter molecules jiggle about and take random walks; they diffuse into their surrounding locale and impart their kinetic energy to molecules they pass by. Hence the faster molecules in $P$ will diffuse across $S$ and impart momentum to the molecules in $P'$. Similarly, slower molecules from $P'$ will diffuse across $S$ to slow the fluid in $P$ down. In regions of the flow where the velocity field changes rapidly over small length scales, this effect is important—see Chorin and Marsden [3, p. 31].

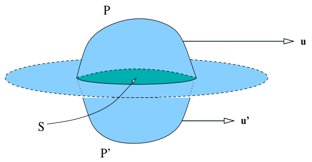


**Fig. 3** Two neighbouring parcels of fluid $P$ and $P'$. Suppose $S$ is the surface of mutual contact between them. Their respective velocities are $\boldsymbol{u}$ and $\boldsymbol{u}'$ and in the same direction and parallel to $S$, but with $|\boldsymbol{u}| \gg |\boldsymbol{u}'|$. The faster molecules in $P$ will diffuse across the surface $S$ and impart momentum to $P'$.

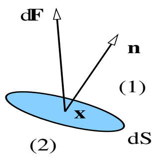


**Fig. 4** The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ is given by $\boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S$.

We now proceed more formally. The force per unit area exerted across a surface (imaginary in the fluid) is called the *stress*. Let $\mathrm{d}S$ be a small imaginary surface in the fluid centred on the point $\boldsymbol{x}$—see Fig. 4. The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ in the fluid/material is given by

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S.$$

Here $\boldsymbol{\Sigma}$ is the stress at the point $\boldsymbol{x}$. It is a function of the normal direction $\boldsymbol{n}$ to the surface $\mathrm{d}S$, in fact it is given by:

$$\boldsymbol{\Sigma}(\boldsymbol{n}) = \sigma(\boldsymbol{x})\,\boldsymbol{n}.$$

Note $\sigma = [\sigma_{ij}]$ is a $3 \times 3$ matrix known as the *stress tensor*. The diagonal components of $\sigma_{ij}$, with $i = j$, generate *normal stresses*, while the off-diagonal components, with $i \neq j$, generate *tangential* or *shear stresses*. Indeed let us decompose the stress tensor $\sigma = \sigma(\boldsymbol{x})$ as follows (here $I$ is the $3 \times 3$ identity matrix):

$$\sigma = -p\,I + \hat{\sigma}.$$

Here the scalar quantity $p = p(\boldsymbol{x}, t)$ is defined to be

$$p := -\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

and represents the fluid *pressure*. The remaining part of the stress tensor $\hat{\sigma} = \hat{\sigma}(\boldsymbol{x})$ is known as the *deviatoric stress tensor*. In this decomposition, the term $-p\,I$ generates the normal stresses, since if this were the only term present,

$$\sigma = -p\,I \qquad \Rightarrow \qquad \boldsymbol{\Sigma}(\boldsymbol{n}) = -p\,\boldsymbol{n}.$$

The deviatoric stress tensor $\hat{\sigma}$ on the other hand, generates the shear stresses. We will discuss them in some detail in Section 14 prior to our derivation of the Navier–Stokes equations in Section 15.

### 10 Euler equations of fluid motion

Consider an arbitrary imaginary subregion $\Omega \subseteq \mathcal{D}$ identified at time $t = 0$, as in Fig. 1. As the fluid flow evolves to some time $t > 0$, let $\Omega_t$ denote the volume of the fluid occupied by particles that originally made up $\Omega$. The total force exerted on the fluid inside $\Omega_t$ through the normal stresses exerted across its boundary $\partial\Omega_t$ is given by

$$\int_{\partial\Omega_t} (-pI)\,\boldsymbol{n}\,\mathrm{d}S \equiv \int_{\Omega_t} (-\nabla p)\,\mathrm{d}V.$$

If $\boldsymbol{f}$ is a body force (external force) per unit mass, which can depend on position and time, then the body force on the fluid inside $\Omega_t$ is

$$\int_{\Omega_t} \rho\,\boldsymbol{f}\,\mathrm{d}V.$$

Thus on any parcel of fluid $\Omega_t$, the total force acting on it is

$$\int_{\Omega_t} -\nabla p + \rho\,\boldsymbol{f}\,\mathrm{d}V.$$

Hence using Newton's 2nd law (force = rate of change of total momentum) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\,\boldsymbol{u}\,\mathrm{d}V = \int_{\Omega_t} -\nabla p + \rho\,\boldsymbol{f}\,\mathrm{d}V.$$

Now we use the Transport Theorem with $F \equiv \boldsymbol{u}$ and that $\Omega$ and thus $\Omega_t$ are arbitrary. We see that for at each $\boldsymbol{x} \in \mathcal{D}$ and $t \geqslant 0$, the following relation must hold—the differential form of the *balance of momentum* in this case:

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho\,\boldsymbol{f}.$$

Thus for an *ideal fluid* for which we only include normal stresses and completely ignore any shear stresses, the fluid flow is governed by the Euler equations of motion (derived by Euler in 1755) given by:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{f},$$

Now that we have partial differential equations that determine how fluid flows evolve, we complement them with *boundary and initial conditions.* The initial condition is the velocity profile $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, 0)$ at time $t = 0$. It is the state in which the flow starts. To have a *well-posed* evolutionary partial differential system for the evolution of the fluid flow, we also need to specify how the flow behaves near boundaries. Here a boundary could be a rigid boundary, for example the walls of the container the fluid is confined to or the surface of an obstacle in the fluid flow. Another example of a boundary is the free surface between two immiscible fluids—such as between seawater and air on the ocean surface. Here we will focus on rigid boundaries.

For ideal fluid flow, i.e. one evolving according to the Euler equations, we simply need to specify that there is no net flow normal to the boundary—the fluid does not cross the boundary but can move tangentially to it. Mathematically this is means that we specify that

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0$$

everywhere on the rigid boundary.

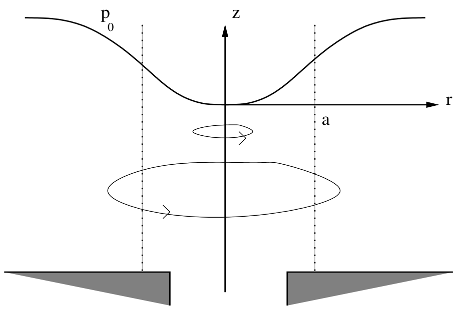


**Fig. 5** Water draining from a bath.

*Remark 3* The incompressible Euler equations of motion can also be derived as a geodesic submanifold flow. The submanifold is the group of measure preserving diffeomorphisms. For a formal discussion see Malham [14].

For many examples of simple Euler flows we refer the reader ahead to Section 17—all the examples there are Euler flows except for the last one. Extending those examples further, we now examine an every day flow.

**Example (sink or bath drain)** As we have all observed when water runs out of a bath or sink, the free surface of the water directly over the drain hole has a depression in it—see Fig. 5. The question is, what is the form/shape of this free surface depression?

The essential idea is we know that the pressure at the free surface is uniform, it is atmospheric pressure, say $P_0$. We need the Euler equations for a homogeneous incompressible fluid in *cylindrical coordinates* $(r, \theta, z)$ with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{T}}$. These are

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r,\\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + f_\theta,\\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z,
\end{aligned}$$

where $p = p(r, \theta, z, t)$ is the pressure, $\rho$ is the uniform constant density and $\boldsymbol{f} = (f_r, f_\theta, f_z)^{\mathrm{T}}$ is the body force per unit mass. Here we also have

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}.$$

Further the incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$ is given in cylindrical coordinates by

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0.$$

We need to make some sensible simplifying assumptions to reduce this system of equations to a set of partial differential equations we might be able to solve analytically. We will assume the fluid has uniform density $\rho$, that the flow is steady, and $u_r = u_z = 0$, i.e. only the azimuthal velocity is non-zero so that the water particles move in horizontal circles—see Fig. 5. We further assume $f_r = f_\theta = 0$. The force due to gravity implies

$f_z = -g$. The whole problem is also symmetric with respect to $\theta$, so we will also assume all partial derivatives with respect to $\theta$ are zero. Combining all these facts reduces Euler's equations above to

$$\begin{aligned} -\frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}, \\ 0 &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \end{aligned}$$

The incompressibility condition is satisfied trivially. The second equation above tells us the pressure $p$ is independent of $\theta$, as we might have already suspected. Hence we assume $p = p(r, z)$ and focus on the first and third equation above.

The question now is can we find functions $u_\theta = u_\theta(r, z)$ and $p = p(r, z)$ that satisfy the first and third partial differential equations above? To help us in this direction we will make some further assumptions. We will suppose that as the water flows out through the hole at the bottom of a bath the residual rotation is confined to a core of radius $a$, so that the water particles may be taken to move on horizontal circles with

$$u_\theta = \begin{cases} \Omega r, & r \leqslant a, \\ \frac{\Omega a^2}{r}, & r > a. \end{cases}$$

The azimuthal flow we assume for $r \leqslant a$ represents solid body rotation in the core region. The flow we assume for $r > a$ represents two-dimensional irrotational flow generated by a point source at the origin. With $u_\theta = u_\theta(r)$ assumed to have this form, the question now is, can we find a corresponding pressure field $p = p(r, z)$ so that the first and third equations above are satisfied?

Assume $r \leqslant a$. Using that $u_\theta = \Omega r$ in the first equation we see that

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r \qquad \Leftrightarrow \qquad p(r, z) = \tfrac{1}{2}\rho \Omega^2 r^2 + C(z),$$

where $C(z)$ is an arbitrary function of $z$. If we then substitute this into the third equation above we see that

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad \Leftrightarrow \qquad C'(z) = -\rho g,$$

and hence $C(z) = -\rho g z + C_0$ where $C_0$ is an arbitrary constant. Thus we now deduce that the pressure function is given by

$$p(r, z) = \tfrac{1}{2}\rho \Omega^2 r^2 - \rho g z + C_0.$$

At the free surface of the water, the pressure is constant atmospheric pressure $P_0$ and so if we substitute this into this expression for the pressure we see that

$$P_0 = \tfrac{1}{2}\rho \Omega^2 r^2 - \rho g z + C_0 \qquad \Leftrightarrow \qquad z = (\Omega^2/2g)\, r^2 - (C_0 - P_0)/\rho g.$$

Hence the depression in the free surface for $r \leqslant a$ is a *parabolic surface* of revolution. Note that pressure is only ever globally defined up to an additive constant so we are at liberty to take $C_0 = 0$ or $C_0 = P_0$ if we like.

For $r > a$ a completely analogous argument using $u_\theta = \Omega a^2/r$ shows that

$$p(r,z) = -\frac{\rho\Omega^2 a^4}{2\,r^2} - \rho g z + K_0,$$

where $K_0$ is an arbitrary constant. Since the pressure must be continuous at $r = a$, we substitute $r = a$ into the expression for the pressure here for $r > a$ and the expression for the pressure for $r \leqslant a$, and equate the two. This gives

$$-\tfrac{1}{2}\rho\Omega^2 a^2 - \rho g z + K_0 = \tfrac{1}{2}\rho\Omega^2\, a^2 - \rho g z \qquad \Leftrightarrow \qquad K_0 = \rho\Omega^2\, a^2.$$

Hence the pressure for $r > a$ is given by

$$p(r,z) = -\frac{\rho\Omega^2 a^4}{2\,r^2} - \rho g z + \rho\Omega^2\, a^2.$$

Using that the pressure at the free surface is $p(r,z) = P_0$, we see that for $r > a$ the free surface is given by

$$z = -\frac{\Omega^2 a^4}{g\,r^2} + \frac{\Omega^2\, a^2}{g}.$$

*Remark 4* The fact that there are no tangential forces in an ideal fluid has some important consequences, quoting from Chorin and Marsden [3, p. 5]:

> ...there is no way for rotation to start in a fluid, nor, if there is any at the beginning, to stop... ... even here we can detect trouble for ideal fluids because of the abundance of rotation in real fluids (near the oars of a rowboat, in tornadoes, etc. ).

In particular see D'Alembert's paradox in Section 12. We discuss some further consequences of Euler flow in Appendix D.

## 11 Bernoulli's Theorem

**Theorem 3 (Bernoulli's Theorem)** *Suppose we have an ideal homogeneous incompressible stationary flow with a conservative body force $\boldsymbol{f} = -\nabla\Phi$, where $\Phi$ is the potential function. Then the quantity*

$$H := \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

*is constant along streamlines.*

*Proof* We need the following identity that can be found in Appendix A:

$$\tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u}).$$

Since the flow is stationary, Euler's equation of motion for an ideal fluid imply

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\Phi.$$

Using the identity above we see that

$$\begin{aligned} & \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\Phi \\ \Leftrightarrow\quad & \nabla\Big(\tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi\Big) = \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) \\ \Leftrightarrow\quad & \nabla H = \boldsymbol{u}\times(\nabla\times\boldsymbol{u}), \end{aligned}$$

using the definition for $H$ given in the theorem. Now let $\boldsymbol{x}(s)$ be a streamline that satisfies $\boldsymbol{x}'(s) = \boldsymbol{u}\big(\boldsymbol{x}(s)\big)$. By the fundamental theorem of calculus, for any $s_1$ and $s_2$,

$$\begin{aligned} H\big(\boldsymbol{x}(s_2)\big) - H\big(\boldsymbol{x}(s_1)\big) &= \int_{s_1}^{s_2} \mathrm{d}H\big(\boldsymbol{x}(s)\big) \\ &= \int_{s_1}^{s_2} \nabla H\cdot\boldsymbol{x}'(s)\,\mathrm{d}s \\ &= \int_{s_1}^{s_2} \big(\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\big)\cdot\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\,\mathrm{d}s \\ &= 0, \end{aligned}$$

where we used that $(\boldsymbol{u}\times\boldsymbol{a})\cdot\boldsymbol{u} \equiv \boldsymbol{0}$ for any vector $\boldsymbol{a}$ (since $\boldsymbol{u}\times\boldsymbol{a}$ is orthogonal to $\boldsymbol{u}$). Since $s_1$ and $s_2$ are arbitrary we deduce that $H$ does not change along streamlines. □

*Remark 5* Note that $\rho H$ has the units of an energy density. Since $\rho$ is constant here, we can interpret Bernoulli's Theorem as saying that energy density is constant along streamlines.

**Example (Torricelli 1643).** Consider the problem of an oil drum full of water that has a small hole punctured into it near the bottom. The problem is to determine the velocity of the fluid jetting out of the hole at the bottom and how that varies with the amount of water left in the tank—the setup is shown in Fig 6. We shall assume the hole has a small cross-sectional area $\alpha$. Suppose that the cross-sectional area of the drum, and therefore of the free surface (water surface) at $z = 0$, is $A$. We naturally assume $A \gg \alpha$. Since the rate at which the amount of water is dropping inside the drum must equal the rate at which water is leaving the drum through the punctured hole, we have

$$\Big(-\frac{\mathrm{d}h}{\mathrm{d}t}\Big)\cdot A = U\cdot\alpha \qquad\Leftrightarrow\qquad \Big(-\frac{\mathrm{d}h}{\mathrm{d}t}\Big) = \Big(\frac{\alpha}{A}\Big)\cdot U.$$

We observe that $A \gg \alpha$, i.e. $\alpha/A \ll 1$, and hence we can deduce

$$\frac{1}{U^2}\Big(\frac{\mathrm{d}h}{\mathrm{d}t}\Big)^2 = \Big(\frac{\alpha}{A}\Big)^2 \ll 1.$$

Since the flow is quasi-stationary, incompressible as it's water, and there is conservative body force due to gravity, we apply Bernoulli's Theorem for one of the typical streamlines shown in Fig. 6. This implies that the quantity $H$ is the same at the free surface and at the puncture hole outlet, hence

$$\tfrac{1}{2}\Big(\frac{\mathrm{d}h}{\mathrm{d}t}\Big)^2 + \frac{P_0}{\rho} = \tfrac{1}{2}U^2 + \frac{P_0}{\rho} - gh.$$

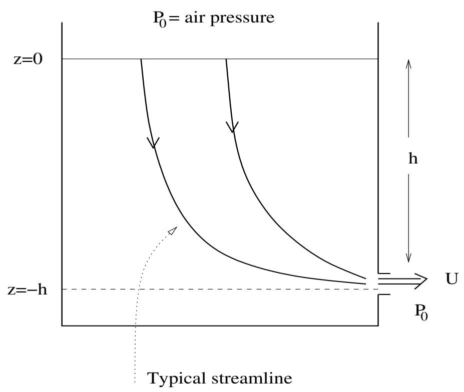


**Fig. 6** *Torricelli problem*: the pressure at the top surface and outside the puncture hole is atmospheric pressure $P_0$. Suppose the height of water above the puncture is $h$. The goal is to determine how the velocity of water $U$ out of the puncture hole varies with $h$.

Thus cancelling the $P_0/\rho$ terms then we can deduce that

$$\begin{aligned} gh &= \tfrac{1}{2}U^2 - \tfrac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 \\ &= \tfrac{1}{2}U^2\left(1 - \frac{1}{U^2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\right) \\ &= \tfrac{1}{2}U^2\left(1 - \left(\frac{\alpha}{A}\right)^2\right) \\ &\sim \tfrac{1}{2}U^2 \end{aligned}$$

for $\alpha/A \ll 1$ with an error of order $(\alpha/A)^2$. Thus in the asymptotic limit $gh = \frac{1}{2}U^2$ so

$$U = \sqrt{2gh}.$$

*Remark 6* Note the pressure inside the container at the puncture hole level is $P_0 + \rho gh$. The difference between this and the atmospheric pressure $P_0$ outside, accelerates the water through the puncture hole.

**Example (Channel flow: Froude number).** Consider the problem of a steady flow of water in a channel over a gently undulating bed—see Fig 7. We assume that the flow is shallow and uniform in cross-section. Upstream the flow is characterized by flow velocity $U$ and depth $H$. The flow then impinges on a gently undulating bed of height $y = y(x)$ as shown in Fig 7, where $x$ measures distance downstream. The depth of the flow is given by $h = h(x)$ whilst the fluid velocity at that point is $u = u(x)$, which is uniform over the depth throughout. Re-iterating slightly, our assumptions are thus,

$$\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| \ll 1 \qquad \text{(bed gently undulating)}$$

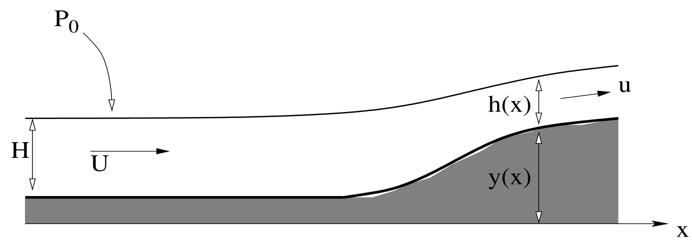


**Fig. 7** *Channel flow problem*: a steady flow of water, uniform in cross-section, flows over a gently undulating bed of height $y = y(x)$ as shown. The depth of the flow is given by $h = h(x)$. Upstream the flow is characterized by flow velocity $U$ and depth $H$.

and

$$\left|\frac{\mathrm{d}h}{\mathrm{d}x}\right| \ll 1 \qquad \text{(small variation in depth).}$$

The continuity equation (incompressibility here) implies that for all $x$,

$$uh = UH.$$

Then Euler's equations for a steady flow imply Bernoulli's theorem which we apply to the surface streamline, for which the pressure is constant and equal to atmospheric pressure $P_0$, hence we have for all $x$:

$$\tfrac{1}{2}U^2 + gH = \tfrac{1}{2}u^2 + g(y + h).$$

Substituting for $u = u(x)$ from the incompressibility condition above, and rearranging, Bernoulli's theorem implies that for all $x$ we have the constraint

$$y = \frac{U^2}{2g} + H - h - \frac{(UH)^2}{2gh^2}.$$

We can think of this as a parametric equation relating the fluid depth $h = h(x)$ to the undulation height $h = h(x)$ where the parameter $x$ runs from $x = -\infty$ far upstream to $x = +\infty$ far downstream. We plot this relation, $y$ as a function of $h$, in Fig 8. Note that $y$ has a unique global maximum $y_0$ coinciding with the local maximum and given by

$$\frac{\mathrm{d}y}{\mathrm{d}h} = 0 \qquad \Leftrightarrow \qquad h = h_0 = \frac{(UH)^{2/3}}{g^{1/3}}.$$

Note that if we set

$$\mathrm{F} \coloneqq U/\sqrt{gH}$$

then $h_0 = H\mathrm{F}^{2/3}$, where F is known as the *Froude number*. It is a dimensionless function of the upstream conditions and represents the ratio of the oncoming fluid speed to the wave (signal) speed in fluid depth $H$.

