An Overview of Theories of Learning in Mathematics Education Research

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This paper is an attempt to provide some background on the various approaches to understanding how people learn and the application of this understanding to teaching. Please note that I only briefly mention the discussion of the theory and associated framework that I personally espouse since it is the topic of another session.

I draw a distinction between learning theories and frameworks used for teaching. Learning theories attempt to explain all the factors involved as an individual grows in knowledge and understanding, while epistemological frameworks guide the researcher in investigating certain aspects of learning. The theories of Piaget, Skinner, Vygotsky, and Lakoff and Núñez are presented briefly to set the stage for the epistemological frameworks. In general, these frameworks are based on (and possibly extend) an existing theory of learning. I describe the frameworks of the leading researchers in mathematics education and, as frameworks tend to evolve, try to present them in a fairly current state. A version of this paper was used as part of a preliminary examination for doctoral research in 1996, so many of the references may seem out of date. While more current citations might be available, I only updated those portions that have been modified in recent years.

An additional resource is *Theories of Mathematical Learning* (1996), edited by Leslie P. Steffe and Perla Nesher, which contains the proceedings of Working Group 4 from the 1992 International Congress on Mathematical Education. That book has a broader scope in addressing learning from pre-kindergarten through post-secondary educational situations. Thus its intersection with the following is rather small.

1 Theories of Learning

1.1 Behaviorism

Behaviorism attempts to explain learning through the observable interactions of the learner with the environment, without inferring anything that is going on inside the learner. It is based on the stimulus-response model of Skinner, which attempts to apply the methods of science to the study of human learning. Learning can be controlled by affecting the variables of the *situation*, the *behavior*, and the *consequences* of the behavior (Bell, 1978). Behaviorism was the prevalent theory of learning from the 1950's through the 1970's, giving rise to use of empirical, quantitative studies of learning. Although the behaviorism movement has largely passed from educational research (some may say proved false), von Glasersfeld (1995) notes "its key notions are still alive and active in the minds of many educators" (p. 178).

1.2 Constructivism

Constructivism differs from behaviorism at the most fundamental question: from where does knowledge come? Constructivists claim that a person's knowledge is constructed by her or himself in the setting of some environment. Piaget's theory of learning is often referred to as a "stage theory" in which the mental development of a child is described. This is a reference to his observations that children demonstrate different modes of thought at different ages and that it seems that they must pass through them in the same order. The stages are pre-operational, concrete operational, and formal operational thinking. Generally, when a child hits adolescence he or she is ready to move to formal operational thinking. In this light, the work of Piaget does not seem applicable to research in undergraduate mathematics education. (Although it may cause despair in those instances when we observe students who have remained at the concrete operational stage.) However, this stage theory was not the theory of learning on which he spent his life's work.

As a biologist, Piaget (1974/1980) developed a theory of evolution which describes phenocopy as the result of establishing equilibrium on the level of the genome. As a parallel construction, he describes cognitive epigenesis occurring through the student establishing equilibrium through the processes of accommodation and assimilation. According to this theory, a student can progress from one level of understanding to a higher level through reflective abstraction. Piaget (1983/1989) also gave stages for this process as he observed intra-, inter-, and trans-operational phases of thought in his experiments with children. The *intraoperational* stage is characterized by the student's focus on the objects of a transformation in isolation from other objects and actions. *Interoperational* thought occurs as the student builds relations between these actions through reflective abstraction. Finally, in the *transoperational* stage the student reflects on these interrelations and is able to transform them as objects in a larger system.

This theory of learning is generally incorporated in an epistemological framework in one of two ways. The framework can be built around finding descriptions of students' understanding which correspond to the levels of thought as described above. Another framework may focus on the movement from one level to the next, as researchers look for the mechanisms of transmission. While the points of view of the frameworks may differ regarding investigations, pedagogically they tend to converge. These recognize that as students construct their knowledge, they need to be placed in situations which allow for the movement from one level to the next.

