Fluid mechanics deals with materials that flow in response to an applied stress. In fluids the stresses are related to rates of strain. The relation can be linear, then the fluid is called Newtonian,

$$\sigma_{ij} = 2\mu\dot{\varepsilon}_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1}$$

where $u$ is velocity, and $\mu$ is viscosity ($[\mu]$ = Pa s).

In glaciology the flow of ice in response to an applied stress in non-linear,

$$\dot{\varepsilon} = A(T)\tau^n, \tag{2}$$

where $n \simeq 3$, it depends on temperature $T$, and the relation above applies homogeneous isotropic ice.

Fluid mechanics are important for convection, glacier flow, aquifer flow, and much more.

### Useful numbers

- $\mu$ Viscosity, $[\mu]$ = Pa s, is a material property, and usually depends on temperature.
- $\nu$ Kinematic viscosity, $\nu = \mu/\rho$, $[\nu] = \mathrm{m^2\ s^{-1}}$. Describes how momentum diffuses.
- $Pr$ Prandl-number, $Pr = \nu/\kappa$, where $\kappa$ is thermal diffusivity. A fluid with a small $Pr$ number diffuses heat more rapidly than it does momentum, and vice versa.
- $Re$ Reynolds number,

$$Re = \frac{\rho u L}{\mu}. \tag{3}$$

  If $Re$ is smaller than about 2200 the fluid flow is laminar, if it is higher, the flow is turbulent.

- $f$ Friction factor,

$$f = \frac{-4R}{\rho\tilde{u}^2}\frac{dp}{dx}, \tag{4}$$

  where $\tilde{u}$ is the mean velocity, $R$ is the pipe radius, and $\rho$ density of the liquid. For laminar flow (calculated), $f = 64/Re$, but for turbulent flow (from measurements) $f = 0.31645Re^{-1/4}$ (Figure 1).

## A Forces

### Pressure forces

Consider a small cube, with side lengths $\delta x_1$, $\delta x_2$, and $\delta x_3$. If the pressure changes along the $\mathbf{x}_1$ direction, we have that the force on an "up-stream" face is,

$$p\delta x_2\delta x_3,$$

and on the "down-stream" face is,

$$\left(p + \frac{\partial p}{\partial x_1}\delta x_1\right)\delta x_2\delta x_3.$$

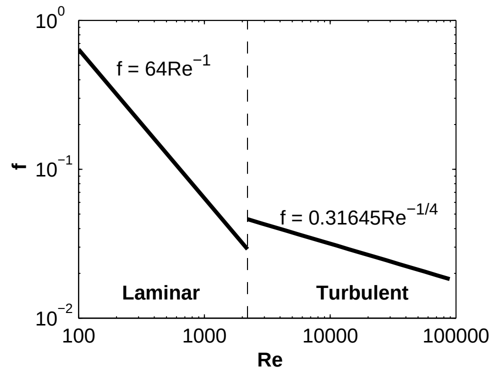


Figure 1: The friction factor $f$ as a function of Reynolds number $Re$.

The difference is then the net force acting in the $\mathbf{x}_1$ direction,

$$F_P = -\frac{\partial p}{\partial x_1}\delta x_1 \delta x_2 \delta x_3 = -\frac{\partial p}{\partial x_1}\delta V \tag{5}$$

### Body forces

In the flow of fluids, generally the only body force is gravity,

$$F_i = -\rho g_i, \tag{6}$$

where $g_i$ is the component of $\mathbf{g}$ parallel to $\mathbf{x}_i$-direction.

### Viscous forces

Viscous forces are given by the divergence of $\sigma_{ij}$ (Eq. 1),

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \mu \left( \frac{\partial^2 u_j}{\partial x_i^2} + \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right), \tag{7}$$

but the second term on the right hand side is $\partial u_i/\partial x_i = 0$ due to incompressibility. The viscous force is therefore,

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \mu \frac{\partial^2 u_j}{\partial x_i^2}. \tag{8}$$

This derivation has assumed incompressible fluids.