Note that when $y = y(x)$ attains its maximum value at $h_0$, then $y = y_0$ where

$$y_0 \coloneqq H\left(1 + \tfrac{1}{2}\mathrm{F}^2 - \tfrac{3}{2}\mathrm{F}^{2/3}\right).$$

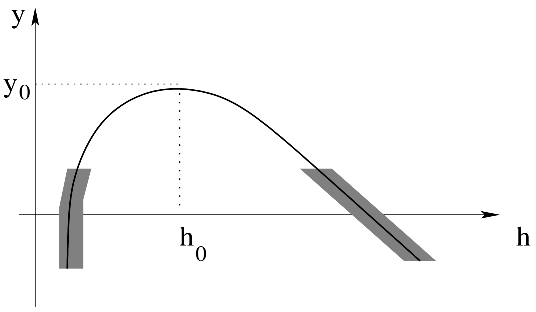


**Fig. 8** *Channel flow problem*: The flow depth $h = h(x)$ and undulation height $y = y(x)$ are related as shown, from Bernoulli's theorem. Note that $y$ has a maximum value $y_0$ at height $h_0 = H\mathrm{F}^{2/3}$ where $\mathrm{F} = U/\sqrt{gH}$ is the Froude number.

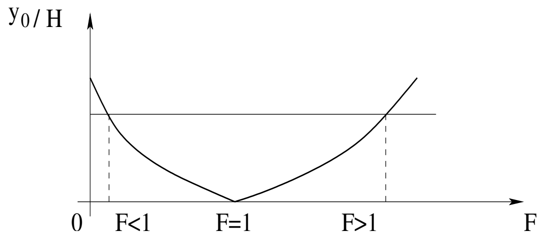


**Fig. 9** *Channel flow problem*: Two different values of the Froude number F give the same maximum permissible undulation height $y_0$. Note we actually plot the normalized maximum possible height $y_0/H$ on the ordinate axis.

This puts a bound on the height of the bed undulation that is compatible with the upstream conditions. In Fig 9 we plot the maximum permissible height $y_0$ the undulation is allowed to attain as a function of the Froude number F. Note that two different values of the Froude number F give the same maximum permissible undulation height $y_0$, one of which is slower and one of which is faster (compared with $\sqrt{gH}$).

Let us now consider and actual given undulation $y = y(x)$. Suppose that it attains an *actual* maximum value $y_{\max}$. There are three cases to consider, in turn we shall consider $y_{\max} < y_0$, the more interesting case, and then $y_{\max} > y_0$. The third case $y_{\max} = y_0$ is an exercise (see the Exercises section at the end of these notes).

In the first case, $y_{\max} < y_0$, as $x$ varies from $x = -\infty$ to $x = +\infty$, the undulation height $y = y(x)$ varies but is such that $y(x) \leqslant y_{\max}$. Refer to Fig. 8, which plots the constraint relationship between $y$ and $h$ resulting from Bernoulli's theorem. Since $y(x) \leqslant y_{\max}$ as $x$ varies from $-\infty$ to $+\infty$, the values of $(h, y)$ are restricted to part of the branches of the graph either side of the global maximum $(h_0, y_0)$. In the figure these parts of the branches are the locale of the shaded sections shown. Note that the derivative $\mathrm{d}y/\mathrm{d}h = 1/(\mathrm{d}h/\mathrm{d}y)$ has the same fixed (and opposite) sign in each of the branches. In the branch for which $h$ is small, $\mathrm{d}y/\mathrm{d}h > 0$, while the branch for which $h$ is larger, $\mathrm{d}y/\mathrm{d}h < 0$. Indeed note the by differentiating the constraint condition, we have

$$\frac{\mathrm{d}y}{\mathrm{d}h} = -\left(1 - \frac{(UH)^2}{gh^3}\right).$$

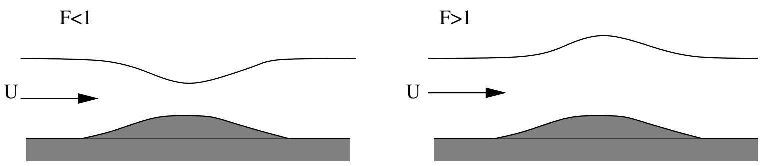


**Fig. 10** *Channel flow problem*: for the case $y_{\max} < y_0$, when $\mathrm{F} < 1$, as the bed height $y$ increases, the fluid depth $h$ decreases and vice-versa. Hence we see a depression in the fluid surface above a bump in the bed. On the other hand, when $\mathrm{F} > 1$, as the bed height $y$ increases, the fluid depth $h$ increases and vice-versa. Hence we see an elevation in the fluid surface above a bump in the bed.

Using the incompressibility condition to substitute for $UH$ we see that this is equivalent to

$$\frac{\mathrm{d}y}{\mathrm{d}h} = -\left(1 - \frac{u^2}{gh}\right).$$

We can think of $u/\sqrt{gh}$ as a local Froude number if we like. In any case, note that since we are in one branch or the other, and in either case the sign of $\mathrm{d}y/\mathrm{d}h$ is fixed, this means that using the expression for $\mathrm{d}y/\mathrm{d}h$ we just derived, for any flow realization the sign of $1 - u^2/gh$ is also fixed. When $x = -\infty$ this quantity has the value $1 - U^2/gH$. Hence the sign of $1 - U^2/gH$ determines the sign of $1 - u^2/gh$. Hence if $\mathrm{F} < 1$ then $U^2/gH = \mathrm{F}^2 < 1$ and therefore for all $x$ we must have $u^2/gh < 1$. And we also deduce in this case that we must be on the branch for which $h$ is relatively large as $\mathrm{d}y/\mathrm{d}h$ is negative. The flow is said to be *subcritical* throughout and indeed we see that

$$\frac{\mathrm{d}h}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{-1} = -\left(1 - \frac{u^2}{gh}\right)^{-1} < -1 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}y}(h+y) < 0.$$

Hence in this case, as the bed height $y$ increases, the fluid depth $h$ decreases and vice-versa. On the other hand if $\mathrm{F} > 1$ then $U^2/gH > 1$ and thus $u^2/gh > 1$. We must be on the branch for which $h$ is relatively small as $\mathrm{d}y/\mathrm{d}h$ is positive. The flow is said to be *supercritical* throughout and we have

$$\frac{\mathrm{d}h}{\mathrm{d}y} = -\left(1 - \frac{u^2}{gh}\right)^{-1} > 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}y}(h+y) > 1.$$

Hence in this case, as the bed height $y$ increases, the fluid depth $h$ increases and vice-versa. Both cases, $\mathrm{F} < 1$ and $\mathrm{F} > 1$, are illustrated by a typical scenario in Fig. 10.

In the second case, $y_{\max} > y_0$, the undulation height is larger than the maximum permissible height $y_0$ compatible with the upstream conditions. Under the conditions we assumed, there is no flow realized here. In a real situation we may imagine a flow impinging on a large barrier with height $y_{\max} > y_0$, and the result would be some sort of *reflection* of the flow occurs to change the upstream conditions in an attempt to make them compatible with the *obstacle*. (Our steady flow assumption obviously breaks down here.)

## 12 Irrotational/potential flow

Many flows have extensive regions where the vorticity is zero; some have zero vorticity everywhere. We would call these, respectively, *irrotational* regions of the flow and irrotational flows. In such regions

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \mathbf{0}.$$

Hence the field $\boldsymbol{u}$ is *conservative* and there exists a scalar function $\phi$ such that

$$\boldsymbol{u} = \nabla \phi.$$

The function $\phi$ is known as the *flow potential*. Note that $\boldsymbol{u}$ is conservative in a region if and only if the circulation

$$\oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 0$$

for all simple closed curves $\mathcal{C}$ in the region.

If the fluid is also incompressible, then $\phi$ is *harmonic* since $\nabla \cdot \boldsymbol{u} = 0$ implies

$$\Delta \phi = 0.$$

Hence for such situations, we in essence need to solve Laplace's equation $\Delta\phi = 0$ subject to certain boundary conditions. For example for an ideal flow, $\boldsymbol{u} \cdot \boldsymbol{n} = \nabla\phi \cdot \boldsymbol{n} = \partial\phi/\partial n$ is given on the boundary, and this would constitute a Neumann problem for Laplace's equation.

**Example (linear two-dimensional flow)** Consider the flow field $\boldsymbol{u} = (kx, -ky)^{\mathrm{T}}$ where $k$ is a constant. It is irrotational. Hence there exists a flow potential $\phi = \frac{1}{2}k(x^2 - y^2)$. Since $\nabla \cdot \boldsymbol{u} = 0$ as well, we have $\Delta\phi = 0$. Further, since this flow is two-dimensional, there also exists a stream function $\psi = kxy$.

**Example (line vortex)** Consider the flow field $(u_r, u_\theta, u_z)^{\mathrm{T}} = (0, k/r, 0)^{\mathrm{T}}$ where $k > 0$ is a constant. This is the idealization of a thin vortex tube. Direct computation shows that $\nabla \times \boldsymbol{u} = \mathbf{0}$ everywhere except at $r = 0$, where $\nabla \times \boldsymbol{u}$ is infinite. For $r > 0$, there exists a flow potential $\phi = k\theta$. For any closed circuit $\mathcal{C}$ in this region the circulation is

$$\oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 2\pi k\, N$$

where $N$ is the number of times the closed curve $\mathcal{C}$ winds round the origin $r = 0$. The circulation will be zero for all circuits reducible continuously to a point without breaking the vortex.

**Example (D'Alembert's paradox)** Consider a uniform flow into which we place an obstacle. We would naturally expect that the obstacle represents an obstruction to the fluid flow and that the flow would exert a force on the obstacle, which if strong enough, might dislodge it and subsequently carry it downstream. However for an ideal flow, as we are just about to prove, this is not the case. There is no net force exerted on an obstacle placed in the midst of a uniform flow.

We thus consider a uniform ideal flow into which is placed a sphere, radius $a$. The set up is shown in Fig. 11. We assume that the flow around the sphere is steady, incompressible and irrotational. Suppose further that the flow is axisymmetric. By this we mean the following. Use spherical polar coordinates to represent the flow with the

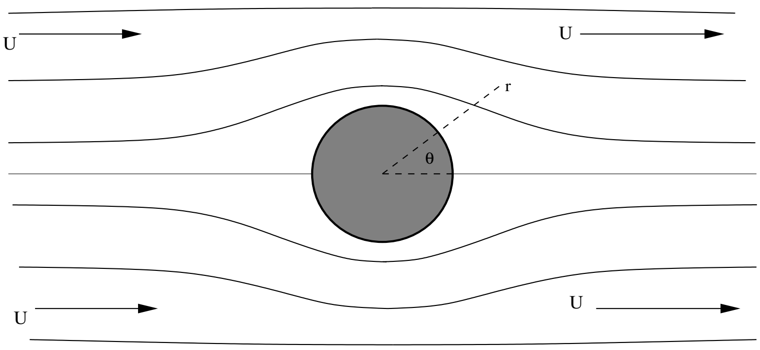


**Fig. 11** Consider an ideal steady, incompressible, irrotational and axisymmetric flow past a sphere as shown. The net force exerted on the sphere (obstacle) in the flow is zero. This is D'Alembert's paradox.

south-north pole axis passing through the centre of the sphere and aligned with the uniform flow $U$ at infinity; see Fig. 11. Then the flow is axisymmetric if it is independent of the azimuthal angle $\varphi$ of the spherical coordinates $(r, \theta, \varphi)$. Further we also assume no swirl so that $u_\varphi = 0$.

Since the flow is incompressible and irrotational, it is a potential flow. Hence we seek a potential function $\phi$ such that $\Delta\phi = 0$. In spherical polar coordinates this is equivalent to

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)\right) = 0.$$

The general solution to Laplace's equation is well known, and in the case of axisymmetry the general solution is given by

$$\phi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

where $P_n$ are the Legendre polynomials; with $P_1(x) = x$. The coefficients $A_n$ and $B_n$ are constants, most of which, as we shall see presently, are zero. For our problem we have two sets of boundary data. First, that as $r \to \infty$ in any direction, the flow field is uniform and given by $\boldsymbol{u} = (0, 0, U)^{\mathrm{T}}$ (expressed in Cartesian coordinates with the $z$-axis aligned along the south-north pole) so that as $r \to \infty$

$$\phi \sim Ur\cos\theta.$$

Second, on the sphere $r = a$ itself we have a no normal flow condition

$$\frac{\partial\phi}{\partial r} = 0.$$

Using the first boundary condition for $r \to \infty$ we see that all the $A_n$ must be zero except $A_1 = U$. Using the second boundary condition on $r = a$ we see that all the $B_n$ must be zero except for $B_1 = \frac{1}{2}Ua^3$. Hence the potential for this flow around the sphere is

$$\phi = U(r + a^3/2r^2)\cos\theta.$$

In spherical polar coordinates, the velocity field $\boldsymbol{u} = \nabla\phi$ is given by

$$\boldsymbol{u} = (u_r, u_\theta) = \big(U(1 - a^3/r^3)\cos\theta, -U(1 + a^3/2r^3)\sin\theta\big).$$

Since the flow is ideal and steady as well, Bernoulli's theorem applies and so along a typical streamline $\frac{1}{2}|\boldsymbol{u}|^2 + P/\rho$ is constant. Indeed since the conditions at infinity are uniform so that the pressure $P_\infty$ and velocity field $U$ are the same everywhere there, this means that for any streamline and in fact everywhere for $r \geqslant a$ we have

$$\frac{1}{2}|\boldsymbol{u}|^2 + P/\rho = \frac{1}{2}U^2 + P_\infty/\rho.$$

Rearranging this equation and using our expression for the velocity field above we have

$$\frac{P - P_\infty}{\rho} = \tfrac{1}{2}U^2\big(1 - (1 - a^3/r^3)^2\cos^2\theta - (1 + a^3/2r^3)^2\sin^2\theta\big).$$

On the sphere $r = a$ we see that

$$\frac{P - P_\infty}{\rho} = \tfrac{1}{2}U^2\big(1 - \tfrac{9}{4}\sin^2\theta\big).$$

Note that on the sphere, the pressure is symmetric about $\theta = 0, \pi/2, \pi, 3\pi/2$. Hence the fluid exerts no net force on the sphere! (There is no drag or lift.) This result, in principle, applies to any shape of obstacle in such a flow. In reality of course this cannot be the case, the presence of viscosity remedies this paradox (and crucially generates vorticity).

## 13 Kelvin's circulation theorem, vortex lines and tubes

We turn our attention to important concepts centred on vorticity in a flow.

**Definition 7 (Circulation)** Let $\mathcal{C}$ be a simple closed contour in the fluid at time $t = 0$. Suppose that $\mathcal{C}$ is carried along by the flow to the closed contour $\mathcal{C}_t$ at time $t$, i.e. $\mathcal{C}_t = \varphi_t(\mathcal{C})$. The *circulation* around $\mathcal{C}_t$ is defined to be the line integral

$$\oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}.$$

Using Stokes' Theorem an equivalent definition for the circulation is

$$\oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n}\,\mathrm{d}S = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\,\mathrm{d}S$$

where $S$ is any surface with perimeter $\mathcal{C}_t$; see Fig. 13. In other words the circulation is equivalent to the flux of vorticity through the surface with perimeter $\mathcal{C}_t$.

**Theorem 4 (Kelvin's circulation theorem (1869))** *For ideal, incompressible flow without external forces, the circulation for any closed contour $\mathcal{C}_t$ is constant in time.*

*Proof* Using a variant of the Transport Theorem and the Euler equations, we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \oint_{\mathcal{C}_t} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{x} = -\oint_{\mathcal{C}_t} \nabla p \cdot \mathrm{d}\boldsymbol{x} = 0,$$

for closed loops of fluid particles $\mathcal{C}_t$. □

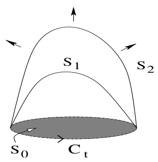


**Fig. 12** Stokes' theorem tells us that the circulation around the closed contour $\mathcal{C}$ equals the flux of vorticity through any surface whose perimeter is $\mathcal{C}$. For example here the flux of vorticity through $S_0$, $S_1$ and $S_2$ is the same.

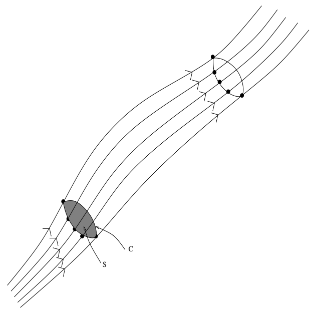

**Fig. 13** The strength of the vortex tube is given by the circulation around any curve $\mathcal{C}$ that encircles the tube once.

**Corollary 3** *The flux of vorticity across a surface moving with the fluid is constant in time.*

**Definition 8 (Vortex lines)** These are the lines that are everywhere parallel to the local vorticity $\boldsymbol{\omega}$, i.e. with $t$ fixed they solve $(\mathrm{d}/\mathrm{d}s)\boldsymbol{x}(s) = \boldsymbol{\omega}(\boldsymbol{x}(s), t)$. These are the trajectories for the field $\boldsymbol{\omega}$ for $t$ fixed.

**Definition 9 (Vortex tube)** This is the surface formed by the vortex lines through the points of a simple closed curve $\mathcal{C}$; see Fig. 13. We can define the *strength* of the vortex tube to be

$$\int_S \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S \equiv \oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}.$$

*Remark 7* This is a good definition because it is independent of the precise cross-sectional area $S$, and the precise circuit $\mathcal{C}$ around the vortex tube taken (because $\nabla \cdot \boldsymbol{\omega} \equiv 0$); see Fig. 13. Vorticity is larger where the cross-sectional area is smaller and vice-versa. Further, for an ideal fluid, vortex tubes move with the fluid and the strength of the vortex tube is constant in time as it does so (Helmholtz's theorem; 1858); see Chorin and Marsden [3, p. 26].

## 14 Shear stresses

Recall our discussion on internal fluid forces in Section 9. We now consider the explicit form of the shear stresses and in particular the deviatoric stress tensor. This is necessary if we want to consider/model any real fluid (i.e. non-ideal fluid). We assume that the deviatoric stress tensor $\hat{\sigma}$ is a function of the rate of strain tensor $\nabla \boldsymbol{u}$. We shall make three assumptions about the deviatoric stress tensor $\hat{\sigma}$ and its dependence on the velocity gradients $\nabla \boldsymbol{u}$. These are that it is:

1. *Linear:* each component of $\hat{\sigma}$ is linearly related to the rate of strain tensor $\nabla \boldsymbol{u}$.
2. *Isotropic:* if $U$ is an orthogonal matrix, then

$$\hat{\sigma}\big(U \cdot \nabla \boldsymbol{u} \cdot U^{-1}\big) \equiv U \cdot \hat{\sigma}(\nabla \boldsymbol{u}) \cdot U^{-1}.$$

   Equivalently we might say that it is invariant under rigid body rotations.
3. *Symmetric*; i.e. $\hat{\sigma}_{ij} = \hat{\sigma}_{ji}$. This can be deduced as a result of balance of angular momentum.

Hence each component of the deviatoric stress tensor $\hat{\sigma}$ is a linear function of each of the components of the velocity gradients $\nabla \boldsymbol{u}$. This means that there is a total of 81 constants of proportionality. We will use the assumptions above to systematically reduce this to 2 constants.

When the fluid performs rigid body rotation, there should be no diffusion of momentum (the whole mass of fluid is behaving like a solid body). Recall our decomposition of the rate of strain tensor, $\nabla \boldsymbol{u} = D + R$, where $D$ is the deformation tensor and $R$ generates rotation. Thus $\hat{\sigma}$ only depends on the symmetric part of $\nabla \boldsymbol{u}$, i.e. it is a linear function of the deformation tensor $D$. Further, since $\hat{\sigma}$ is symmetric, we can restrict our attention to linear functions from symmetric matrices to symmetric matrices. We now lean heavily on the isotropy assumption 2; see Gurtin [7, Section 37] for more details.