1.3 Socio-culturalism

The work of Vygotsky has gained increased recognition in the mathematics education community. His theory states that the development of a student's intelligence "results from social interaction in the world and that speech, social interaction, and co-operative activity are all important aspects of this social world" (Sutherland, 1993, p. 104). The student uses language to build cognitive tools over which he or she has conscious control. The role of the teacher is central to this theory in that the teacher must convey the relationship between the sign and the meaning of the sign. Vygotsky described a *Zone of Proximal Development* as the distance between the level of development of a student (working on problems) and her or his level of potential development (working with an adult). This zone allows the adult to be the "tool holder," that is, having conscious control of the concept, for the child until he or she is able to internalize external knowledge. This process is referred to as *scaffolding* (Vygotsky, 1978, 1986).

It seems well to note here two comparisons of Vygotsky and Piaget. Sutherland (1993) notes the similarity of views in which the child has an active role in learning. Indeed, Vygotsky's notion the child needing to internalize external knowledge is very constructive. However, Sutherland cites a lengthy passage from Vygotsky addressing his differences with Piaget's views on the role of speech. She also noted their different emphases on the role of the teacher, implying that Piaget does not address teaching specifically. Confrey, in a three-part article (1994, 1995a, 1995b), seeks to combine these two theories, supplementing them with feminist scholarship in her revised theory. She describes at length both theories of learning and itemizes their respective strengths and limitations. Confrey seems to exaggerate Piaget's view of the student constructing understanding for her or himself to the view that she or he must do it in isolation. Also, while Vygotsky values the role of social interaction, his socio-cultural perspective can limit diversity in the classroom. These authors are comparing different things. While Sutherland looks at the theories themselves, Confrey compares implementations of the theory.

1.4 Theory of Embodied Mathematics

Lakoff and Núñez (2000) offer a view of learning mathematics based on the notion "that conceptual metaphor plays a central, defining role in mathematical ideas within the cognitive unconscious". Their work extends findings in cognitive science to describe how many mathematical concepts arise in the minds of learners through the use of metaphor and blending.

2 Epistemological Frameworks

The following set of frameworks is my list of those authors who are leaders in mathematics education research and should not be interpreted as complete. I describe each framework briefly, giving its distinctive characteristics and theoretical background. While there is no explicit attempt to compare and contrast frameworks, I have categorized them into subsections.

2.1 Actions, Processes and Objects

Dubinsky (1991) has developed an epistemological framework referred to as Action-Process-Object-Schema, or APOS. The framework considers the development of a mathematical concept as moving from an action (intraoperational) to a process (interoperational) via a type of reflective abstraction called interiorization. The resulting process can be encapsulated into an object (transoperational). The framework notes that objects constructed in this manner can be de-encapsulated back to the process when needed. Schemas are constructed by coordinating processes and actions and can also be thematized into objects (Asiala et al., 1996).

This framework results in descriptions of the mental constructions a student makes to come to understand a concept. These descriptions are called genetic decompositions. Instructional treatments are devised which may bring the student to make the constructions described in the genetic decomposition. These treatments generally involve the use of a mathematical programming language on computers, cooperative learning strategies, and alternatives to lecturing (Asiala et al., 1996).

Sfard (1991) also employs a framework which seeks to describe mental constructions in two ways: structurally (as objects) or operationally (as processes). Her description of the reflective abstractions necessary to move from process to object are interiorization, condensation, and reification. This framework posits that objects and processes have a dual nature, rather than a dichotomous relation.

Thompson (1994a) also describes the development of concepts in terms of processes and objects. He describes the student's image of a concept as figural knowledge or a metaphor. He distinguishes this image from the concept image of Vinner (discussed below) and from schemas (although not as explicitly):

> Vinner's idea of concept image focuses on the coalescence of mental pictures into categories corresponding to conventional mathematical vocabulary, while the notion of image I've attempted to develop focuses on the dynamics of mental operations. The two notions of image are not inconsistent, they merely have a different focus. (Thompson, 1994a, p. 231)

Thompson's framework proposes the development of instruction which nurtures and extends students' images in mathematics.

2.2 Concept Image and Concept Definition

Vinner (1992), along with Dreyfus and Tall, has distinguished the concept image and the concept definition of a student. The concept image is a collection of all the objects, processes, and schemas possessed by a student which are associated with the concept. This may include mental pictures, misconceptions and properties (Dreyfus, 1990). The tension between this image and the mathematical definition is the focus of the framework. By describing a student's concept image, one is able to help the student reorganize it into a more coherent structure which is consonant with the concept definition (Thompson, 1994b).