## B One dimensional channel flow

Consider a one-dimensional channel flow, where the fluid moves with velocity $u$ in the $\mathbf{x}$-direction, and the vertical coordinate is $\mathbf{z}$. There is no vertical velocity, and the horizontal velocity is a function of the vertical coordinate only, $u = u(z)$. The flow occurs due to 1) an applied pressure gradient, and/or 2) pre-described motion of one of the walls.

For a Newtonian fluid with constant viscosity $\mu$ the shear stress at any location in the channel is given by,

$$\tau = \mu \frac{\partial u}{\partial z}. \tag{9}$$

We can relate the change in shear stress to changes in pressure,

$$\frac{d\tau}{dz} = \frac{dp}{dx}. \tag{10}$$

This follows from the fact that the momentum of the element $\delta V$ is not changing, and thus the sum of the viscous and pressure force has to be zero. Substituting into Equation (1) we find,

$$\mu \frac{d^2u}{dz^2} = \frac{dp}{dx}, \tag{11}$$

which integrated twice gives,

$$u = \frac{1}{2\mu}\frac{dp}{dx} z^2 + c_1 z + c_2. \tag{12}$$

### Solutions

The solution when $u = 0$ at $z = h$, and $u = u_0$ at $z = 0$, we get,

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(z^2 - hz\right) - \frac{u_0 z}{h} + u_0. \tag{13}$$

The solution when $dp/dx = 0$, $u_0 \neq 0$, that is, the velocity of the upper plate is $u_0$, is called *Couette* flow,

$$u = u_0 \left(1 - \frac{z}{h}\right). \tag{14}$$

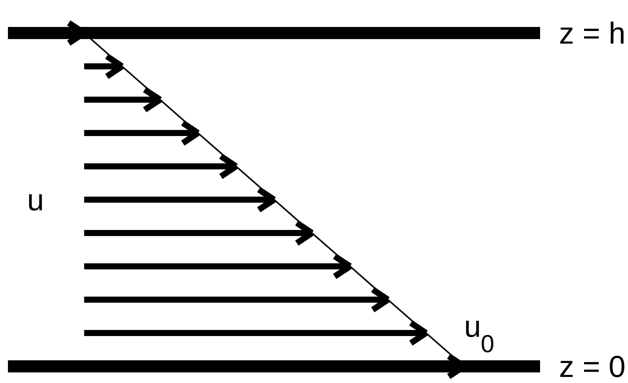


Figure 2: Coutte flow.

If $dp/dx \neq 0$, but $u_0 = 0$, we get,

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(z^2 - hz\right), \tag{15}$$

where $h$ is the height of the channel (Figure 3). This is called plane Poiseuille flow (Segel, 1987).

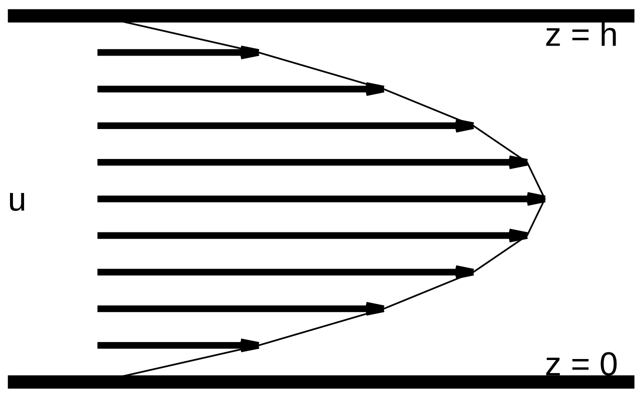


Figure 3: Plane Poiseuille flow.

## C Pipe flow

Consider a pipe of radius $R$, and length $l$. We will only consider changes as a function of radius $r$, assuming that the density, and velocity depend only on $r$.

Force balance considerations lead to,

$$\tau = \frac{r}{2}\frac{dp}{dx}. \tag{16}$$

In cylindrical coordinates the shear stress is proportional to the radial gradient of velocity,

$$\tau = \mu\frac{du}{dr}. \tag{17}$$

Using Equations (16) and (17) we get,

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx}, \tag{18}$$

which can be integrated to give,

$$u = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2), \tag{19}$$

where we have assumed that $u = 0$ at $r = R$. The velocity profile thus obtained is known as *Poiseuille flow* (Figure 4).