First, we have the transfer theorem.

**Theorem 5 (Transfer theorem)** *Let $\hat{\sigma}$ be an endomorphism on the set of $3 \times 3$ symmetric matrices. Then if $\hat{\sigma}$ is isotropic, the symmetric matrices $D$ and $\hat{\sigma}(D)$ are simultaneously diagonalizable.*

*Proof* Let $\boldsymbol{e}$ be an eigenvector of $D$ and let $U$ be the orthogonal matrix denoting reflection in the plane perpendicular to $\boldsymbol{e}$, so that $U\boldsymbol{e} = -\boldsymbol{e}$, while any vector perpendicular to $\boldsymbol{e}$ is invariant under $U$. The eigenstructure of $D$ is invariant to such a transformation so that $UDU^{-1} = D$. Thus, since $\hat{\sigma} = \hat{\sigma}(D)$ is isotropic, we have $U\hat{\sigma}U^{-1} = \hat{\sigma}(UDU^{-1}) = \hat{\sigma}(D)$ and thus $U\hat{\sigma} = \hat{\sigma}U$. Any such commuting matrices share eigenvectors since $U\hat{\sigma}\boldsymbol{e} = \hat{\sigma}U\boldsymbol{e} = -\hat{\sigma}\boldsymbol{e}$. Thus $\hat{\sigma}\boldsymbol{e}$ is also an eigenvector of the reflection transformation $U$ corresponding to the same eigenvalue $-1$. Thus $\hat{\sigma}\,\boldsymbol{e}$ is proportional to $\boldsymbol{e}$ and so $\boldsymbol{e}$ is an eigenvector of $\hat{\sigma}$. Since $\boldsymbol{e}$ was any eigenvector of $D$, the statement of the theorem follows. □

Second, for any $3 \times 3$ matrix $A$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, the three scalar functions

$$I_1(A) := \operatorname{Tr} A, \qquad I_2(A) := \tfrac{1}{2}\big((\operatorname{Tr}A)^2 - \operatorname{Tr}(A^2)\big) \qquad \text{and} \qquad I_2(A) := \det A,$$

are isotropic. This can be checked by direct computation. Indeed these three functions are the elementary symmetric functions of the eigenvalues of $A$:

$$I_1(A) = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2(A) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_2\lambda_3 \quad \text{and} \quad I_2(A) = \lambda_1\lambda_2\lambda_3.$$

We have the following representation theorem for isotropic functions.

**Theorem 6 (Representation theorem)** *An endomorphism $\hat{\sigma}$ on the set of $3 \times 3$ symmetric matrices is isotropic if and only if it has the form*

$$\hat{\sigma}(D) = \alpha_0 I + \alpha_1 D + \alpha_2 D^2,$$

*for every symmetric matrix $D$, where $\alpha_0$, $\alpha_1$ and $\alpha_2$ are scalar functions that depend only on the isotropic invariants $I_1(D), I_2(D)$ and $I_3(D)$.*

*Proof* Scalar functions $\alpha = \alpha(D)$ are isotropic if and only if they are functions of the isotropic invariants of $D$ only. The 'if' part of this statement follows trivially as the isotropic invariants are isotropic. The 'only if' statement is established if, assuming $\alpha$ is isotropic, we are able to show that

$$I_i(D) = I_i(D') \text{ for } i = 1, 2, 3 \qquad \Longrightarrow \qquad \alpha(D) = \alpha(D').$$

Since the map between the eigenvalues of $D$ and its isotropic invariants is bijective, if $I_i(D) = I_i(D')$ for $i = 1, 2, 3$, then $D$ and $D'$ have the same eigenvalues. Since the isospectral action $UDU^{-1}$ of orthogonal matrices $U$ on symmetric matrices $D$ is transitive, there exists an orthogonal matrix $U$ such that $D' = UDU^{-1}$. Since $\alpha$ is isotropic, $\alpha(UDU^{-1}) = \alpha(D)$, i.e. $\alpha(D') = \alpha(D)$.

Now let us consider the symmetric matrix valued function $\hat{\sigma}$. The 'if' statement of the theorem follows by direct computation and the result we just established for scalar isotropic functions. The 'only if' statement is proved as follows. Assume $\hat{\sigma}$ has three distinct eigenvalues (we leave the other possibilities as an exercise). Using the transfer theorem and the Spectral Theorem (see for example Meyer [16, p. 517]) we have

$$\hat{\sigma}(D) = \sum_{i=1}^{3} \hat{\sigma}_i E_i$$

where $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ are the eigenvalues of $\hat{\sigma}$ and the projection matrices $E_1$, $E_2$ and $E_3$ have the properties $E_i E_j = O$ when $i \neq j$ and $E_1 + E_2 + E_3 = I$. Since we have

$$\text{span}\{I, D, D^2\} = \text{span}\{E_1, E_2, E_3\},$$

there exist scalars $\alpha_0$, $\alpha_1$ and $\alpha_2$ depending on $D$ such that

$$\hat{\sigma}(D) = \alpha_0 I + \alpha_1 D + \alpha_2 D^2.$$

We now have to show that $\alpha_0$, $\alpha_1$ and $\alpha_2$ are isotropic. This follows by direct computation, combining this last representation with the property that $\hat{\sigma}$ is isotropic. □

*Remark 8* Note that neither the transfer theorem nor the representation theorem require that the endomorphism $\hat{\sigma}$ is linear.

Third, now suppose that $\hat{\sigma}$ is a linear function of $D$. Thus for any symmetric matrix $D$ it must have the form

$$\hat{\sigma}(D) = \lambda I + 2\mu D,$$

where the scalars $\lambda$ and $\mu$ depend on the isotropic invariants of $D$. By the Spectral Theorem we have

$$D = \sum_{i=1}^{3} d_i E_i,$$

where $d_1$, $d_2$ and $d_3$ are the eigenvalues of $D$ and $E_1$, $E_2$ and $E_3$ are the corresponding projection matrices—in particular each $E_i$ is symmetric with an eigenvalue 1 and double eigenvalue 0. Since $\hat{\sigma}$ is linear we have

$$\begin{aligned}\hat{\sigma}(D) &= \sum_{i=1}^{3} d_i\, \hat{\sigma}(E_i) \\ &= \sum_{i=1}^{3} d_i\, \big(\lambda\, I + 2\mu\, E_i\big).\end{aligned}$$

where for each $i = 1, 2, 3$ the only non-zero isotropic invariant is $I_1(E_i) = 1$ so that $\lambda$ and $\mu$ are simply constant scalars. Using that $E_1 + E_2 + E_3 = I$ we have

$$\hat{\sigma} = \lambda(d_1 + d_2 + d_3)I + 2\mu D.$$

Recall that $d_1 + d_2 + d_3 = \nabla \cdot \boldsymbol{u}$. Thus we have

$$\hat{\sigma} = \lambda(\nabla \cdot \boldsymbol{u})I + 2\mu D.$$

If we set $\zeta = \lambda + \frac{2}{3}\mu$ this last relation becomes

$$\hat{\sigma} = 2\mu\big(D - \tfrac{1}{3}(\nabla \cdot \boldsymbol{u})I\big) + \zeta(\nabla \cdot \boldsymbol{u})I,$$

where $\mu$ and $\zeta$ are the first and second *coefficients of viscosity*, respectively.

*Remark 9* Note that if $\nabla \cdot \boldsymbol{u} = 0$, then the linear relation between $\hat{\sigma}$ and $D$ is homogeneous, and we have the key property of what is known as a *Newtonian fluid*: the stress is proportional to the rate of strain.

## 15 Navier–Stokes equations

Consider again an arbitrary imaginary subregion $\Omega$ of $\mathcal{D}$ identified at time $t = 0$, as in Fig. 1. As in our derivation of the Euler equations, let $\Omega_t$ denote the volume of the fluid occupied by the particles at $t > 0$ that originally made up $\Omega$. The total force exerted on the fluid inside $\Omega_t$ through the stresses exerted across its boundary $\partial\Omega_t$ is given by

$$\int_{\partial\Omega_t} (-pI + \hat{\sigma})\, \boldsymbol{n}\, \mathrm{d}S \equiv \int_{\Omega_t} (-\nabla p + \nabla \cdot \hat{\sigma})\, \mathrm{d}V,$$

where (for convenience here set $(x_1, x_2, x_3)^{\mathrm{T}} \equiv (x, y, z)^{\mathrm{T}}$ and $(u_1, u_2, u_3)^{\mathrm{T}} \equiv (u, v, w)^{\mathrm{T}}$)

$$\begin{aligned}[\nabla \cdot \hat{\sigma}]_i &= \sum_{j=1}^{3} \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \\ &= \lambda[\nabla(\nabla \cdot \boldsymbol{u})]_i + 2\mu \sum_{j=1}^{3} \frac{\partial D_{ij}}{\partial x_j} \\ &= \lambda[\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= \lambda[\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\ &= (\lambda + \mu)[\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \nabla^2 u_i.\end{aligned}$$

If $\boldsymbol{f}$ is a body force (external force) per unit mass, which can depend on position and time, then on any parcel of fluid $\Omega_t$, the total force acting on it is

$$\int_{\Omega_t} -\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}\, \mathrm{d}V.$$

Hence using Newton's 2nd law we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho\, \boldsymbol{u}\, \mathrm{d}V = \int_{\Omega_t} -\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}\, \mathrm{d}V.$$

Using the Transport Theorem with $F \equiv \boldsymbol{u}$ and that $\Omega$ and thus $\Omega_t$ are arbitrary, we see for each $\boldsymbol{x} \in \mathcal{D}$ and $t \geqslant 0$, we can deduce the following relation known as Cauchy's equation of motion:

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}.$$

Combining this with the form for $\nabla \cdot \hat{\sigma}$ we deduced above, we arrive at

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \Delta \boldsymbol{u} + \rho\, \boldsymbol{f},$$

where $\Delta = \nabla^2$ is the Laplacian operator. These are the *Navier–Stokes equations*. If we assume we are in three dimensional space so $d = 3$, then together with the continuity equation we have four equations, but five unknowns—namely $\boldsymbol{u}$, $p$ and $\rho$. Thus for a *compressible* fluid flow, we cannot specify the fluid motion completely without specifying one more condition/relation. (We could use the principle of conservation of energy to establish as additional relation known as the *equation of state*; in simple scenarios this takes the form of relationship between the pressure $p$ and density $\rho$ of the fluid.)

For an *incompressible homogeneous* flow for which the density $\rho$ is constant, we get a complete set of equations known as the *Navier–Stokes equations for an incompressible flow*:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu\, \Delta \boldsymbol{u} - \frac{1}{\rho} \nabla p + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\nu = \mu/\rho$ is the coefficient of *kinematic viscosity*. Note that we have a closed system of equations: we have four equations in four unknowns, $\boldsymbol{u}$ and $p$.

*Remark 10* Often the factor $1/\rho$ is scaled into the pressure and thus explicitly omitted: since $\rho$ is constant $(\nabla p)/\rho \equiv \nabla(p/\rho)$, and we re-label the term $p/\rho$ to be $p$.

As for the Euler equations of motion for an ideal fluid, we need to specify *initial and boundary conditions*. For viscous flow we specify an additional boundary condition to that we specified for the Euler equations. This is due to the inclusion of the extra term $\nu \Delta \boldsymbol{u}$ which increases the number of spatial derivatives in the governing evolution equations from one to two. Mathematically, we specify that

$$\boldsymbol{u} = \boldsymbol{0}$$

everywhere on the rigid boundary, i.e. in addition to the condition that there must be no net normal flow at the boundary, we also specify there is no tangential flow there. The fluid velocity is simply zero at a rigid boundary; it is sometimes called *no-slip* boundary conditions. Experimentally this is observed as well, to a very high degree of precision; see Chorin and Marsden [3, p. 34]. (Dye can be introduced into a flow near a boundary and how the flow behaves near it observed and measured very accurately.) Further, recall that in a viscous fluid flow we are incorporating the effect of molecular diffusion between neighbouring fluid parcels—see Fig. 3. The rigid non-moving boundary should impart a zero tangential flow condition to the fluid particles right up against it. The no-slip boundary condition crucially represents the mechanism for vorticity production in nature that can be observed everywhere. Just look at the flow of a river close to the river bank.

*Remark 11* At a material boundary (or free surface) between two immiscible fluids, we would specify that there is no jump in the velocity across the surface boundary. This is true if there is no surface tension or at least if it is negligible—for example at the seawater-air boundary of the ocean. However at the surface of melting wax at the top of a candle, there is surface tension, and there is a jump in the stress $\sigma\,\boldsymbol{n}$ at the boundary surface. Surface tension is also responsible for the phenomenon of being able to float a needle on the surface of a bowl of water as well as many other interesting effects such as the shape of water drops.

## 16 Evolution of vorticity

Recall from our discussion in Section 8, that the *vorticity field* of a flow with velocity field $\boldsymbol{u}$ is defined as

$$\boldsymbol{\omega} := \nabla \times \boldsymbol{u}.$$

It encodes the magnitude of, and direction of the axis about which, the fluid rotates, locally. Note that $\nabla \times \boldsymbol{u}$ can be computed as follows

$$\nabla \times \boldsymbol{u} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{pmatrix} = \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix}.$$

Using the Navier–Stokes equations for a homogeneous incompressible fluid, we can in fact derive a closed system of equations governing the evolution of vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ as follows. Using the identity $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ we see that we can equivalently represent the Navier–Stokes equations in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \times \boldsymbol{\omega} = \nu\,\Delta \boldsymbol{u} - \nabla\Big(\frac{p}{\rho}\Big) + \boldsymbol{f}.$$

If we take the curl of this equation and use the identity

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u}\,(\nabla \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}\,(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega},$$

noting that $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0$, we find that we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\,\Delta \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nabla \times \boldsymbol{f}.$$

Note that we can recover the velocity field $\boldsymbol{u}$ from the vorticity $\boldsymbol{\omega}$ by using the identity $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}$. This implies

$$\Delta \boldsymbol{u} = -\nabla \times \boldsymbol{\omega},$$

and closes the system of partial differential equations for $\boldsymbol{\omega}$ and $\boldsymbol{u}$. However, we can also simply observe that

$$\boldsymbol{u} = \big(-\Delta\big)^{-1}(\nabla \times \boldsymbol{\omega}).$$

If the body force is conservative so that $\boldsymbol{f} = \nabla \Phi$ for some potential $\Phi$, then $\nabla \times \boldsymbol{f} \equiv \boldsymbol{0}$.

*Remark 12* We can replace the 'vortex stretching' term $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$ in the evolution equation for the vorticity by $D\boldsymbol{\omega}$, where $D$ is the $3 \times 3$ deformation matrix, since

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = (\nabla \boldsymbol{u})\boldsymbol{\omega} = D\boldsymbol{\omega} + R\boldsymbol{\omega} = D\boldsymbol{\omega},$$

as direct computation reveals that $R\boldsymbol{\omega} \equiv \boldsymbol{0}$.

## 17 Simple example flows

We roughly follow an illustrative sequence of examples given in Majda and Bertozzi [13, pp. 8–15]. The first few are example flows of a class of exact solutions to both the Euler and Navier–Stokes equations.

**Lemma 1 (Majda and Bertozzi, p. 8)** *Let $D = D(t)$ be a real symmetric $3 \times 3$ matrix such that $\mathrm{Tr}(D) = 0$ (representing the deformation matrix). Suppose that the vorticity $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ solves the ordinary differential system*

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = D(t)\,\boldsymbol{\omega}$$

*for some initial data $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$. If the three components of vorticity are thus $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$, set*

$$R := \tfrac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

*Then we have that*

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \tfrac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x} + D(t)\,\boldsymbol{x}, \\ p(\boldsymbol{x}, t) &= -\tfrac{1}{2}\left(\frac{\mathrm{d}D}{\mathrm{d}t} + D^2(t) + R^2(t)\right)\boldsymbol{x}\cdot\boldsymbol{x}, \end{aligned}$$

*are exact solutions to the incompressible Euler and Navier–Stokes equations.*

*Remark 13* Since the pressure is a quadratic function of the spatial coordinates $\boldsymbol{x}$, these solutions only have meaningful interpretations locally. Note the pressure field here has been rescaled by the constant mass density $\rho$—see Remark 10. Further note that the velocity solution field $\boldsymbol{u}$ only depends linearly on the spatial coordinates $\boldsymbol{x}$; this explains why once we established these are exact solutions of the Euler equations, they are also solutions of the Navier–Stokes equations.

*Proof* Recall that $\nabla \boldsymbol{u}$ is the rate of strain tensor. It can be decomposed into a direct sum of its symmetric and skew-symmetric parts which are the $3 \times 3$ matrices

$$\begin{aligned} D &:= \tfrac{1}{2}\big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}}\big), \\ R &:= \tfrac{1}{2}\big((\nabla \boldsymbol{u}) - (\nabla \boldsymbol{u})^{\mathrm{T}}\big). \end{aligned}$$

We can determine how $\nabla \boldsymbol{u}$ evolves by taking the gradient of the homogeneous (no body force) Navier–Stokes equations so that

$$\frac{\partial}{\partial t}(\nabla \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla(\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^2 = \nu\, \Delta(\nabla \boldsymbol{u}) - \nabla\nabla p.$$

Note here $(\nabla \boldsymbol{u})^2 = (\nabla \boldsymbol{u})(\nabla \boldsymbol{u})$ is simply matrix multiplication. By direct computation

$$(\nabla \boldsymbol{u})^2 = (D + R)^2 = (D^2 + R^2) + (DR + RD),$$

where the first term on the right is symmetric and the second is skew-symmetric. Hence we can decompose the evolution of $\nabla \boldsymbol{u}$ into the coupled evolution of its symmetric and skew-symmetric parts

$$\begin{aligned} \frac{\partial D}{\partial t} + \boldsymbol{u} \cdot \nabla D + D^2 + R^2 &= \nu\, \Delta D - \nabla\nabla p, \\ \frac{\partial R}{\partial t} + \boldsymbol{u} \cdot \nabla R + DR + RD &= \nu\, \Delta R. \end{aligned}$$

Directly computing the evolution for the three components of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$ from the second system of equations we would arrive at the following equation for vorticity,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega},$$

which we derived more directly in Section 16.

Thus far we have not utilized the ansatz (form) for the velocity or pressure we assume in the statement of the theorem. Assuming $\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x} + D(t)\, \boldsymbol{x}$, for a given deformation matrix $D = D(t)$, then $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}(t)$, independent of $\boldsymbol{x}$, and substituting this into the evolution equation for $\boldsymbol{\omega}$ above we obtain the following system of ordinary differential equations governing the evolution of $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$:

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = D(t)\boldsymbol{\omega}.$$

Now the symmetric part governing the evolution of $D = D(t)$, which is independent of $\boldsymbol{x}$, reduces to the system of differential equations

$$\frac{\mathrm{d}D}{\mathrm{d}t} + D^2 + R^2 = -\nabla\nabla p.$$

Note that $R = R(t)$ only as well, since $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$, and thus $\nabla\nabla p$ must be a function of $t$ only. Hence $p = p(\boldsymbol{x}, t)$ can only quadratically depend on $\boldsymbol{x}$. Indeed after integrating we must have $p(\boldsymbol{x}, t) = -\frac{1}{2}(\mathrm{d}D/\mathrm{d}t + D^2 + R^2)\, \boldsymbol{x} \cdot \boldsymbol{x}$. □

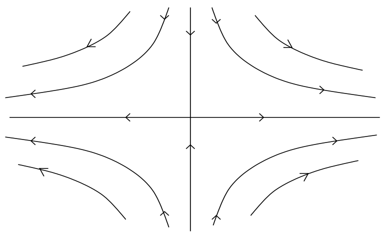

**Fig. 14** Strain flow example.

**Example (jet flow)** Suppose the initial vorticity $\boldsymbol{\omega}_0 = \mathbf{0}$ and $D = \mathrm{diag}\{d_1, d_2, d_3\}$ is a constant diagonal matrix where $d_1 + d_2 + d_3 = 0$ so that $\mathrm{Tr}(D) = 0$. Then from Lemma 1, we see that the flow is irrotational, i.e. $\boldsymbol{\omega}(t) = \mathbf{0}$ for all $t \geqslant 0$. Hence the velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = D(t)\boldsymbol{x} = \begin{pmatrix} d_1\, x \\ d_2\, y \\ d_3\, z \end{pmatrix}.$$

The particle path for a particle at $(x_0, y_0, z_0)^{\mathrm{T}}$ at time $t = 0$ is given by: $x(t) = \mathrm{e}^{d_1 t} x_0$, $y(t) = \mathrm{e}^{d_2 t} y_0$ and $z(t) = \mathrm{e}^{d_3 t} z_0$. If $d_1 < 0$ and $d_2 < 0$, then $d_3 > 0$ and we see the flow resembles two jets streaming in opposite directions away from the $z = 0$ plane.