Tall (1989) uses a framework based on the concept image which includes the use of generic organizers and computer modeling. A *generic organizer* is

> an environment (or microworld) which enables the learner to manipulate *examples* and (if possible) *non-examples* of a specific mathematical concept or a related system of concepts." (Tall, 1989, p. 39 [emphasis in original])

Pedagogically, this framework allows the learner to build from cognitive bases rather than mathematical foundations.

2.3 Multiple Representations

Kaput (1985) explores student understanding of mathematical concepts by observing the student's ability to represent the concept. Five uses of representation in mathematics are (a) mental, (b) computer, (c) explanatory, (d) mathematical, and (e) symbolic—as with external mathematical notation. Kaput (1991) extends the idea of the student's notational system as an architecture which organizes her or his mathematical experience. The notational system is also used to represent the mental structures in the physical world. The analogy to architecture is extended to say that, as an educator, one can design elegant and functional representational systems, possibly with computer contexts, for students. He describes the existence of three "worlds"; material, mental, and social (or consensual). The mathematics classroom is located in the consensual world where change can take place rapidly. Thus learning takes place in the relationship between the student representing and interpreting mathematics and in the shared meaning with others.

A related framework is that of Greeno, who also considers a conceptual domain for mathematical thinking. It should be noted that these frameworks are not explicitly based on any of the three theories previously given (neither is that of Schoenfeld, below). Battista (1994) describes Greeno's environmental/model view in which the conceptual domain is considered an environment with spatial properties. Reasoning is accomplished by interacting with mental models in this environment. Battista comments on the analogous nature of research into spatial thinking as a way to see this view of Greeno.

2.4 Problem-solving

Schoenfeld (1985) perceives a student's mathematical understanding as the ability to solve problems. He has identified four categories of knowledge which influence this ability. The first is *resources,* the student's foundation of basic mathematical knowledge. The student also needs *heuristics* which are a set of broad problem-solving techniques. Third is *control* over the resources, i.e., whether or not a student selects necessary resources. Finally, the student brings *belief systems* to bear on the problem situation. All four of these categories must be taken into account when explaining a student's behavior in mathematics.

2.5 Problématique

Balacheff (1990) defines a *problématique* as "a set of research questions related to a specific theoretical framework" (p. 258). The theoretical framework is based on constructivism and the hypothesis that the source of mathematical knowledge is problem situations. This epistemological framework also takes into account the role of the teacher in the classroom and the social aspect of mathematics. Pedagogically, it proposes the use of a didactical process of placing the student in various situations. These include situations of institutionalization, of validation, of formulation, for decision, for communication, and for action. The key to this framework is to bring the student to *epistemological obstacles* which he or she must overcome to (re-)construct mathematical knowledge (Artigue, 1992). Sierpinska (1995) addresses both the role of the teacher and some social aspects of learning in a discussion of transfer of knowledge and contextual problem situations.

2.6 Radical Constructivism

> What is radical constructivism? It is an unconventional approach to the problems of knowledge and knowing. It starts from the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. What we make of experience constitutes the only world we consciously live in. It can be sorted into many kinds, such as things, self, others, and so on. But all kinds of experience are essentially subjective, and though I may find reasons to believe that my experience may not be unlike yours, I have no way of knowing that it is the same. The experience and interpretation of language are no exception. (von Glasersfeld, 1995, p. 1)

Von Glasersfeld has taken the theory of Piaget and extended it. He and Steffe have built an epistemological framework for research in mathematics education based on this view. The framework involves using teaching experiments and cooperative learning to explore the ways children learn. They propose building models of the students' mathematical knowledge in terms of schemes of actions and operations (Cobb & Steffe, 1983).

2.7 Theoretical Mixtures

Cobb (1995) has adopted the language of Vygotsky's theory into his framework based on the work of Steffe and von Glasersfeld. As mentioned above, Confrey (1995b) has also combined the theories of Piaget and Vygotsky and extended them into a framework which addresses issues of gender equity along with mathematical knowledge.

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