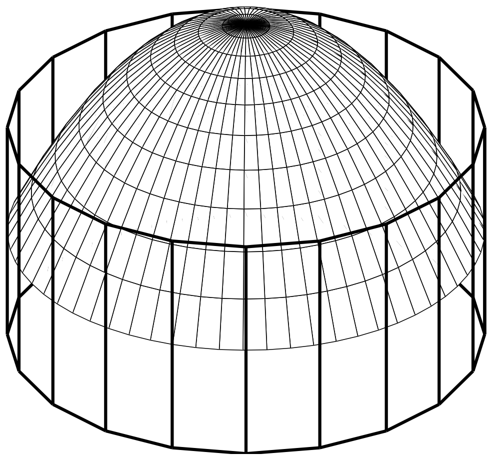

Figure 4: Velocity surface in pipe flow.

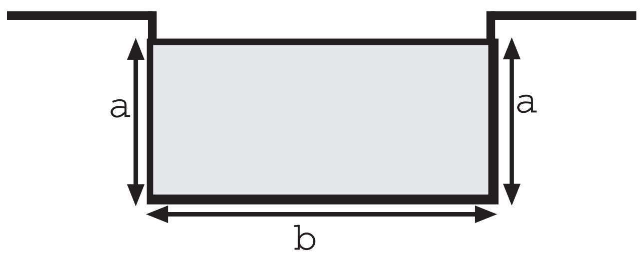


Figure 5: The hydraulic radius for this open channel is $R = \frac{a \cdot b}{a+b+a}$.

## D River flow

River flow is rarely laminar, more often close to, or fully, turbulent.

*Manning equation* gives the average velocity of flow of water in an open-channel,

$$\nu = \frac{1}{n} R^{2/3} S^{1/2}, \tag{20}$$

where, $\nu$ is the average velocity (m s$^{-1}$), $R$ the hydraulic radius (m), $S$ the slope of the water surface, and $n$ is the Manning roughness coefficient. The hydraulic radius is the cross-sectional area divided by the wetted perimeter, as shown in Figure 5. The Manning coefficient $n$ varies from 0.012 for smooth concrete to 0.05 for mountain streams with rocky beds (Fetter, 2001, p. 59).

### Short derivation

Consider a channel segment as shown in Figure 5 with length $L$. The force acting on the unit of water in the direction of flow is the downslope component of its weight,

$$F_g = \rho g a L b \sin\alpha.$$

For small angles, $\sin\alpha \simeq$ slope $S$. The resisting force is the stress per unit area, $\tau$, times the boundary area over which the stress is applied,

$$F_s = \tau(2a + b)L.$$

Since there is no acceleration, $F_g = F_s$, or

$$\rho g a L b S = \tau(2a + b)L. \tag{21}$$

Since $A = a \cdot b$ is the area of the cross section, $\rho g A S = \tau(2a + b)$, or

$$\tau = \rho g S \frac{A}{2a + b}. \tag{22}$$

The ratio

$$\frac{A}{2a + b},$$

is defined as the hydraulic radius $R$.

It has been shown experimentally that in turbulent flow the resistance to flow is proportional to the square of the mean flow velocity, $\tau \propto v^2$, provided that the boundary does not change as velocity is varied.

Using this with Eq. 22 from above we get,

$$v^2 = kg\rho RS.$$

The contant $\sqrt{k\rho g}$ is called $C$, the Chezy coefficient (Leopold and others, 1995). Thus the velocity is,

$$v = C\sqrt{RS}. \tag{23}$$

A variation of this, the Manning equation (Eq. 20), is based on field and experimental observations.