**Example (strain flow)** Suppose the initial vorticity $\boldsymbol{\omega}_0 = \mathbf{0}$ and $D = \mathrm{diag}\{d_1, d_2, 0\}$ is a constant diagonal matrix such that $d_1 + d_2 = 0$. Then as in the last example, the flow is irrotational with $\boldsymbol{\omega}(t) = 0$ for all $t \geqslant 0$ and

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} d_1\, x \\ d_2\, y \\ 0 \end{pmatrix}.$$

The particle path for a particle at $(x_0, y_0, z_0)^{\mathrm{T}}$ at time $t = 0$ is given by: $x(t) = \mathrm{e}^{d_1 t} x_0$, $y(t) = \mathrm{e}^{d_2 t} y_0$ and $z(t) = z_0$. Since $d_2 = -d_1$, the flow forms a strain flow as shown in Fig. 14—neighbouring particles are pushed together in one direction while being pulled apart in the other orthogonal direction.

**Example (vortex)** Suppose the initial vorticity $\boldsymbol{\omega}_0 = (0, 0, \omega_0)^{\mathrm{T}}$ and $D = O$. Then from Lemma 1 the velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x} = \begin{pmatrix} -\frac{1}{2}\omega_0 y \\ \frac{1}{2}\omega_0 x \\ 0 \end{pmatrix}.$$

The particle path for a particle at $(x_0, y_0, z_0)^{\mathrm{T}}$ at time $t = 0$ is given by: $x(t) = \cos(\frac{1}{2}\omega_0 t)x_0 - \sin(\frac{1}{2}\omega_0 t)y_0$, $y(t) = \sin(\frac{1}{2}\omega_0 t)x_0 + \cos(\frac{1}{2}\omega_0 t)y_0$ and $z(t) = z_0$. These are circular trajectories, and indeed the flow resembles a solid body rotation; see Fig. 15.

**Example (jet flow with swirl)** Now suppose the initial vorticity $\boldsymbol{\omega}_0 = (0, 0, \omega_0)^{\mathrm{T}}$ and $D = \mathrm{diag}\{d_1, d_2, d_3\}$ is a constant diagonal matrix where $d_1 + d_2 + d_3 = 0$. Then from

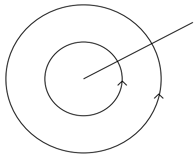

**Fig. 15** When a fluid flow is a rigid body rotation, the fluid particles flow on circular streamlines. The fluid particles on paths further from the origin or axis of rotation, circulate faster at just the right speed that they remain alongside their neighbours on the paths just inside them.

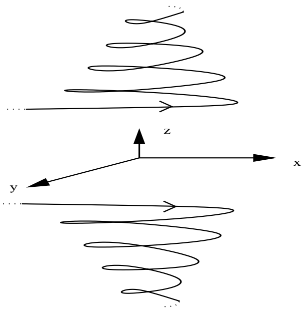


**Fig. 16** Jet flow with swirl example. Fluid particles rotate around and move closer to the $z$-axis whilst moving further from the $z = 0$ plane.

Lemma 1, we see that the only non-zero component of vorticity is the third component, say $\omega = \omega(t)$, where

$$\omega(t) = \omega_0 \mathrm{e}^{d_3 t}.$$

The velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} d_1\, x - \frac{1}{2}\omega(t) y \\ d_2\, y + \frac{1}{2}\omega(t) x \\ d_3\, z \end{pmatrix}.$$

The particle path for a particle at $(x_0, y_0, z_0)^{\mathrm{T}}$ at $t = 0$ can be described as follows. We see that $z(t) = z_0 \mathrm{e}^{d_3 t}$ while $x = x(t)$ and $y = y(t)$ satisfy the coupled system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 & -\frac{1}{2}\omega(t) \\ \frac{1}{2}\omega(t) & d_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we assume $d_1 < 0$ and $d_2 < 0$ then the particles spiral around the $z$-axis with decreasing radius and increasing angular velocity $\frac{1}{2}\omega(t)$. The flow thus resembles a rotating jet flow; see Fig. 16.

We now derive a simple class of solutions that retain the three underlying mechanisms of Navier–Stokes flows: convection, vortex stretching and diffusion.

**Example (shear-layer flows)** Recall the vorticity $\boldsymbol{\omega}$ evolves according to the partial differential system

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D \boldsymbol{\omega},$$

with $\Delta \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}$. The material derivative term $\partial \boldsymbol{\omega}/\partial t + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega}$ convects vorticity along particle paths, while the term $\nu\, \Delta \boldsymbol{\omega}$ is responsible for the diffusion of vorticity and $D\boldsymbol{u}$ represents vortex stretching—the vorticity $\boldsymbol{\omega}$ increases/decreases when aligns along eigenvectors of $D$ corresponding to positive/negative eigenvalues of $D$.

We seek an exact solution to the incompressible Navier–Stokes equations of the following form (the first two velocity components represent a strain flow)

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} -\gamma x \\ \gamma y \\ w(x,t) \end{pmatrix}$$

where $\gamma$ is a constant, with $p(\boldsymbol{x}, t) = -\frac{1}{2}\gamma^2\big(x^2 + y^2\big)$. This represents a solution to the Navier–Stokes equations if we can determine the solution $w = w(x,t)$ to the linear diffusion equation

$$\frac{\partial w}{\partial t} - \gamma x \frac{\partial w}{\partial x} = \nu \frac{\partial^2 w}{\partial x^2},$$

with $w(x,0) = w_0(x)$. Computing the vorticity directly we get

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \begin{pmatrix} 0 \\ -\big(\partial w/\partial x\big)(x,t) \\ 0 \end{pmatrix}.$$

If we differentiate the equation above for the velocity field component $w$ with respect to $x$, then if $\omega := -\partial w/\partial x$, we get

$$\frac{\partial \omega}{\partial t} - \gamma x \frac{\partial \omega}{\partial x} = \gamma \omega + \nu \frac{\partial^2 \omega}{\partial x^2},$$

with $\omega(x,0) = \omega_0(x) = -(\partial w_0/\partial x)(x)$. For this simpler flow we can see simpler signatures of the three effects we want to isolate: there is the convecting velocity $-\gamma x$; vortex stretching from the term $\gamma\omega$ and diffusion in the term $\nu \partial^2 \omega/\partial x^2$. Note that as in the general case, the velocity field $w$ can be recovered from the vorticity field $\omega$ by

$$w(x,t) = -\int_{-\infty}^{x} \omega(\xi, t)\, \mathrm{d}\xi.$$

Let us consider a special case: the viscous shear-layer solution where $\gamma = 0$. In this case we see that the partial differential equation above for $\omega$ reduces to the simple heat equation with solution

$$\omega(x,t) = \int_{\mathbb{R}} G(x - \xi, \nu t)\, \omega_0(\xi)\, \mathrm{d}\xi,$$

where $G$ is the Gaussian heat kernel

$$G(\xi, t) := \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\xi^2/4t}.$$

Indeed the velocity field $w$ is given by

$$w(x,t) = \int_{\mathbb{R}} G(x - \xi, \nu t)\, w_0(\xi)\, \mathrm{d}\xi,$$

so that both the vorticity $\omega$ and velocity $w$ fields diffuse as time evolves; see Fig. 17.

It is possible to write down the exact solution for the general case in terms of the Gaussian heat kernel, indeed, a very nice exposition can be found in Majda and Bertozzi [13, p. 18].

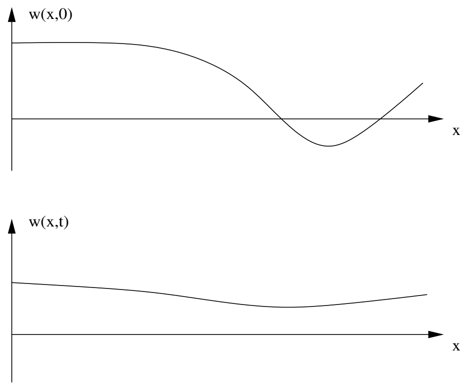


**Fig. 17** Viscous shear flow example. The effect of diffusion on the velocity field $w = w(x,t)$ is to smooth out variations in the field as time progresses.

## 18 Dynamical similarity and Reynolds number

Our goal in this section is to demonstrate an important scaling property of the Navier–Stokes equations for a homogeneous incompressible fluid without body force:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu\, \Delta \boldsymbol{u} - \frac{1}{\rho} \nabla p,$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

Note that two physical properties inherent to the fluid modelled are immediately apparent, the mass density $\rho$, which is constant throughout the flow, and the kinematic viscosity $\nu$. Suppose we consider such a flow which is characterized by a typical length scale $L$ and velocity $U$. For example we might imagine a flow past an obstacle such a sphere whose diameter is characterized by $L$ and the impinging/undisturbed far-field flow is uniform and given by $U$. These two scales naturally determine a typically time scale $T = L/U$. Using these scales we can introduce the dimensionless variables

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{L}, \qquad \boldsymbol{u}' = \frac{\boldsymbol{u}}{U} \qquad \text{and} \qquad t' = \frac{t}{T}.$$

Directly substituting for $\boldsymbol{u} = U\boldsymbol{u}'$ and using the chain rule to replace $t$ by $t'$ and $\boldsymbol{x}$ by $\boldsymbol{x}'$ in the Navier–Stokes equations, we obtain:

$$\frac{U}{T}\frac{\partial \boldsymbol{u}'}{\partial t'} + \frac{U^2}{L}\boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'} \boldsymbol{u}' = \frac{\nu U}{L^2}\, \Delta_{\boldsymbol{x}'} \boldsymbol{u}' - \frac{1}{\rho L} \nabla_{\boldsymbol{x}'} p.$$

The incompressibility condition becomes $\nabla_{\boldsymbol{x}'} \cdot \boldsymbol{u}' = 0$. Using that $T = L/U$ and dividing through by $U^2/L$ we get

$$\frac{\partial \boldsymbol{u}'}{\partial t'} + \boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'} \boldsymbol{u}' = \frac{\nu}{UL}\, \Delta_{\boldsymbol{x}'} \boldsymbol{u}' - \frac{1}{\rho U^2} \nabla_{\boldsymbol{x}'} p.$$

If we set $p' = p/\rho U^2$ and then drop the primes, we get

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{\mathrm{Re}}\, \Delta \boldsymbol{u} - \nabla p,$$

which is the representation for the Navier–Stokes equations in dimensionless variables. The dimensionless number

$$\mathrm{Re} := \frac{UL}{\nu}$$

is the *Reynolds number*. Its practical significance is as follows. Suppose we want to design a jet plane (or perhaps just a wing). It might have a characteristic scale $L_1$ and typically cruise at speeds $U_1$ with surrounding air having viscosity $\nu_1$. Rather than build the plane to test its airflow properties it would be cheaper to build a scale model of the aircraft—with exactly the same shape/geometry but smaller, with characteristic scale $L_2$. Then we could test the airflow properties in a wind tunnel for example, by using a driving impinging wind of characteristic velocity $U_2$ and air of viscosity $\nu_2$ so that

$$\frac{U_1 L_1}{\nu_1} = \frac{U_2 L_2}{\nu_2}.$$

The Reynolds number in the two scenarios are the same and the dimensionless Navier–Stokes equations for the two flows identical. Hence the shape of the flows in the two scenarios will be the same. We could also for example, replace the wind tunnel by a water tunnel: the viscosity of air is $\nu_1 = 0.15\,\mathrm{cm}^2/\mathrm{s}$ and of water $\nu_2 = 0.0114\,\mathrm{cm}^2/\mathrm{s}$, i.e. $\nu_1/\nu_2 \approx 13$. Hence for the same geometry and characteristic scale $L_1 = L_2$, if we choose $U_1 = 13\,U_2$, the Reynolds numbers for the two flows will be the same. Such flows, with the same geometry and the same Reynolds number are said to be *similar*.

*Remark 14* Some typical Reynolds are as follows: aircraft: $10^8$ to $10^9$; cricket ball: $10^5$; blue whale: $10^8$; cruise ship: $10^9$; canine artery: $10^3$; nematode: 0.6; capillaries: $10^{-3}$.

## 19 Stokes flow

Consider the individual terms in the incompressible three-dimensional Navier–Stokes equations with no body force (in dimensionless form):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{\boldsymbol{u}\cdot\nabla\boldsymbol{u}}_{\substack{\text{inertia or}\\ \text{convective terms}}} = \underbrace{\mathrm{Re}^{-1}\Delta\boldsymbol{u}}_{\substack{\text{diffusion or}\\ \text{dissipation term}}} - \nabla p,$$

where Re is the Reynolds number. We wish to consider the small Reynolds number limit $\mathrm{Re} \to 0$. Naively this means that the diffusion/dissipation term will be the dominant term in the equations above. However we also want to maintain incompressibility, i.e. $\nabla \cdot \boldsymbol{u} = 0$. Since the pressure field is the Lagrange multiplier term that maintains the incompressibility constraint, we should attempt to maintain it in the limit $\mathrm{Re} \to 0$. Hence we suppose

$$p = \mathrm{Re}^{-1} q,$$

for a scaled pressure $q$. Further the flow may evolve on a slow timescale so that

$$t = \mathrm{Re}\,\tau.$$

Making these changes of variables in the Navier–Stokes equations and taking the limit $\mathrm{Re} \to 0$, we obtain

$$\begin{aligned} & \frac{\partial \boldsymbol{u}}{\partial \tau} = \Delta \boldsymbol{u} - \nabla q, \\ \Leftrightarrow \qquad & \frac{\partial \boldsymbol{u}}{\partial t} = \mathrm{Re}^{-1} \Delta \boldsymbol{u} - \nabla p. \end{aligned}$$

In dimensional variables we have thus derived the equations for *Stokes flow*:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \mu \Delta \boldsymbol{u} - \nabla p.$$

More commonly, the stationary version of these equations are denoted *Stokes flow*. Some immediate consequences are useful. If we respectively take the curl and divergence of the Stokes equations we get

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} &= \mu \Delta \boldsymbol{\omega}, \\ \Delta p &= 0. \end{aligned}$$

Suppose we know a *stream function* exists for the flow under consideration—so the flow is incompressible and an additional symmetry allows us to eliminate one spatial coordinate and one velocity component. For example suppose we have a stationary two-dimensional flow $\boldsymbol{u} = (u, v)^{\mathrm{T}}$ in cartesian coordinates $\boldsymbol{x} = (x, y)^{\mathrm{T}}$. Then there exists a stream function $\psi = \psi(x, y)$ given by

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}.$$

Hence the vorticity of such a flow is given by $\boldsymbol{\omega} = (0, 0, -\Delta \psi)^{\mathrm{T}}$. Since for a stationary Stokes flow $\Delta \boldsymbol{\omega} = 0$, we must have

$$\Delta(\Delta \psi) = 0.$$

In other words the stream function satisfies the *biharmonic equation*.

In cylindrical polar coordinates assuming a stationary axisymmetric flow with no swirl, i.e. no $\theta$ dependence and $u_\theta = 0$, the stream function $\psi = \psi(r, z)$ is given by

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial z} \qquad \text{and} \qquad u_z = -\frac{1}{r} \frac{\partial \psi}{\partial r}.$$

(We could use $\psi \to -r\psi$.) Direct computation reveals that the vorticity is given by $\boldsymbol{\omega} = (0, \frac{1}{r} \mathrm{D}^2 \psi, 0)^{\mathrm{T}}$, where $\mathrm{D}^2$ is the second order partial differential operator defined by

$$\mathrm{D}^2 := r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}.$$

Again, for a stationary Stokes flow $\Delta \boldsymbol{\omega} = 0$. We use that for a divergence-free vector field $\boldsymbol{\omega}$ we have $\Delta \boldsymbol{\omega} = -\nabla \times \nabla \times \boldsymbol{\omega}$ to show that the stream function satisfies

$$\mathrm{D}^2(\mathrm{D}^2 \psi) = 0.$$

In spherical polar coordinates assuming a stationary axisymmetric flow with no swirl, i.e. no $\varphi$ dependence and $u_\varphi = 0$, the stream function $\psi = \psi(r, \theta)$ is given by

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}.$$

Direct computation implies that the vorticity is $\boldsymbol{\omega} = (0, 0, -\frac{1}{r\sin\theta}\mathrm{D}^2\psi)^{\mathrm{T}}$, where $\mathrm{D}^2$ is now the second order partial differential operator given by

$$\mathrm{D}^2 := \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big).$$

For a stationary Stokes flow $\Delta\boldsymbol{\omega} = 0$. We again use that $\Delta\boldsymbol{\omega} = -\nabla\times\nabla\times\boldsymbol{\omega}$ to show that the only non-zero component of $\Delta\boldsymbol{\omega}$ is its third component given by $-\frac{1}{r\sin\theta}\mathrm{D}^2(\mathrm{D}^2\psi)$, where $\mathrm{D}^2$ in both cases is the operator just quoted. Hence the stream function satisfies

$$\mathrm{D}^2(\mathrm{D}^2\psi) = 0.$$

**Example (Viscous drag on sphere)** Consider a uniform incompressible *viscous* flow of velocity $U$, into which we place a spherical obstacle, radius $a$. The physical set up is similar to that shown in Fig. 11. We assume that the flow around the sphere is steady. Use spherical polar coordinates $(r, \theta, \varphi)$ to represent the flow with the south-north pole axis passing through the centre of the sphere and aligned with the uniform flow $U$ at infinity. We assume that the flow is axisymmetric, i.e. independent of the azimuthal angle $\varphi$, and there is no swirl so that $u_\varphi = 0$. Further, take the flow around the sphere to be a Stokes flow, i.e. we have

$$\nabla p = \mu\Delta\boldsymbol{u},$$

where $p = p(r, \theta)$ is the pressure field, $\boldsymbol{u}$ the velocity field and $\mu$ is the viscosity (a constant parameter). Also assume the stream function is given by $\psi = \psi(r, \theta)$. For such a flow, as we have seen, the stream function satisfies

$$\mathrm{D}^2(\mathrm{D}^2\,\psi) = 0,$$

where $\mathrm{D}^2$ is the second order partial differential operator

$$\mathrm{D}^2 := \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

*Step 1: Determine the boundary conditions.* The stream function $\psi$ for the flow prescribed is given by

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

First let us consider the boundary conditions as $r \to \infty$. Decomposing the far-field axial directed velocity field of speed $U$ into components along $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ we get $U\cos\theta$ and $-U\sin\theta$, respectively. Hence in the far-field limit we have

$$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \sim U\cos\theta \qquad \text{and} \qquad \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \sim U\sin\theta.$$

The solution to this pair of first order partial differential equations is

$$\psi \sim \tfrac{1}{2}Ur^2\sin^2\theta,$$

generating the far-field boundary condition in terms of $\psi$.

Second consider the boundary conditions on the surface of the sphere. The no-slip condition on $r = a \implies$

$$\frac{1}{a^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = 0 \qquad \text{and} \qquad \frac{1}{a\sin\theta}\frac{\partial\psi}{\partial r} = 0.$$

Hence $\psi$ is independent of $r$ and $\theta$ along the boundary $r = a$; we can therefore take $\psi = 0$ and $\partial\psi/\partial r = 0$ to be the boundary conditions on $r = a$. Thus, to summarize, the boundary conditions for this problem are

$$\psi \to \tfrac{1}{2}Ur^2\sin^2\theta \;\text{ as }\; r\to\infty \qquad \text{and} \qquad \psi = \frac{\partial\psi}{\partial r} = 0 \;\text{ on }\; r = a.$$

*Step 2: Solve the modified biharmonic equation.* Motivated by the boundary conditions, we look for a solution to the modified biharmonic equation $\mathrm{D}^2(\mathrm{D}^2\,\psi) = 0$ of the form $\psi = Uf(r)\sin^2\theta$. First computing $\mathrm{D}^2\,\psi$ gives

$$\mathrm{D}^2\,\psi = U\Big(f''(r) - \frac{2}{r^2}f(r)\Big)\sin^2\theta,$$

in which we set

$$F(r) := f''(r) - \frac{2}{r^2}f(r).$$

Now compute $\mathrm{D}^2(\mathrm{D}^2\,\psi)$ which gives

$$\begin{aligned}\mathrm{D}^2(\mathrm{D}^2\,\psi) &= \mathrm{D}^2(UF(r)\sin^2\theta)\\ &= U\Big(F''(r) - \frac{2}{r^2}F(r)\Big)\sin^2\theta.\end{aligned}$$

Hence $\mathrm{D}^2(\mathrm{D}^2\,\psi) = 0$ if and only if

$$F''(r) - \frac{2}{r^2}F(r) = 0.$$

This is a linear second order ordinary differential equation whose two independent solutions are $r^2$ and $1/r$; thus $F(r)$ is a linear combination of these two solutions. However we require $f(r)$ which satisfies

$$f''(r) - \frac{2}{r^2}f(r) = F(r).$$

This is a non-homogeneous linear second order ordinary differential equation, again whose two independent homogeneous solutions are $r^2$ and $1/r$; while the particular integral for the non-homogeneous component $F(r)$, which is a linear combination of $r^2$ and $1/r$, has the form $Ar^4 + Cr$ for some constants $A$ and $C$. Hence $f = f(r)$ necessarily has the form

$$f(r) = Ar^4 + Br^2 + Cr + \frac{D}{r},$$

where $B$ and $D$ are two further constants.

*Step 3: Substitute the boundary conditions.* First we see as $r\to\infty$ we must have

$$U\Big(Ar^4 + Br^2 + Cr + \frac{D}{r}\Big)\sin^2\theta = \tfrac{1}{2}Ur^2\sin^2\theta$$

so that $A = 0$ and $B = \frac{1}{2}$. Second we see that on $r = a$ we must have

$$U\Big(\frac{a^2}{2} + Ca + \frac{D}{a}\Big)\sin^2\theta = 0 \qquad \text{and} \qquad U\Big(a + C - \frac{D}{a^2}\Big)\sin^2\theta = 0.$$

Hence we get two simultaneous equations for $C$ and $D$, namely $a^2/2 + Ca + D/a = 0$ and $a + C - D/a^2 = 0$, whose solution is $C = -3a/4$ and $D = a^3/4$. We thus have

$$\psi = \tfrac{1}{4}Ua^2\left(\frac{2r^2}{a^2} - \frac{3r}{a} + \frac{a}{r}\right)\sin^2\theta.$$

*Step 4: Compute the pressure.* Using the stationary Stokes equations $\nabla p = \mu\varDelta\boldsymbol{u}$ we first compute $\boldsymbol{u}$, i.e. the components $u_r$ and $u_\theta$ from the stream function $\psi$. Using the relations given

$$u_r = \tfrac{1}{4}Ua^2\cdot 2\Big(\frac{2}{a^2} - \frac{3}{ar} + \frac{a}{r^3}\Big)\cos\theta \quad \text{and} \quad u_\theta = -\tfrac{1}{4}Ua^2\cdot\frac{1}{r}\Big(\frac{4r}{a^2} - \frac{3}{a} - \frac{a}{r^2}\Big)\sin\theta.$$

Then second, using the hint given, compute $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ and then $\nabla\times\boldsymbol{\omega}$. Note $\boldsymbol{\omega}$ has only one non-zero component, namely

$$\begin{aligned}\omega_\varphi &= \frac{1}{r}\frac{\partial}{\partial r}\big(ru_\theta\big) - \frac{1}{r}\frac{\partial}{\partial\theta}\big(u_r\big)\\ &= -\frac{3Ua}{2r^2}\sin\theta,\end{aligned}$$

with a lot of terms cancelling. Then we have

$$-\nabla\times\boldsymbol{\omega} = \frac{3Ua}{2r^3}\begin{pmatrix}2\cos\theta\\ \sin\theta\\ 0\end{pmatrix}.$$

Thus to find the pressure $p$ we must solve the pair of first order partial differential equations:

$$\begin{pmatrix}\frac{\partial p}{\partial r}\\ \frac{1}{r}\frac{\partial p}{\partial\theta}\end{pmatrix} = -\frac{3Ua\mu}{2r^3}\begin{pmatrix}2\cos\theta\\ \sin\theta\end{pmatrix}$$

These give (here $p_\infty$ is the constant ambient pressure as $r\to\infty$)

$$p = p_\infty - \frac{3Ua\mu}{2r^2}\cos\theta.$$

*Step 5: Compute the axial force.* For a small patch of area $\mathrm{d}S$ on the surface of the sphere the force is given by $\mathrm{d}\boldsymbol{F} = \sigma(\boldsymbol{x})\hat{\boldsymbol{n}}\,\mathrm{d}S$. Since the stress tensor $\sigma = -p\mathrm{I} + \hat{\sigma}$ where $p$ is the pressure and $\hat{\sigma}$ is the deviatoric stress tensor, the total force on the surface of the sphere $r = a$ is given by

$$\boldsymbol{F} = \int\!\!\int(-p\mathrm{I} + \hat{\sigma})\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

Now use that the deviatoric stress is a linear function of the deformation tensor $D$, indeed $\hat{\sigma} = 2\mu D$. Further the normal $\hat{\boldsymbol{n}}$ to the surface $r = a$ is simply $\hat{\boldsymbol{r}}$, the unit normal in the $r$ coordinate direction, and so

$$\begin{aligned} \hat{\sigma}\hat{\boldsymbol{n}} &= 2\mu D\hat{\boldsymbol{r}} \\ &= 2\mu \begin{pmatrix} D_{rr} & D_{r\theta} & D_{r\varphi} \\ D_{r\theta} & D_{\theta\theta} & D_{\theta\varphi} \\ D_{r\varphi} & D_{\theta\varphi} & D_{\varphi\varphi} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 2\mu \begin{pmatrix} D_{rr} \\ D_{r\theta} \\ 0 \end{pmatrix} \\ &= 2\mu\big(D_{rr}\hat{\boldsymbol{r}} + D_{r\theta}\hat{\boldsymbol{\theta}}\big), \end{aligned}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the $\theta$ coordinate direction. Now note that the area integral over the sphere surface $r = a$ can be split into a single integral of concentric rings of radius $a\sin\theta$ on the sphere surface of area '$2\pi a \cdot a\sin\theta\,\mathrm{d}\theta$'. Thus we get

$$\boldsymbol{F} = 2\pi a^2 \int_0^\pi \big(-p\hat{\boldsymbol{r}} + 2\mu D_{rr}\hat{\boldsymbol{r}} + 2\mu D_{r\theta}\hat{\boldsymbol{\theta}}\big)\sin\theta\,\mathrm{d}\theta,$$

The *axial* component of the force $\boldsymbol{F}_{\mathrm{ax}}$ (i.e. along the direction of the far-field flow), since the components of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ in the axial direction are given by $\hat{\boldsymbol{r}}\cos\theta$ and $-\hat{\boldsymbol{\theta}}\sin\theta$, respectively, is thus given by

$$\boldsymbol{F}_{\mathrm{ax}} = 2\pi a^2 \int_0^\pi \big((-p + 2\mu D_{rr})\cos\theta - 2\mu D_{r\theta}\sin\theta\big)\sin\theta\,\mathrm{d}\theta.$$

From the formulae sheet we find

$$D_{rr} = \frac{\partial u_r}{\partial r} \qquad \text{and} \qquad D_{r\theta} = \tfrac{1}{2}\Big(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\Big).$$

The no-slip boundary conditions on the sphere surface $r = a$ implies $u_r = u_\theta = 0$. Further from our expressions for $u_r$ and $u_\theta$ in part (d) above we see that $\partial u_r/\partial r = \partial u_r/\partial\theta = 0$ and $\partial u_\theta/\partial r = -(3U/2a)\sin\theta$ on $r = a$. Substituting these into the expressions for the total force above, as well as using the expression for the pressure $p$ from Step 4, we get

$$\begin{aligned} \boldsymbol{F}_{\mathrm{ax}} &= 2\pi a^2 \int_0^\pi \Big(\frac{3U\mu}{2a}\Big)\big(\cos^2\theta\sin\theta + \sin^3\theta\big)\,\mathrm{d}\theta \\ &= 3\pi U\mu a \int_0^\pi \sin\theta\,\mathrm{d}\theta \\ &= 6\pi U\mu a. \end{aligned}$$

Hence the *axial* component of the force which corresponds to the *drag* on the sphere is given by $6\pi\mu U a$.

**Example (Viscous corner flow)** Consider a steady incompressible *viscous* corner flow as shown in Fig. 18. The fluid is trapped between two plates, one is horizontal, while the other plate lies above the horizontal plate at an acute angle $\alpha$. The flat edge of the upper plate almost touches the horizontal plate; there is a small gap between the two. The horizontal plate moves with a speed $U$ to the left perpendicular to the imaginary line of intersection between the two plates; the upper plate remains fixed. We assume the trapped flow between the two plates to be a Stokes flow, i.e. we have

$$\nabla p = \mu \Delta \boldsymbol{u},$$

where $p$ is the pressure field, $\boldsymbol{u}$ the velocity field and $\mu$ is the viscosity. Further we assume the flow is uniform in the direction given by the imaginary line of intersection between the plates—denote this the $z$-axis. Hence in cylindrical polar coordinates there exists a stream function $\psi = \psi(r, \theta)$ such that

$$\boldsymbol{u} = \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi(r,\theta) \end{pmatrix}.$$

Taking the curl of the Stokes flow equation we see that

$$\begin{aligned} \mathbf{0} &= \nabla \times \big(\mu \Delta \boldsymbol{u}\big) \\ &= \mu \Delta \big(\nabla \times \boldsymbol{u}\big) \\ &= \mu \Delta \left( \nabla \times \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi(r,\theta) \end{pmatrix} \right) \\ &= \mu \Delta \, (-\Delta) \begin{pmatrix} 0 \\ 0 \\ \psi(r,\theta) \end{pmatrix}. \end{aligned}$$

Hence the the stream function $\psi$ for this Stokes flow satisfies the biharmonic equation

$$\Delta(\Delta \psi) = 0.$$

Next we determine the boundary conditions. Explicitly in plane polar coordinates (we henceforth drop the third $z$ coordinate with respect to which the corner flow is uniform), the relations between the velocity components $u_r$ and $u_\theta$ and the stream function $\psi$ are

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial \psi}{\partial r}.$$

First, using the no-slip boundary conditions on the plate along $\theta = 0$ which is moving towards the origin at speed $U$ we immediately see that

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U \quad \text{on} \quad \theta = 0.$$

Second, for the plate at the angle $\theta = \alpha$, the no-slip boundary conditions imply

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} = 0 \quad \text{on} \quad \theta = \alpha.$$

Given the form of the boundary conditions, we look for a solution to the biharmonic equation $\Delta(\Delta\psi) = 0$ of the form $\psi = Urf(\theta)$. Directly computing, we have

$$\begin{aligned} \Delta\big(Urf(\theta)\big) &= U\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big)\big(rf(\theta)\big) \\ &= \frac{U}{r}\big(f(\theta) + f''(\theta)\big). \end{aligned}$$

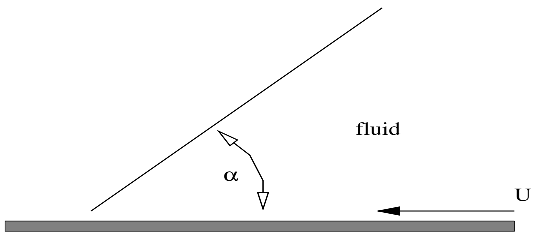


**Fig. 18** Corner flow: An incompressible viscous fluid is trapped between two plates, one is horizontal, while the other plate lies above at an acute angle $\alpha$. The flat edge of the upper plate almost touches the horizontal plate; there is a small gap between the two. The horizontal plate moves with a speed $U$ to the left perpendicular to the imaginary line of intersection between the two plates; the upper plate remains fixed.

Then we directly compute

$$\begin{aligned}\Delta\Big(\Delta\big(Urf(\theta)\big)\Big) &= \Delta\Big(\frac{U}{r}\big(f(\theta)+f''(\theta)\big)\Big)\\ &= U\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big(\frac{1}{r}\big(f(\theta)+f''(\theta)\big)\Big)\\ &= \frac{U}{r^3}\big(f(\theta)+2f''(\theta)+f''''(\theta)\big).\end{aligned}$$

Hence $\Delta(\Delta\psi)=0$ if and only if $f=f(\theta)$ satisfies

$$f''''+2f''+f=0.$$

This is a linear homogeneous constant coefficient fourth order ordinary differential equation. Looking for solutions of the form $f(\theta)=\exp(\lambda\theta)$ we obtain the polynomial auxiliary equation $(\lambda^2+1)^2=0$ whose roots are $\pm\mathrm{i}$ (each repeated). Hence the general solution has the form

$$f(\theta)=A\sin\theta+B\cos\theta+C\theta\sin\theta+D\theta\cos\theta,$$

for some constants $A$, $B$, $C$ and $D$.

Now for the boundary conditions, note that the form of the solution assumed implies $u_r=Uf'(\theta)$ and $u_\theta=-Uf(\theta)$. First consider the boundary conditions on $\theta=0$:

$$u_r=-U \quad\Leftrightarrow\quad f'(0)=-1 \qquad\text{and}\qquad u_\theta=0 \quad\Leftrightarrow\quad f(0)=0.$$

Second for the boundary conditions on $\theta=\alpha$:

$$u_r=0 \quad\Leftrightarrow\quad f'(\alpha)=0 \qquad\text{and}\qquad u_\theta=0 \quad\Leftrightarrow\quad f(\alpha)=0.$$

In other words $f=f(\theta)$ must satisfy

$$f(0)=f(\alpha)=f'(\alpha)=0 \qquad\text{and}\qquad f'(0)=-1.$$

We can use these four boundary conditions to determine the constants $A$, $B$, $C$ and $D$, indeed we get

$$f(\theta)=\frac{\theta\sin\alpha\sin(\alpha-\theta)-\alpha(\alpha-\theta)\sin\theta}{\alpha^2-\sin^2\alpha}.$$

Finally we should ask ourselves, what is the distance from the origin within which the solution we sought is consistent with our assumption of Stokes flow? Since the inertia terms $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ scale like $U^2/r$ while the viscous terms $\nu\Delta\boldsymbol{u}$ scale like $\nu U/r^2$. Our Stokes flow assumption was that

$$\frac{U^2/r}{\nu U/r^2} \ll 1 \qquad \Leftrightarrow \qquad \frac{Ur}{\nu} \ll 1.$$

Hence our solution is valid provided $r \ll \nu/U$.

## 20 Lubrication theory

In *lubrication theory* we consider the following scenario. Suppose an incompressible homogeneous viscous fluid occupies a *shallow layer* whose typical depth is $H$ and horizontal extent is $L$; see Fig. 19 for the set up. Let $x$ and $y$ denote horizontal Cartesian coordinates and $z$ is the vertical coordinate. Suppose $(u, v, w)$ are the fluid velocity components in the three coordinate directions $x$, $y$ and $z$, respectively. As usual, $p$ denotes the pressure, while $\rho$ denotes the constant density, of the fluid. We also assume a typical horizontal velocity scale for $(u, v)$ is $U$ and a typical vertical velocity scale for $w$ is $W$. As we have already hinted, we assume $H \ll L$. Our goal is to systematically derive a reduced set of equations from the incompressible Navier–Stokes equations that provide a very accurate approximation under the conditions stated. Note that the natural time scale for this problem is $T = \mathcal{O}(L/U)$.

First consider the continuity equation (incompressibility condition) which reveals the scaling

$$\underbrace{\frac{\partial u}{\partial x}}_{U/L} + \underbrace{\frac{\partial v}{\partial y}}_{U/L} + \underbrace{\frac{\partial w}{\partial z}}_{W/H} = 0.$$

To maintain incompressibility we deduce we must have $W = \mathcal{O}\big(U(H/L)\big)$. Since $H \ll L$ we conclude $W \ll U$.

Second consider the Navier–Stokes equation for the velocity components $u$ and $v$ and the corresponding scaling of each of the terms therein:

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{U^2/L} = \nu\Big(\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\nu U/L^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{\nu U/H^2}\Big) - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\underbrace{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{U^2/L} = \nu\Big(\underbrace{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}}_{\nu U/L^2} + \underbrace{\frac{\partial^2 v}{\partial z^2}}_{\nu U/H^2}\Big) - \frac{1}{\rho}\frac{\partial p}{\partial y}.$$

In the first equation, comparing the viscous terms $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ to $\partial^2 u/\partial z^2$ we see that the ratio of their scaling is

$$\frac{\nu U/L^2}{\nu U/H^2} = \frac{H^2}{L^2} \ll 1.$$

Thus we omit the $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ terms. Now compare the inertia terms (those on the left-hand side) to the viscosity term $\nu\partial^2 u\partial z^2$, the ratio of their scaling is

$$\frac{U^2/L}{\nu U/H^2} = \frac{UH^2}{\nu L} = \frac{UH}{\nu}\frac{H}{L} \ll 1,$$

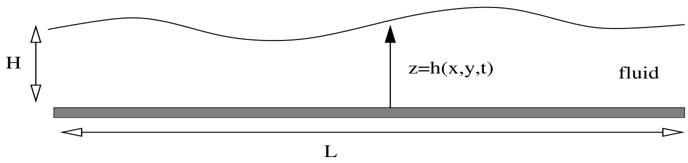


**Fig. 19** Lubrication theory: an incompressible homogeneous viscous fluid occupies a shallow layer whose typical depth is $H$ and horizontal extent is $L$. In the asymptotic limit $H \ll L$, the Navier–Stokes equations reduce to the shallow layer equations.

and hence we omit all the inertia terms—under the assumption that the modified Reynolds number $\mathrm{R_m} = UH/\nu$ is small or order one. Finally to keep the pressure term (which mediates the incompressibility condition) it must have the scaling

$$\partial_x P/\rho = \mathcal{O}\big(\nu U/H^2\big) \qquad \Longrightarrow \qquad P = \mathcal{O}\big(\mu UL/H^2\big).$$

An exactly analogous scaling argument applies to the second equation above and thus we see that the final equations for these two components are (with $\mu = \rho\nu$)

$$\begin{aligned}\frac{\partial p}{\partial x} &= \mu\frac{\partial^2 u}{\partial z^2},\\ \frac{\partial p}{\partial y} &= \mu\frac{\partial^2 v}{\partial z^2}.\end{aligned}$$

Third consider the Navier–Stokes equation for the velocity component $w$. The scaling of the individual terms reveals:

$$\underbrace{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{UW/L=U^2H/L^2} = \nu\Big(\underbrace{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}}_{\nu W/L^2=\nu UH/L^3} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{\nu W/H^2=\nu U/LH}\Big) - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\nu UL/H^3}.$$

The largest scaling term appears to be the pressure term so we compare the other terms to that. The ratio of the inertia terms to the pressure term in terms of their scaling is

$$\frac{U^2H/L^2}{\nu UL/H^3} = \frac{UH^4}{\nu L^3} = \frac{UH}{\nu}\Big(\frac{H}{L}\Big)^3 \ll 1,$$

again under the assumption the modified Reynolds number $\mathrm{R_m} = UH/\nu$ is small or order one. The ratio of the viscous terms $\nu\partial^2 w/\partial x^2$ and $\nu\partial^2 w/\partial y^2$ to the pressure term in terms of their scaling is

$$\frac{\nu UH/L^3}{\nu UL/H^3} = \Big(\frac{H}{L}\Big)^4 \ll 1.$$

Finally the ratio of the viscous term $\nu\partial^2 w/\partial z^2$ to the pressure term is

$$\frac{\nu U/LH}{\nu UL/H^3} = \Big(\frac{H}{L}\Big)^2 \ll 1.$$

Hence the third equation reduces to $\partial p/\partial z = 0$. The complete shallow layer equations are thus

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2},$$
$$\frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2},$$
$$\frac{\partial p}{\partial z} = 0,$$

together with the incompressibility condition. Note that from the third equation we deduce the pressure $p = p(x, y, t)$ only.