## READING ASSIGNMENT

Good sections to read are T+S 6.1 to 6.8 (including 6.8)

# PROBLEMS

**1** Consider the steady, unidirectional flow of a viscous fluid down the upper face of an inclinded plane. Assume that the flow occurs in a layer of constant thickness $h$, as shown in Figure 6. Show that the velocity profile is given by,

$$u = \frac{\rho g \sin\alpha}{2\mu}(h^2 - z^2),$$

where $z$ is the coordinate measured perpendicular to the inclined plane, $\alpha$ is the inclination of the plane to the horizontal, and $g$ is the acceleration of gravity. First show that,

$$\frac{d\tau}{dz} = -\rho g \sin\alpha,$$

and then apply the no-slip condition at $z = h$ and the free-surface condition, $\tau = 0$, at $z = 0$. What is the mean velocity in the layer? What is the thickness of a layer whose rate of flow down the incline is $Q$?

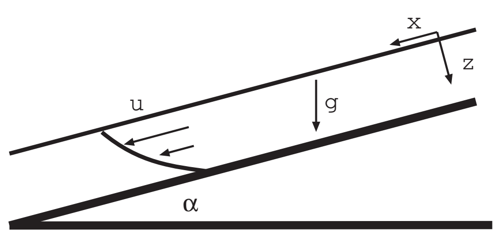


Figure 6: Viscous flow down inclined plane.

**2** Show that the mean velocity in a one dimensional channel flow is given by

$$\overline{u} = -\frac{h^2}{12\mu}\frac{dp}{dx} + \frac{u_0}{2}.$$

**3** Derive a general expression for the shear stress $\tau$ at any location $y$ in the channel. What are the simplified forms of $\tau$ for Couette flow and for the case $u_0 = 0$?

**4** Find the point in the channel at which the velocity is a maximum.

**5** For an asthenosphere with a viscosity $\mu = 4 \cdot 10^{19}$ Pa s and a thickness $h = 200$ km, what is the shear stress on the base of the lithosphere if there is no counterflow ($\partial p/\partial x = 0$)? Assume $u_0 = 50$ mm a$^{-1}$ and that the base of the asthenosphere has zero velocity.

**6** Assume that the base stress obtained in the previous problem is acting on 6000 km of lithosphere with a thickness of 100 km. What tensional stress in the lithosphere ($h_L = 100$ km) must be applied at a trench to overcome this basal drag?

**7** Determine the Reynolds number for the asthenosphereic flow considered in the previous two problems. Base the Reynolds number on the thickness of the flowing layer and the mean velocity ($u_0 = 50$ mm a$^{-1}$ and $\rho = 3200$ kg m$^{-3}$). This problem illustrates that the viscosity of mantle rock is so high that the Reynolds number is generally small. Thus mantle flows are laminar.

**8** A spring has a flow of 100 liters per minute. The entrance to the spring lies 2 km away from the outlet and 50 m above it. If the aquifer supplying the spring is modeled according to Figure 7, find its cross-sectional radius. What is the average velocity? Is the flow laminar or turbulent?

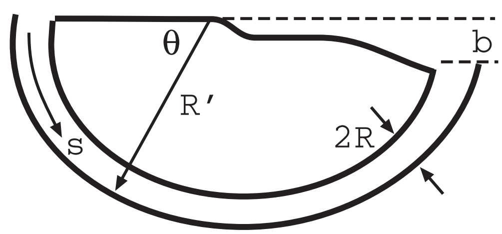


Figure 7: A semicircular aquifer with a circular cross section. A hydrostatic head $b$ is available to drive the flow.

**9** a) Find the maximum velocity $u_{max}$ in Poiseuille flow, b) find the flux, using $Q = \int_0^R 2\pi r u(r) dr$, c) find the mean velocity $\tilde{u}$, d) show that $\tilde{u} = u_{max}/2$.

**10** Explain "Aquifer flows" and "Volcanic pipe flow"

**11** Explain "Conservation of fluid and force balance"

Fetter, C. W. 2001. *Applied Hydrogeology*. Prentice Hall, New Jersey, fourth edition.

Leopold, L. B., M. G. Wolman and J. P. Miller. 1995. *Fluvial Processes in Geomorphology*. Dover, New York.

Segel, L. A. 1987. *Mathematics applied to continuum mechanics*. Dover, New York, 1st edition.