As shown in Fig. 19 suppose that the fluid occupies a region between a lower rigid plate at $z = 0$ and a top surface at $z = h(x, y, t)$. We assume here for the moment that the top surface is free, so perhaps it represents a fluid-air boundary, however we can easily specialize our subsequent analysis to a rigid lid upper boundary. Importantly though, we assume no-slip boundary conditions at $z = 0$ while we suppose the velocity components at the surface height $z = h(x, y, t)$ are $u(x, y, h, t) = U$ and $v(x, y, h, t) = V$ with $U$ and $V$ given. For this scenario, we can derive a closed form equation for how the surface height $z = h(x, y, t)$ evolves in time as follows. Since $p = p(x, y, t)$ only we deduce that from the first two shallow layer equations that $u$ and $v$ are quadratic functions of $z$. Indeed if we integrate the equations for $u$ and $v$ with respect to $z$ twice and apply the boundary conditions we find

$$u = \frac{Uz}{h} + \frac{1}{2\mu} \cdot z(z - h) \cdot \frac{\partial p}{\partial x}$$
$$v = \frac{Vz}{h} + \frac{1}{2\mu} \cdot z(z - h) \cdot \frac{\partial p}{\partial y}.$$

However incompressibility implies we have

$$w(x, y, h, t) - w(x, y, 0, t) = -\int_0^h \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \, \mathrm{d}z.$$

Recall we have no-slip boundary conditions at $z = 0$. Further note that at $z = h$ the vertical velocity component is

$$w = \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + V \frac{\partial h}{\partial y}.$$

Also we have the calculus identity

$$\frac{\partial}{\partial x} \int_0^h u \, \mathrm{d}z = \int_0^h \frac{\partial u}{\partial x} \, \mathrm{d}\eta + U \frac{\partial h}{\partial x}\bigg|_{z=h},$$

with a similar result for the $\partial v/\partial y$ component in the incompressibility constraint just above. Putting all this together we find that

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x} \int_0^h u \, \mathrm{d}z - \frac{\partial}{\partial y} \int_0^h v \, \mathrm{d}z.$$

If we substitute our expressions for $u$ and $v$ above into the right-hand side and integrate we arrive at the following closed form evolution equation for $h$:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} - \tfrac{1}{2} U h\Big) + \frac{\partial}{\partial y}\Big(\frac{h^3}{12\mu} \frac{\partial p}{\partial y} - \tfrac{1}{2} V h\Big) = 0,$$

or equivalently

$$12\mu\frac{\partial h}{\partial t}+6\mu\Big(\frac{\partial(Uh)}{\partial x}+\frac{\partial(Vh)}{\partial y}\Big)=\frac{\partial}{\partial x}\Big(h^3\frac{\partial p}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(h^3\frac{\partial p}{\partial y}\Big).$$

**Example (Squeeze film)** An incompressible homogeneous viscous fluid occupies a region between two parallel plates that are very close together—a squeeze film. The upper plate is a circular disc with radius $L$. Assume that the volume occupied by the liquid is a flat cylindrical shape with circular cross-section of radius $L$ and height $H$—see Fig. 20 for the set up. Suppose $(r,\theta,z)$ are cylindrical polar coordinates relative to the origin which is on the lower plate at the centre of the disc of fluid. Let $(u_r,u_\theta,u_z)$ be the fluid velocity components in the three coordinate directions $r$, $\theta$ and $z$, respectively. We assume throughout that the flow is axisymmetric (independent of $\theta$) and there is no swirl (the velocity component $u_\theta=0$). Further we assume that the lower plate remains fixed at $z=0$ while the height of the parallel upper disc plate given by $z=h(t)$ changes with time. No-slip boundary conditions apply on both plates, i.e. $u_r=0$ on $z=0$ and $z=h(t)$, while $u_z=0$ on $z=0$ and $u_z=h'(t)$ on $z=h(t)$. Our goal in this example is to compute the total force on the upper disc plate. Since the relative scaling of the terms in the incompressible Navier–Stokes equations here concern the separation of scales between the vertical $(z,u_z)$ and horizontal $(r,\theta,u_r,u_\theta)$ coordinates and velocities in cylindrical polar coordinates, we can equally carry out the scale analysis in Cartesian vertical $(z,w)$ and horizontal $(x,y,u,v)$ coordinates and velocities, as we did above, assuming the same scaling for the two, namely $(H,W)$ and $(L,U)$ for the vertical and horizontal coordinates and velocities. Hence in this example, the scale analysis above assuming $H\ll L$ generates the shallow layer equations:

$$\frac{\partial p}{\partial r}=\mu\frac{\partial^2u_r}{\partial z^2},$$
$$\frac{\partial p}{\partial z}=0,$$

together with incompressibility (note $u_\theta\equiv0$). We deduce $p=p(r)$ only and

$$\frac{\partial p}{\partial r}=\mu\frac{\partial^2u_r}{\partial z^2}.$$

Integrating and using the no-slip boundary conditions on both plates, we get

$$u_r=\frac{1}{2\mu}\cdot\frac{\partial p}{\partial r}\cdot z(z-h).$$

The flow out of the volume of fluid (cylinder) is the integral quantity

$$\int_0^h\int_0^{2\pi}u_r\,r\,\mathrm{d}\theta\,\mathrm{d}z,$$

since '$r\,\mathrm{d}\theta\,\mathrm{d}z$' is a small patch of area on the sides of the cylinder and $u_r$ is the normal velocity there. Hence the rate of change of total volume of the fluid between the parallel plates is

$$2\pi r\int_0^h u_r\,\mathrm{d}z,$$

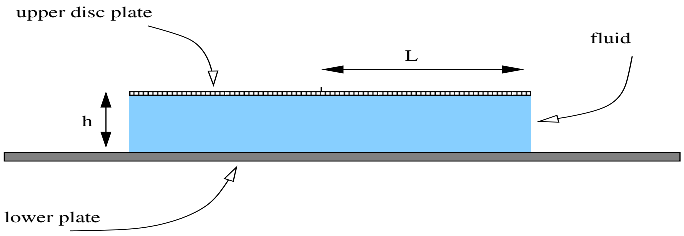


**Fig. 20** Lubrication theory: an incompressible homogeneous viscous fluid occupies a region between two plates that are very close together—a squeeze film. The upper plate is a circular disc with radius $L$. We assume that the volume occupied by the liquid is a flat cylindrical shape with circular cross-section of radius $L$ and height $H$.

Substituting the expression for the velocity field $u_r$ above we find

$$\begin{aligned}2\pi r\int_0^h u_r\,\mathrm{d}z &= \frac{\pi r}{\mu}\frac{\partial p}{\partial r}\int_0^h z(z-h)\,\mathrm{d}z\\ &= -\frac{\pi r}{\mu}\cdot\frac{\partial p}{\partial r}\cdot\frac{h^3}{6}.\end{aligned}$$

If the upper disc plate moves with velocity $h'(t)$ then the rate of change of total volume of the fluid between the parallel plates is also given by the cross-sectional area times that velocity, i.e. $\pi r^2 h'(t)$. Equating this expression with that above we get

$$\pi r^2 h'(t) = -\frac{\pi r}{\mu}\cdot\frac{\partial p}{\partial r}\cdot\frac{h^3}{6} \qquad \Leftrightarrow \qquad \frac{\partial p}{\partial r} = -\frac{6\mu r}{h^3}h'.$$

Assuming that the pressure at the $r = L$ boundary of the cylindrical volume of fluid is zero, the total force on the disc plate is given by

$$\begin{aligned}p(r) &= \frac{6\mu}{h^3}h'\int_r^L r\,\mathrm{d}r\\ &= \frac{3\mu}{h^3}h'(L^2-r^2).\end{aligned}$$

Hence the total force on the disc is given by

$$\frac{3\mu}{h^3}h'\int_0^L (L^2-r^2)\,2\pi r\,\mathrm{d}r = \frac{3\pi\mu L^4}{2h^3}h'.$$

Finally, we can approximate time for a constant force $F$ to pull the parallel plates apart if the initial separation is $h_0$. Assuming the mass of the upper disc plate is negligible, we have

$$F = \frac{3\pi\mu L^4}{2h^3}h' \qquad \Leftrightarrow \qquad h^{-2} = h_0^{-2} - \frac{4Ft}{3\pi\mu L^4}.$$

Hence $h\to\infty$ when $t\to 3\pi\mu L^4/4Fh_0^2$; which is the time it takes to pull the parallel plates apart.

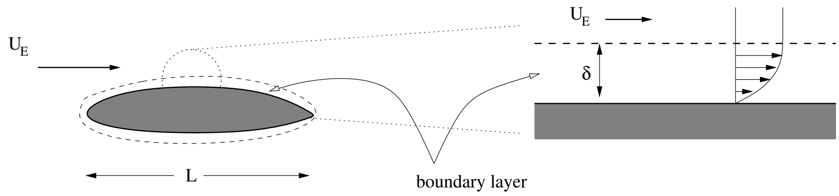


**Fig. 21** Boundary layer theory: the flow over the wing is well approximated by an Euler flow as it is a high Reynolds number flow. However the fluid is ultimately viscous and fluid particles at the wing surface must adhere to it. Hence there is a boundary layer between the wing and bulk flow across which the velocity field rapidly changes from zero velocity relative to the wing to the bulk velocity $U_{\mathrm{E}}$ past the wing.

## 21 Boundary layer theory

The Reynolds numbers associated with flows past aircraft or ships are typically large, indeed of the order $10^8$ of $10^9$—recall the remark at the end of Section 18 on Dynamical similarity and Reynolds numbers. For individual wings or fins the Reynolds number may be an order of magnitude or two smaller. However such Reynolds numbers are still large and the flow around wings for example would be well approximated by Euler flow. We can imagine the flow over the top of the wing of an aircraft has a high relative velocity tangential to the surface wing directed towards the rear edge of the wing. This would appear to be consistent with no flux boundary conditions we apply for the Euler equations—in particular there is no boundary restriction on the tangential component of the fluid velocity field. However air or water flow is viscous. The fluid particles on the wing must satisfy viscous boundary conditions, i.e. exactly at the surface they must adhere to the wing and thus have zero velocity relative to the wing. The reconciliation of this conundrum is that there must be a *boundary layer* on the surface of the wing. By this we mean a special thin fluid layer exists between the wing and the fast moving Euler flow past the wing. The velocity profile of the flow past the wing across the boundary layer as one measures continuously from the wing surface to the top of the boundary layer must change extremely rapidly. Indeed it must change from zero velocity relative to the wing surface to fast relative velocity (the speed of the aircraft) towards the rear of the wing—see Fig. 21.

We shall now derive an accurate model, reduced (or even deduced!) from the full Navier–Stokes equations, for this scenario. We will assume a two dimensional flow around an object of typical size $L$ (for example the length from wing tip to rear edge). We assume that the bulk flow is governed by the Euler equations, which is a good approximation given it is a very high Reynolds number flow (as discussed above). We further assume it is one-dimensional and given by a horizontal velocity field to the right $U_{\mathrm{E}} = U_{\mathrm{E}}(x,t)$ which depends on the horizontal parameter $x$ which is positive towards the right—see Fig. 21. This is consistent with no normal flow close to the boundary layer. We return to discussing the bulk flow once we have derived the boundary layer equations. If $U$ represents the typical bulk fluid velocity (relative speed of the object) then the underlying Reynolds number is

$$\mathrm{Re} = \frac{UL}{\nu}.$$

A typical time scale is thus $T = \mathcal{O}(L/U)$.

Let us now focus on the boundary layer itself. We assume the flow is two-dimensional incompressible homogeneous Navier–Stokes flow. Suppose the velocity components $u = u(x, y, t)$ and $v = v(x, y, t)$ in the horizontal and vertical directions, respectively, with typical scale $U$ and $V$ . Let $\delta$ denote the typical width of the boundary layer. The continuity equation reveals the scaling

$$\underbrace{\frac{\partial u}{\partial x}}_{U/\ell} + \underbrace{\frac{\partial v}{\partial y}}_{V/\delta} = 0.$$

To maintain incompressibility we deduce that $V = \mathcal{O}(U\delta/\ell)$. We now consider the Navier–Stokes equation for the first velocity component $u$ which is given by

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}_{U^2/L} + \underbrace{v\frac{\partial u}{\partial y}}_{VU/\delta} = \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\nu U/L^2} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\nu U/\delta^2} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{P/\rho L},$$

where we suppose the pressure has typical scale $P$. Note since $V = \mathcal{O}(U\delta/\ell)$, all the inertia terms have the same scaling. Our goal in the boundary layer is to keep the inertia terms and a viscous term. In particular the horizontal velocity field in the boundary layer varies rapidly with $y$. We thus want to retain the $\nu\partial^2 u/\partial y^2$ term. This means we must have

$$\frac{U^2}{L} = \frac{\nu U}{\delta^2} \qquad \Leftrightarrow \qquad \delta = \left(\frac{\nu L}{U}\right)^{1/2} \qquad \Leftrightarrow \qquad \delta = \frac{L}{(\mathrm{Re})^{1/2}}.$$

The other viscous term $\nu\partial^2 u/\partial x^2$ thus has typical scaling $\nu U/L^2 = (\mathrm{Re})^{-1}U^2/L$ which is thus asymptotically small compared to the inertia terms since we are assuming the Reynolds number Re is very large. We thus neglect this viscous term. If we now consider the second velocity component $v$, we find

$$\underbrace{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x}}_{UV/L} + \underbrace{v\frac{\partial v}{\partial y}}_{V^2/\delta} = \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\nu V/L^2} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\nu V/\delta^2} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{P/\rho\delta}.$$

Using the equivalent scaling we already established above we observe that

$$\frac{UV}{L} = \frac{U^2\delta}{L^2} = (\mathrm{Re})^{-1/2}\frac{U^2}{L} \qquad \text{and} \qquad \frac{V^2}{\delta} = \frac{U^2\delta}{L^2} = (\mathrm{Re})^{-1/2}\frac{U^2}{L}.$$

We further observe that

$$\frac{\nu V}{L^2} = \frac{\nu U\delta}{L^3} = (\mathrm{Re})^{-1/2}\frac{\nu U}{L^2} = (\mathrm{Re})^{-1}\frac{U^2}{L}$$

and

$$\frac{\nu V}{\delta^2} = \frac{\nu U}{\delta L} = (\mathrm{Re})^{1/2}\frac{\nu U}{L^2} = (\mathrm{Re})^{-1/2}\frac{U^2}{L}.$$

Hence we see the viscous term $\nu\partial^2 v/\partial y^2$ is very small and we immediately neglect it. Further we observe that the pressure term $-(1/\rho)\partial p/\partial y$, to balance the remaining terms in the equation for the field $v = v(x, y, t)$, must have the scaling $(\mathrm{Re})^{-1/2}U^2/L$.

Since these terms are asymptotically small compared to the terms we have retained in the evolution equation for horizontal velocity field $u = u(x, y, t)$, we deduce that

$$\frac{\partial p}{\partial y} = 0.$$

Consequently we only retain the evolution equation for the $u = u(x, y, t)$ with the viscous term $\nu \partial^2 u / \partial x^2$ neglected. Let us now non-dimensionalize our variables as follows

$$x' = \frac{x}{L}, \quad y' = \frac{y}{\delta}, \quad u' = \frac{u}{U}, \quad v' = \frac{v}{V}, \quad t' = \frac{tU}{L} \quad \text{and} \quad p' = \frac{p}{\rho U^2}.$$

Using the natural scaling above, in these non-dimensional variables after dropping the primes, we obtain the *Prandtl boundary layer equations* (derived in 1904):

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial p}{\partial y} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned}$$

These equations are supplemented with the boundary conditions

$$u = 0 \qquad \text{and} \qquad v = 0$$

when $y = 0$. The second equation above implies that the pressure field in the boundary layer is independent of $y$ so that $p = p(x, t)$ only. Hence if the pressure field or more particularly $\partial p / \partial x$ can be determined at the top boundary of the boundary layer, then it is determined inside the boundary layer. With this knowledge, we note that the system of equations represented by the first and third Prandtl equations above, is third order with respect to $y$—the first equation is a second order partial differential equation with respect to $y$ while the third equation is first order. We thus require an additional boundary condition in the $y$ direction. This is naturally provided by matching the boundary layer flow with the bulk Euler flow outside the boundary layer. See Chorin and Marsden [**?**, Section 2.2] for an in depth discussion of possible matching strategies. Here we will simply match horizontal boundary layer velocity field $u = u(x, y, t)$ with the far field Euler flow $U_{\mathrm{E}} = U_{\mathrm{E}}(x, t)$. In terms of the non-dimensionalized $y$ coordinate, temporarily reverting back to the primed notation for them, we have

$$y' = \frac{y}{\delta} = (\mathrm{Re})^{1/2} \frac{y}{L}.$$

In the limit of large Reynolds number, the top boundary corresponds to $y' \to \infty$. Thus, dropping primes again, the third boundary condition we require is, as $y \to \infty$, that

$$u \sim U_{\mathrm{E}}(x, t).$$

Thus we can in principle solve the Prandtl boundary layer equations if $U_{\mathrm{E}} = U_{\mathrm{E}}(x, t)$ is known, in which case

$$\frac{\partial p}{\partial x} = \frac{\partial U_{\mathrm{E}}}{\partial t} + U_{\mathrm{E}} \frac{\partial U_{\mathrm{E}}}{\partial x}.$$

This can be directly substituted into the first Prandtl boundary layer equation above.

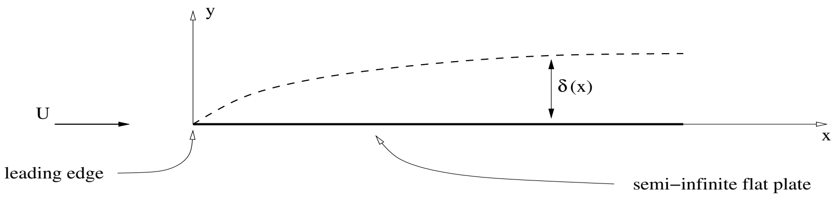


**Fig. 22** Blasius problem: steady boundary layer flow over a semi-infinite flat plate, uniform in the $z$ direction.

*Remark 15* We have implicitly assumed that the lower surface is flat. If lower surface has curvature then $\partial p/\partial y$ is not zero, corresponding to some centripetal acceleration.

**Example (Blasius problem, 1908)** Consider a steady boundary layer flow on a semi-infinite flat plate as shown in Fig. 22. The plate and flow is assumed to be uniform in the $z$ direction. The leading edge of the plate coincides the origin while the plate itself lies along the positive $x$-axis. The $y$-axis is orthogonal to the plate as shown in Fig. 22. We will focus in the flow in the $x \geqslant 0$ and $y \geqslant 0$ region. We suppose that a uniform horizontal flow of velocity $U$ in the positive $x$ direction washes over the plate. We assume that a steady boundary layer flow develops on the plate as shown. Since the Euler flow outside the boundary is uniformly $U$ there is no pressure gradient with respect to $x$ so that $\partial p/\partial x$ is zero outside the boundary, as thus by the arguments in the general theory above, also zero inside the boundary. Hence the Prandtl boundary layer equations in dimensional form are

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned}$$

The boundary conditions for $u = u(x, y)$ and $v = v(x, y)$ on the flat plate are

$$u(x,0) = 0 \qquad \text{and} \qquad v(x,0) = 0,$$

for all $x > 0$. Since the plate is semi-infinite there is no imposed horizontal scale. Following the arguments in the general theory above, but with $L$ replace by $x$ we deduce that $V = U\delta/x$ and thus also that $\delta = \delta(x)$ where

$$\delta(x) := \left(\frac{\nu x}{U}\right)^{1/2}.$$

In the boundary layer, the natural non-dimensional vertical coordinate is

$$\eta := \frac{y}{\delta(x)}.$$

As the flow is incompressible and two dimensional there is a stream function $\psi = \psi(x, y)$ satisfying $\partial\psi/\partial y = u$ and $\partial\psi/\partial y = -v$. We seek a similarity solution of the form

$$\psi = U\,\delta(x)\,f(\eta).$$

Directly computing the partial derivatives we find that

$$u = U\, f'(\eta) \qquad \text{and} \qquad v = -\tfrac{1}{2}\Big(\frac{\nu U}{x}\Big)^{1/2}(f - \eta f').$$

Substituting these forms for $u$ and $v$ into the first Prandtl boundary layer equation above we find that $f = f(\eta)$ satisfies the third order ordinary differential equation

$$f''' + \tfrac{1}{2} f f'' = 0.$$

This is supplemented with the boundary conditions on the flat plate that correspond to $f(0) = 0$ and $f'(0) = 0$. The final boundary condition should be that $u \sim U$ as $y' \to \infty$. Indeed we see that this boundary condition corresponds to $f' \sim 1$ as $\eta \to \infty$. This boundary value problem can be numerically computed.

## 22 Exercises

**Exercise (trajectories and streamlines: expanding jet)** Find the trajectories and streamlines when $(u, v, w)^{\mathrm{T}} = (x\mathrm{e}^{2t-z}, y\mathrm{e}^{2t-z}, 2\mathrm{e}^{2t-z})^{\mathrm{T}}$. What is the track of the particle passing through $(1, 1, 0)^{\mathrm{T}}$ at time $t = 0$?

**Exercise (trajectories and streamlines: three dimensions)** Suppose a velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)^{\mathrm{T}}$ is given for $t > -1$ by

$$u = \frac{x}{1+t}, \qquad v = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad w = z.$$

Find the particle paths and streamlines for a particle starting at $(x_0, y_0, z_0)^{\mathrm{T}}$.

**Exercise (streamlines: plane/cylindrical polar coordinates)** Sketch streamlines for the steady flow field $(u, v, w)^{\mathrm{T}} = \alpha(t) \cdot (x - y, x + y, 0)^{\mathrm{T}}$—show that the streamlines are exponential spirals. Here $\alpha = \alpha(t)$ is an arbitrary function of $t$. (*Hint:* convert to cylindrical polar coordinates $(r, \theta, z)$ first. Note that in these coordinates the equations for trajectories are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_r, \qquad r\frac{\mathrm{d}\theta}{\mathrm{d}t} = u_\theta, \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z,$$

where $u_r$, $u_\theta$ and $u_z$ are the velocity components in the corresponding coordinate directions.)

**Exercise (steady oscillating channel flow)** An incompressible fluid is in steady two-dimensional flow in the channel $-\infty < x < \infty$, $-\pi/2 < y < \pi/2$, with velocity $\boldsymbol{u} = (1 + x\sin y, \cos y)^{\mathrm{T}}$. Find the equation of the streamlines and sketch them. Show that the flow has stagnation points at $(1, -\pi/2)$ and $(-1, \pi/2)$.

**Exercise (channel shear flow)** Consider the two-dimensional channel flow (with $U$ a given constant)

$$\boldsymbol{u} = \begin{pmatrix} 0 \\ U(1 - x^2/a^2) \\ 0 \end{pmatrix},$$

between the two walls $x = \pm a$. Show that there is a *stream function* and find it. (*Hint:* a stream function $\psi$ exists for a velocity field $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$ when $\nabla \cdot \boldsymbol{u} = 0$ and we

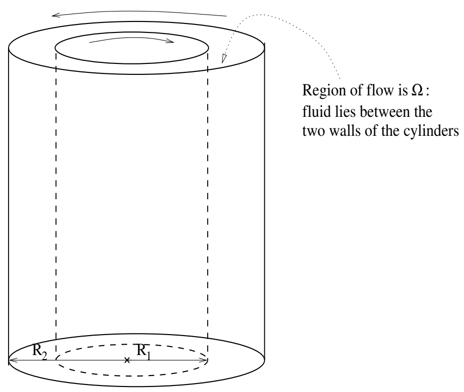


**Fig. 23** Couette flow between two concentric cylinders of radii $R_1 < R_2$.

have an additional symmetry. Here the additional symmetry is uniformity with respect to $z$. You thus need to verify that if $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, then $\nabla \cdot \boldsymbol{u} = 0$ and then solve this system of equations to find $\psi$.)

Show that approximately 91% of the volume flux across $y = y_0$ for some constant $y_0$ flows through the central part of the channel $|x| \leqslant \frac{3}{4}a$.

**Exercise (flow inside and around a disc)** Calculate the stream function $\psi$ for the flow field $\boldsymbol{u} = \big(U\cos\theta\cdot(1-a^2/r^2), -U\sin\theta\cdot(1+a^2/r^2)-\Gamma/2\pi r\big)^{\mathrm{T}}$ in plane polar coordinates, where $U, a, \gamma$ are constants.

**Exercise (Couette flow)** (From Chorin and Marsden, p. 31.) Let $\Omega$ be the region between two concentric cylinders of radii $R_1$ and $R_2$, where $R_1 < R_2$. Suppose the velocity field in cylindrical coordinates $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{T}}$ of the fluid flow inside $\Omega$, is given by $u_r = 0$, $u_z = 0$ and

$$u_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2 R_2^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \qquad \text{and} \qquad B = -\frac{R_1^2\omega_1 - R_2^2\omega_2}{R_2^2 - R_1^2}.$$

This is known as a *Couette flow*—see Fig. 23. Show that the:

(a) velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{T}}$ is a stationary solution of Euler's equations of motion for an ideal fluid with density $\rho \equiv 1$ (*hint:* you need to find a pressure field $p$ that is consistent with the velocity field given. Indeed the pressure field should be $p = -A^2/2r^2 + 2AB\log r + B^2r^2/2 + C$ for some arbitrary constant $C$.);
(b) angular velocity of the flow (i.e. the quantity $u_\theta/r$) is $\omega_1$ on the cylinder $r = R_1$ and $\omega_2$ on the cylinder $r = R_2$.
(c) the vorticity field $\nabla \times \boldsymbol{u} = (0, 0, 2B)$.

**Exercise (hurricane)** We devise a simple model for a hurricane.

(a) Using the Euler equations for an ideal incompressible flow in cylindrical coordinates (see the bath or sink drain problem in the main text) show that at position $(r, \theta, z)$, for a flow which is independent of $\theta$ with $u_r = u_z = 0$, we have

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p}{\partial r},$$
$$0 = \frac{1}{\rho_0}\frac{\partial p}{\partial z} + g,$$

where $p = p(r,z)$ is the pressure and $g$ is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) In a *simple* model for a hurricane the air is taken to have uniform constant density $\rho_0$ and each fluid particle traverses a horizontal circle whose centre is on the fixed vertical $z$-axis. The (angular) speed $u_\theta$ at a distance $r$ from the axis is

$$u_\theta = \begin{cases} \Omega r, & \text{for } 0 \leqslant r \leqslant a, \\ \Omega \frac{a^{3/2}}{r^{1/2}}, & \text{for } r > a, \end{cases}$$

where $\Omega$ and $a$ are known constants.

(i) Now consider the flow given above in the inner region $0 \leqslant r \leqslant a$. Using the equations in part (a) above, show that the pressure in this region is given by

$$p = c_0 + \tfrac{1}{2}\rho_0 \Omega^2 r^2 - g\rho_0 z,$$

where $c_0$ is a constant. A free surface of the fluid is one for which the pressure is constant. Show that the shape of a free surface for $0 \leqslant r \leqslant a$ is a paraboloid of revolution, i.e. it has the form

$$z = Ar^2 + B,$$

for some constants $A$ and $B$. Specify the exact form of $A$ and $B$.

(ii) Now consider the flow given above in the outer region $r > a$. Again using the equations in part (a) above, and that the pressure must be continuous at $r = a$, show that the pressure in this region is given by

$$p = c_0 - \frac{\rho_0}{r}\Omega^2 a^3 - g\rho_0 z + \tfrac{3}{2}\rho_0 \Omega^2 a^2,$$

where $c_0$ is the same constant (reference pressure) as that in part (i) above.

**Exercise (Venturi tube)** Consider the Venturi tube shown in Fig. 24. Assume that the ideal fluid flow through the construction is homogeneous, incompressible and steady. The flow in the wider section of cross-sectional area $A_1$, has velocity $u_1$ and pressure $p_1$, while that in the narrower section of cross-sectional area $A_2$, has velocity $u_2$ and pressure $p_2$. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

(a) Why does the relation $A_1 u_1 = A_2 u_2$ hold? Why is the flow faster in the narrower region of the tube compared to the wider region of the tube?

(b) Use Bernoulli's theorem to show that

$$\tfrac{1}{2}u_1^2 + \frac{p_1}{\rho_0} = \tfrac{1}{2}u_2^2 + \frac{p_2}{\rho_0},$$

where $\rho_0$ is the constant uniform density of the fluid.

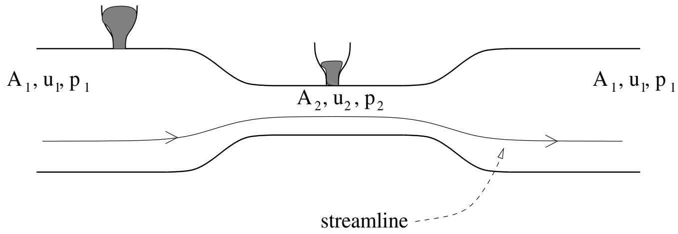


**Fig. 24** *Venturi tube*: the flow in the wider section of cross-sectional area $A_1$ has velocity $u_1$ and pressure $p_1$, while that in the narrower section of cross-sectional area $A_2$ has velocity $u_2$ and pressure $p_2$. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

(c) Using the results in parts (a) and (b), compare the pressure in the narrow and wide regions of the tube.
(d) Give a practical application where the principles of the Venturi tube is used or might be useful.

**Exercise (Clepsydra or water clock)** A clepsydra has the form of a surface of revolution containing water and the level of the free surface of the water falls at a *constant* rate, as the water flows out through a small hole in the base. The basic setup is shown in Fig. 25.

(a) Apply Bernoulli's theorem to one of the typical streamlines shown in Fig. 25 to show that

$$\frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = \tfrac{1}{2}U^2 - gz$$

where $z$ is the height of the free surface above the small hole in the base, $U$ is the velocity of the water coming out of the small hole and $g$ is the acceleration due to gravity.
(b) If $S$ is the cross-sectional area of the hole in the bottom, and $A$ is the cross-sectional area of the free surface, explain why we must have

$$A\,\frac{\mathrm{d}z}{\mathrm{d}t} = S\,U.$$

(c) Assuming that $S \ll A$, combine parts (a) and (b) to explain why we can deduce

$$U \sim \sqrt{2gz}.$$

(d) Now combine the results from (b) and (c) above, to show that the shape of the container that guarantees that the free surface of the water drops at a constant rate must have the form $z = C\,r^4$ in cylindrical polars, where $C$ is a constant.

**Exercise (coffee in a mug)** A coffee mug in the form of a right circular cylinder (diameter $2a$, height $h$), closed at one end, is initially filled to a depth $d > \frac{1}{2}h$ with static inviscid coffee. Suppose the coffee is then made to rotate inside the mug—see Fig. 26.

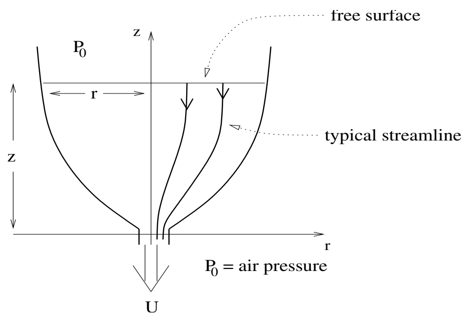


**Fig. 25** Clepsydra (water clock).

(a) Using the Euler equations for an ideal incompressible homogeneous flow in cylindrical coordinates show that at position $(r, \theta, z)$, for a flow which is independent of $\theta$ with $u_r = u_z = 0$, the Euler equations reduce to

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r},$$
$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z} + g,$$

where $p = p(r, z)$ is the pressure, $\rho$ is the constant uniform fluid density and $g$ is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) Assume that the coffee in the mug is rotating as a solid body with constant angular velocity $\Omega$ so that the velocity component $u_\theta$ at a distance $r$ from the axis of symmetry for $0 \leqslant r \leqslant a$ is

$$u_\theta = \Omega r.$$

Use the equations in part (a), to show that the pressure in this region is given by

$$p = \tfrac{1}{2}\rho\Omega^2 r^2 - g\rho z + C,$$

where $C$ is an arbitrary constant. At the free surface between the coffee and air, the pressure is constant and equal to the atmospheric pressure $P_0$. Use this to show that the shape of the free surface has the form

$$z = \frac{\Omega^2}{2g} r^2 + \frac{C - P_0}{\rho g}.$$

(c) Note the we are free to choose $C = P_0$ in the equation of the free surface so that it is described by $z = \Omega^2 r^2 / 2g$. This is equivalent to choosing the origin of our cylindrical coordinates to be the centre of the dip in the free surface. Suppose that this origin is a distance $z_0$ from the bottom of the mug.

(i) Explain why the total volume of coffee is $\pi a^2 d$. Then by using incompressibility, explain why the following constraint must be satisfied:

$$\pi a^2 z_0 + \int_0^a \frac{\Omega^2 r^2}{2g} \cdot 2\pi r \,\mathrm{d}r = \pi a^2 d.$$

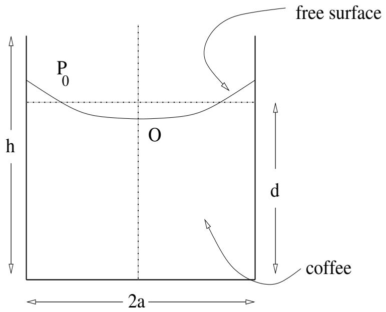


**Fig. 26** Coffee mug: we consider a mug of coffee of diameter $2a$ and height $h$, which is initially filled with coffee to a depth $d$. The coffee is then made to rotate about the axis of symmetry of the mug. The free surface between the coffee and the air takes up the characteristic shape shown, dipping down towards the middle (axis of symmetry). The goal is to specify the shape of the free surface.

(ii) By computing the integral in the constraint in part (i), show that some coffee will be spilled out of the mug if $\Omega^2 > 4g(h-d)/a^2$. Explain briefly why this formula does not apply when the mug is initially less than half full.

**Exercise (Channel flow: Froude number)** Recall the scenario of the steady channel flow over a gently undulating bed given in Section 11. Consider the case when the maximum permissible height $y_0$ compatible with the upstream conditions, and the actual maximum height $y_{\max}$ of the undulation are exactly equal, i.e. $y_{\max} = y_0$. Show that the flow becomes locally critical immediately above $y_{\max}$ and, by a local expansion about that position, show that there are subcritical and supercritical flows downstream consistent with the continuity and Bernoulli equations (friction in a real flow leads to the latter being preferred).

**Exercise (Bernoulli's Theorem for irrotational unsteady flow)** Consider Euler's equations of motion for an ideal homogeneous incompressible fluid, with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ denoting the fluid velocity at position $\boldsymbol{x}$ and time $t$, $\rho$ the uniform constant density, $p = p(\boldsymbol{x}, t)$ the pressure, and $\boldsymbol{f}$ denoting the body force per unit mass. Suppose that the flow is unsteady, but irrotational, i.e. we know that $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ throughout the flow. This means that we know there exists a scalar potential function $\varphi = \varphi(\boldsymbol{x}, t)$ such that $\boldsymbol{u} = \nabla\varphi$. Also suppose that the body force is conservative so that $\boldsymbol{f} = -\nabla\Phi$ for some potential function $\Phi = \Phi(\boldsymbol{x}, t)$.

(a) Using the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2}\nabla(|\boldsymbol{u}|^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

show from Euler's equations of motion that the Bernoulli quantity

$$H := \frac{\partial \varphi}{\partial t} + \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

satisfies $\nabla H = \boldsymbol{0}$ throughout the flow.

(b) From part (a) above we can deduce that $H$ can only be a function of $t$ throughout the flow, say $H = f(t)$ for some function $f$. By setting

$$V := \varphi - \int_0^t f(\tau)\,\mathrm{d}\tau,$$

show that the Bernoulli quantity

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

is constant throughout the flow.

**Exercise (rigid body rotation)** An ideal fluid of constant uniform density $\rho_0$ is contained within a fixed right-circular cylinder (with symmetry axis the $z$-axis). The fluid moves under the influence of a body force field $\boldsymbol{f} = (\alpha x + \beta y, \gamma x + \delta y, 0)^{\mathrm{T}}$ per unit mass, where $\alpha$, $\beta$, $\gamma$ and $\delta$ are independent of the space coordinates. Use Euler's equations of motion to show that a rigid body rotation of the fluid about the $z$-axis, with angular velocity $\omega(t)$ given by $\dot{\omega} = \frac{1}{2}(\gamma - \beta)$ is a possible solution of the equation and boundary conditions. Show that the pressure is given by

$$p = p_0 + \frac{1}{2}\rho_0\big((\omega^2 + \alpha)x^2 + (\beta + \gamma)xy + (\omega^2 + \delta)y^2\big),$$

where $p_0$ is the pressure at the origin.

**Exercise (vorticity and streamlines)** An inviscid incompressible fluid of uniform density $\rho$ is in steady two-dimensional horizontal motion. Show that the Euler equations are equivalent to

$$\frac{\partial H}{\partial x} = v\omega \qquad \text{and} \qquad \frac{\partial H}{\partial y} = -u\omega,$$

where $H = p/\rho + \frac{1}{2}(u^2 + v^2)$, where $p$ is the dynamical pressure, $(u, v)^{\mathrm{T}}$ is the velocity field and $\omega$ is the vorticity. Deduce that $\omega$ is constant along streamlines and that this is in accord with Kelvin's theorem.

**Exercise (vorticity, streamlines with gravity)** An incompressible inviscid fluid, under the influence of gravity, has the velocity field $\boldsymbol{u} = (2\alpha y, -\alpha x, 0)^{\mathrm{T}}$ with the $z$-axis vertically upwards; and $\alpha$ is constant. Also the density $\rho$ is constant. Verify that $\boldsymbol{u}$ satisfies the governing equations and find the pressure $p$. Show that the Bernoulli function $H = p/\rho + \frac{1}{2}|\boldsymbol{u}|^2 + \Phi$ is constant on streamlines and vortex lines, where $\Phi$ is the gravitational potential.

**Exercise (Flow in an infinite pipe: Poiseuille flow)** (From Chorin and Marsden, pp. 45-6.) Consider an infinite pipe with circular cross-section of radius $a$, whose centre line is aligned along the $z$-axis. Assume *no-slip* boundary conditions at $r = a$, for all $z$, i.e. on the inside surface of the cylinder. Using cylindrical polar coordinates, look for a stationary solution to the fluid flow in the pipe of the following form. Assume there is no radial flow, $u_r = 0$, and no swirl, $u_\theta = 0$. Further assume there is a constant pressure gradient down the pipe, i.e. that $p = -Cz$ for some constant $C$. Lastly, suppose that the flow down the pipe, i.e. the velocity component $u_z$, has the form $u_z = u_z(r)$ (it is a function of $r$ only).

(a) Using the Navier–Stokes equations, show that

$$C = -\rho\nu\Delta u_z = -\rho\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right).$$

(b) Integrating the equation above yields

$$u_z = -\frac{C}{4\rho\nu} r^2 + A \log r + B,$$

where $A$ and $B$ are constants. We naturally require that the solution be bounded. Explain why this implies $A = 0$. Now use the no-slip boundary condition to determine $B$. Hence show that

$$u_z = \frac{C}{4\rho\nu}(a^2 - r^2).$$

(c) Show that the *mass-flow rate* across any cross section of the pipe is given by

$$\int \rho u_z \, \mathrm{d}S = \pi C a^4 / 8\nu.$$

This is known as the *fourth power law.*

**Exercise (Elliptic pipe flow)** Consider an infinite horizontal pipe with constant elliptical cross-section, whose centre line is aligned along the $z$-axis. Assume *no-slip* boundary conditions at

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a$ and $b$ are the semi-axis lengths of the elliptical cross-section. Using the incompressible Navier–Stokes equations for a homogeneous fluid in Cartesian coordinates, look for a stationary solution to the fluid flow in the pipe of the following form. Assume there is no flow transverse to the axial direction of the pipe, so that if $u$ and $v$ are the velocity components in the coordinate $x$ and $y$ directions, respectively, then $u = 0$ and $v = 0$. Further assume there is a constant pressure gradient down the pipe, i.e. the pressure $p = -Gz$ for some constant $G$, and there is no body force. Lastly, suppose that for the flow down the pipe the velocity component $w = w(x, y)$ only.

(a) Using the Navier–Stokes equations, show that $w$ must satisfy

$$\frac{\partial^2 w}{\partial^2 x} + \frac{\partial^2 w}{\partial^2 y} = -\frac{G}{\nu}.$$

(b) Show that, assuming $A$, $B$ and $C$ are constants,

$$w = Ax^2 + By^2 + C$$

is a solution to the partial differential equation for $w$ in part (a) above provided

$$A + B = -\frac{G}{2\nu}.$$

(c) Use the no-slip boundary condition to show that

$$A = -\frac{G}{2\nu} \cdot \frac{b^2}{a^2 + b^2}, \qquad B = -\frac{G}{2\nu} \cdot \frac{a^2}{a^2 + b^2} \qquad \text{and} \qquad C = \frac{G}{2\nu} \cdot \frac{a^2 b^2}{a^2 + b^2}.$$

(d) Explain why the *volume flux* across any cross section of the pipe is given by

$$\int w \, \mathrm{d}S$$

and then show that it is given by

$$\frac{\pi a^3 b^3 G}{4(a^2+b^2)\nu}.$$

(*Hint:* to compute the integral you may find the substitutions $x = ar\cos\theta$ and $y = br\sin\theta$, together with the fact that the infinitesimal area element '$\mathrm{d}x\,\mathrm{d}y$' transforms to '$ab\,r\mathrm{d}r\,\mathrm{d}\theta$', useful.)

(e) Explain why for a given elliptical cross-sectional area, the optimal choice of $a$ and $b$ to maximize the volume flow-rate is $a = b$.

**Exercise (Wind blowing across a lake)** Wind blowing across the surface of a lake of uniform depth $d$ exerts a constant and uniform tangential stress $S$. The water is initially at rest. Find the water velocity at the surface as a function of time for $\nu t \ll d^2$. (*Hint:* solve for the vorticity using the vorticity equation for a very deep lake.)

Suppose now that the wind has been blowing for a sufficiently long time to establish a steady state. Assuming that the water velocity can be taken to be almost uni-directional and that the horizontal dimensions of the lake are large compared with $d$, show that the water velocity at the surface is $Sd/4\mu$. (*Hint:* A pressure gradient would be needed to ensure no net flux (why?) in the steady state, and this pressure gradient leads to a small rise in the surface elevation of the lake in the direction of the wind.)

**Exercise (Stokes flow: between hinged plates)** Starting with the continuity and Navier–Stokes equations for a steady incompressible two-dimensional flow, show that for Stokes flow the stream function satisfies $\nabla^4\psi = 0$. Two identical rigid plates are hinged together along their line of intersection O, and have a relative angular velocity $\Omega$. Find the stream function representing the (two-dimensional) Stokes flow near O, and estimate the distance within O within which the solution is self-consistent.

**Exercise (Stokes flow: rotating sphere)** A rigid sphere of radius $a$ is rotating with angular velocity $\boldsymbol{\Omega}$ in a fluid at rest at infinity. Show that when $\rho a^2\Omega/\mu \ll 1$ the couple exerted on the fluid by the sphere is $8\pi\mu a^3\boldsymbol{\Omega}$. (Use that in spherical polar coordinates $(r,\theta,\varphi)$ for a velocity field $\boldsymbol{u} = (0,0,u_\varphi)^{\mathrm{T}}$, the relevant component of the deformation matrix is $D_{r\varphi} = \frac{1}{2}r\partial(u_\varphi/r)/\partial r$.)

**Exercise (Lubrication theory: shear stress)** Incompressible fluid of viscosity $\mu$ is contained between $y = 0$ and $y = h(x)$ for $0 \leqslant x \leqslant a$ where $h \ll a$. The fluid pressure at $x = 0$ exceeds that at $x = a$ by an amount $\hat{p}$. Using lubrication theory show that

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{A}{\big(h(x)\big)^3}$$

where $A$ is a constant. For the special case where $h = C\mathrm{e}^{-Bx}$, determine $\mathrm{d}p/\mathrm{d}x$, where $B$ and $C$ are positive constants. Hence show that the maximum shear stress on the plane is

$$\tfrac{3}{2}\hat{p}\left(\frac{BC\mathrm{e}^{2Ba}}{\mathrm{e}^{3Ba}-1}\right).$$

**Exercise (Lubrication theory: shallow layer)** An incompressible homogeneous viscous fluid occupies a shallow layer whose typical depth is $H$ and horizontal extent is $L$; see Fig. 19 for the set up. Suppose that for this question $x$ and $y$ are horizontal Cartesian coordinates and $z$ is the vertical coordinate. Let $u$, $v$ and $w$ be the fluid velocity components in the three coordinate directions $x$, $y$ and $z$, respectively. Suppose $p$ is the pressure. Assume throughout that a typical horizontal velocity scale for $u$ and $v$ is $U$ and a typical vertical velocity scale for $w$ is $W$. We denote by $\rho$, the constant density of the fluid.

(a) Assume throughout that $H \ll L$.
  (i) Using that the fluid is incompressible, explain how we can deduce that $W$ is asymptotically smaller than $U$.
  (ii) The body force $\rho g$ in this example is due to gravity $g$ which we suppose acts in the negative $z$ direction (i.e. downwards). Using part (i), show the three dimensional Navier–Stokes equations reduce to the *shallow layer* equations:

$$\begin{aligned}\frac{\partial P}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2},\\ \frac{\partial P}{\partial y} &= \mu \frac{\partial^2 v}{\partial z^2},\\ \frac{\partial P}{\partial z} &= 0,\end{aligned}$$

  where $P := p + \rho g z$ is the modified pressure and $\mu = \rho\nu$ is the first coefficient of viscosity

(b) For the shallow fluid layer shown in Fig. 19, assume no-slip boundary conditions on the rigid lower layer. Further suppose that the pressure $p$ is constant (indeed take it to be zero) along the top surface of the layer at $z = h(x, y, t)$. Further suppose that surface stress forces are applied to the surface $z = h(x, y, t)$ so that

$$\begin{aligned}\mu \frac{\partial u}{\partial z} &= \frac{\partial \Gamma}{\partial x},\\ \mu \frac{\partial v}{\partial z} &= \frac{\partial \Gamma}{\partial y},\end{aligned}$$

where $\Gamma = \Gamma(x, y)$ is a given function. Using the shallow layer equations in part (a) and the boundary conditions, show that the height function $h(x, y, t)$ satisfies the partial differential equation

$$2\mu \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^2 \frac{\partial}{\partial x}\big(\Gamma - \tfrac{1}{3}\rho g h^2\big)\right) + \frac{\partial}{\partial y}\left(h^2 \frac{\partial}{\partial y}\big(\Gamma - \tfrac{1}{3}\rho g h^2\big)\right) = 0.$$

**Exercise (Lubrication theory: Hele–Shaw cell)** An incompressible homogeneous fluid occupies the region between two horizontal rigid parallel planes, which are a distance $h$ apart, and outside a rigid cylinder which intersects the planes normally; see Fig. 27 for the set up. Suppose that for this question $x$ and $y$ are horizontal coordinates and $z$ is the vertical coordinate. Assume throughout that a typical horizontal velocity scale for $u$ and $v$ is $U$ and a typical vertical velocity scale for $w$ is $W$.

(a) Explain very briefly why $a$ is a typical horizontal scale for $(x, y)$ and $h$ a typical vertical scale for $z$.

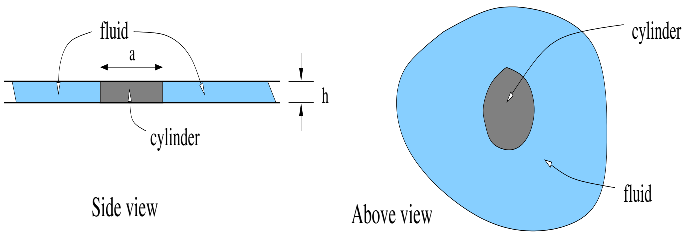


**Fig. 27** Hele–Shaw cell: an incompressible homogeneous fluid occupies the region between two parallel planes (a distance $h$ apart) and outside the cylinder of radius $a$ (normal to the planes).

(b) Hereafter further assume that $h \ll a$ and further that

$$\rho U h^2 \ll a\mu$$

where $\mu = \rho\nu$ is the first coefficient of viscosity. Using these assumptions, show that the Navier–Stokes equations for an incompressible homogeneous fluid reduce to the system of equations

$$\begin{aligned} \mu\frac{\partial^2 u}{\partial z^2} &= \frac{\partial p}{\partial x}, \\ \mu\frac{\partial^2 v}{\partial z^2} &= \frac{\partial p}{\partial y}, \\ \frac{\partial p}{\partial z} &= 0, \end{aligned}$$

for a steady flow, where $p = p(x,y,z)$ is the pressure.

(c) We define the vertically averaged velocity components $(\overline{u},\overline{v})^{\mathrm{T}} = \big(\overline{u}(x,y),\overline{v}(x,y)\big)^{\mathrm{T}}$ for the flow in part (b) by

$$\big(\overline{u}(x,y),\overline{v}(x,y)\big)^{\mathrm{T}} := \frac{1}{h}\int_0^h \big(u(x,y),v(x,y)\big)^{\mathrm{T}}\,\mathrm{d}z.$$

Show that the vertically averaged velocity field $\overline{\boldsymbol{u}} = (\overline{u},\overline{v})^{\mathrm{T}}$ is both *incompressible* and *irrotational*.

(d) What is an appropriate boundary condition for $\overline{\boldsymbol{u}}$ on the cylinder?

**Exercise (Boundary layer theory: axisymmetric flow)** In an axisymmetric flow the velocity components corresponding to cylindrical polar coordinates $(r,\theta,z)$ are $(u_r,u_\theta,u_z)^{\mathrm{T}}$. If $u_r = -\alpha r/2$ and $u_z = \alpha z$, where $\alpha$ is a constant, verify that the continuity equation is satisfied. If the swirl velocity $u_\theta$ is assumed independent of $z$, show that the vorticity has the form $\boldsymbol{\omega} = \big(0,0,\omega(r,t)\big)^{\mathrm{T}}$.

(a) At $t = 0$ the vorticity is given by $\omega = \omega_0 f(r)$, where $\omega_0$ is a constant. Verify from the dynamical *inviscid* vorticity equation that at a later time $t$,

$$\omega = \omega_0 \exp(\alpha t) f\big(r\exp(\alpha t/2)\big),$$

and interpret this result in terms of stretching of material lines.

(b) Consider now steady *viscous* flow. Write down the governing equation for $\omega$ and show that it is satisfied by

$$\omega = \omega_0 \exp(-\alpha r^2/4\nu).$$

Why is a steady state possible in this case but not in (a)? What is the dominant physical balance in the flow?

**Exercise (Boundary layer theory: rigid wall)** Seek similarity solutions of the boundary layer equation

$$\psi_x \psi_{xy} - \psi_x \psi_{yy} = UU_x + \nu \psi_{yyy}$$

in the form $\psi = U(x)\delta(x)f(\eta)$ where $\eta = y/\delta(x)$. Show that $f$ satisfies the equation

$$f''' + \alpha f f'' + \beta\big(1 - (f')^2\big) = 0,$$

and explain why $\alpha$ and $\beta$ must be constants. Give $\alpha$ and $\beta$ in terms of $U(x)$ and $\delta(x)$, and hence determine the possible forms of $U(x)$ and $\delta(x)$. State the boundary conditions on $f$ if there is a rigid wall at $y = 0$ and an outer flow with velocity $U(x)$ as $y \to \infty$.

## A Multivariable calculus identities

We provide here some useful multivariable calculus identities. Here $\phi$ and $\psi$ are generic scalars, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are generic vectors.

1. $\nabla \times \boldsymbol{u} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}.$
2. $\nabla \cdot (\nabla\phi) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}.$
3. $\nabla \times (\nabla\phi) \equiv \boldsymbol{0}.$
4. $\nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0.$
5. $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}.$
6. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi.$
7. $\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}).$
8. $\nabla \cdot (\phi\boldsymbol{u}) = \phi(\nabla \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla\phi.$
9. $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}).$
10. $\nabla \times (\phi\boldsymbol{u}) = \phi\nabla \times \boldsymbol{u} + \nabla\phi \times \boldsymbol{u}.$
11. $\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}\,(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}\,(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}.$

## B Navier–Stokes equations in cylindrical polar coordinates

The incompressible Navier–Stokes equations in *cylindrical polar coordinates* $(r, \theta, z)$ with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{T}}$ are

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + f_r, \\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + f_\theta, \\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta u_z + f_z,
\end{aligned}$$

where $p = p(r, \theta, z, t)$ is the pressure, $\rho$ is the mass density and $\boldsymbol{f} = (f_r, f_\theta, f_z)^{\mathrm{T}}$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

and

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Further the gradient operator and the divergence of a vector field $\boldsymbol{u}$ are given in cylindrical coordinates, respectively, by

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

and

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

In cylindrical coordinates $\nabla \times \boldsymbol{u}$ is given by

$$\nabla \times \boldsymbol{u} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \end{pmatrix}.$$

Lastly the diagonal components of the deformation matrix $D$ are

$$D_{rr} = \frac{\partial u_r}{\partial r}, \qquad D_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \text{and} \qquad D_{zz} = \frac{\partial u_z}{\partial z},$$

while the off-diagonal components are given by

$$2D_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta}, \quad 2D_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \quad \text{and} \quad 2D_{\theta z} = \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

## C Navier–Stokes equations in spherical polar coordinates

The incompressible Navier–Stokes equations in *spherical polar coordinates* $(r, \theta, \varphi)$ with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_\varphi)^{\mathrm{T}}$ are (note $\theta$ is the angle to the south-north pole axis and $\varphi$ is the azimuthal angle)

$$\begin{aligned}
\frac{\partial u_r}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} - \frac{u_\varphi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \\
&+ \nu \left( \Delta u_r - 2\frac{u_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (u_\theta \sin\theta) - \frac{2}{r^2 \sin\theta} \frac{\partial u_\varphi}{\partial \varphi} \right) + f_r, \\
\frac{\partial u_\theta}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cos\theta}{r \sin\theta} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \\
&+ \nu \left( \Delta u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - 2\frac{\cos\theta}{r^2 \sin^2\theta} \frac{\partial u_\varphi}{\partial \varphi} \right) + f_\theta, \\
\frac{\partial u_\varphi}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_\varphi + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi \cos\theta}{r \sin\theta} = -\frac{1}{\rho r \sin\theta} \frac{\partial p}{\partial \varphi} \\
&+ \nu \left( \Delta u_\varphi + \frac{2}{r^2 \sin\theta} \frac{\partial u_r}{\partial \varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2\theta} \right) + f_z,
\end{aligned}$$

where $p = p(r, \theta, \varphi, t)$ is the pressure, $\rho$ is the mass density and $\boldsymbol{f} = (f_r, f_\theta, f_\varphi)^{\mathrm{T}}$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

and

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Further the gradient operator and the divergence of a vector field $\boldsymbol{u}$ are given in spherical coordinates, respectively, by

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \end{pmatrix}$$

and

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}.$$

In spherical coordinates $\nabla\times\boldsymbol{u}$ is given by

$$\nabla\times\boldsymbol{u} = \begin{pmatrix}\omega_r\\ \omega_\theta\\ \omega_\varphi\end{pmatrix} = \begin{pmatrix} \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\varphi) - \frac{\partial u_\theta}{\partial\varphi} \\ \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\varphi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} \end{pmatrix}.$$

Lastly the diagonal components of the deformation matrix $D$ are

$$D_{rr} = \frac{\partial u_r}{\partial r}, \qquad D_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} \qquad \text{and} \qquad D_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

while the off-diagonal components are given by

$$2D_{r\theta} = r\frac{\partial}{\partial r}\Big(\frac{u_\theta}{r}\Big) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}, \qquad 2D_{r\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + r\frac{\partial}{\partial r}\Big(\frac{u_\varphi}{r}\Big)$$

and

$$2D_{\theta\varphi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big(\frac{u_\varphi}{\sin\theta}\Big) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}.$$

## D Ideal fluid flow and conservation of energy

We show that an ideal flow that conserves energy is necessarily incompressible. We have derived two conservation laws thus far, first, conservation of mass,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0,$$

and second, balance of momentum,

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho\,\boldsymbol{f}.$$

If we are in three dimensional space so $d = 3$, we have four equations, but five unknowns—namely $\boldsymbol{u}$, $p$ and $\rho$. We cannot specify the fluid motion completely without specifying one more condition.

**Definition 10 (Kinetic energy)** The *kinetic energy* of the fluid in the region $\Omega \subseteq \mathcal{D}$ is

$$E := \tfrac{1}{2}\int_\Omega \rho\,|\boldsymbol{u}|^2\,\mathrm{d}V.$$

The rate of change of the kinetic energy, using the Transport Theorem, is given by

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\int_{\Omega_t} \rho\,|\boldsymbol{u}|^2\,\mathrm{d}V\right)\\ &= \tfrac{1}{2}\int_{\Omega_t} \rho\frac{\mathrm{D}|\boldsymbol{u}|^2}{\mathrm{D}t}\,\mathrm{d}V\\ &= \tfrac{1}{2}\int_{\Omega_t} \rho\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{u}\cdot\boldsymbol{u})\,\mathrm{d}V\\ &= \int_{\Omega_t} \rho\boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V\\ &= \int_{\Omega_t} \boldsymbol{u}\cdot\left(\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)\mathrm{d}V.\end{aligned}$$

Here we assume that *all* the energy is kinetic. The principal of conservation of energy states (from Chorin and Marsden, page 13):

> *the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work.*

In other words we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_{\partial\Omega_t} p\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\Omega_t} \rho\,\boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}V.$$

We compare this with our expression above for the rate of change of the kinetic energy. Equating the two expressions, using Euler's equation of motion, and noticing that the body force term immediately cancels, we get

$$\begin{aligned} & \int_{\partial\Omega_t} p\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{\Omega_t} \boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow\quad & \int_{\Omega_t} \nabla\cdot(\boldsymbol{u}p)\,\mathrm{d}V = \int_{\Omega_t} \boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow\quad & \int_{\Omega_t} \boldsymbol{u}\cdot\nabla p + (\nabla\cdot\boldsymbol{u})\,p\,\mathrm{d}V = \int_{\Omega_t} \boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow\quad & \int_{\Omega_t} (\nabla\cdot\boldsymbol{u})\,p\,\mathrm{d}V = 0.\end{aligned}$$

Since $\Omega$ and therefore $\Omega_t$ is arbitrary we see that the assumption that all the energy is kinetic implies

$$\nabla\cdot\boldsymbol{u} = 0.$$

Hence our third conservation law, conservation of energy has lead to the *equation of state*, $\nabla\cdot\boldsymbol{u} = 0$, i.e. that an ideal flow is *incompressible*.

Hence the *Euler equations* for a homogeneous incompressible flow in $\mathcal{D}$ are

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\frac{1}{\rho}\nabla p + \boldsymbol{f},\\ \nabla\cdot\boldsymbol{u} &= 0,\end{aligned}$$

together with the boundary condition on $\partial\mathcal{D}$ which is $\boldsymbol{u}\cdot\boldsymbol{n} = 0$.

## E Isentropic flows

A compressible flow is *isentropic* if there is a function $\pi$, called the *enthalpy*, such that

$$\nabla\pi = \frac{1}{\rho}\nabla p.$$

The Euler equations for an isentropic flow are thus

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\pi + \boldsymbol{f}$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0,$$

in $\mathcal{D}$, and on $\partial\mathcal{D}$, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ (or matching normal velocities if the boundary is moving).

For compressible ideal gas flow, the pressure is often proportional to $\rho^\gamma$, for some constant $\gamma \geqslant 1$, i.e.

$$p = C\,\rho^\gamma,$$

for some constant $C$. This is a special case of an isentropic flow, and is an example of an *equation of state*. In fact we can actually compute

$$\pi = \int^\rho \frac{p'(z)}{z}\,\mathrm{d}z = \frac{\gamma\,C\,\rho^\gamma}{\gamma-1},$$

and the *internal energy* (see Chorin and Marsden, pages 14 and 15)

$$\epsilon = \pi - (p/\rho) = \frac{C\,\rho^\gamma}{\gamma-1}.$$

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