

Chapter 2: HOW DID THE ROMANS DO MULTIPLICATION? (44 B.C.)

The Romans had a clumsy way of writing numbers that made multiplication next to impossible to carry out. Just try to verify the result of the multiplication

$$\mathbf{XIX \times XXVII = DXIII}$$

and see for yourself what the problem is. But human ingenuity came to rescue and they invented a method whereby multiplication could be reduced to addition, doubling and halving — something that the Romans could cope with.

This method was forgotten and re-invented later; it is also known as the “Russian Method”.

For example, you want to find 58×249 . Write the two numbers next to one another; then double one and halve the other. (It is preferable, though not mandatory, to halve the smaller and double the larger.) If in halving there is a remainder, ignore it. Repeat the process until the number in the halving column is reduced to 1. Then in the doubling column cross out every number that stands opposite to an even number in the halving column, and sum up the rest. The sum is exactly the required product:

| | |
|----|-------------------------|
| 58 | 249 |
| 29 | 498 |
| 14 | 996 |
| 7 | 1992 |
| 3 | 3984 |
| 1 | <u>7968</u> |
| | 14442 = 58×249 |

Exercises:

1. By way of checking, do the calculation again, this time halving the larger and doubling the smaller number. Do you see now why it is preferable to halve the smaller number, while doubling the larger?
2. Use the Roman method of multiplication to calculate the squares of consecutive integers. Fill the blanks in the table

| | | | | | | | | | | | | | | | | | |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| n^2 | 1 | 4 | 9 | | | | | | | | | | | | | | |

We have the important identity

$$a^2 - b^2 = (a + b)(a - b)$$

We check its validity by using the Roman way of multiplication and the table of Exercise 2, for $a = 10, b = 7$. $(a + b)(a - b)$:

$$\begin{array}{r} 3 \quad 17 \\ 1 \quad \underline{34} \\ \hline 51 \end{array}$$

$$a^2 - b^2 : 100 - 49 = 51.$$

Exercises:

3. Check the validity of the identity also for
 - (i) $a = 9, b = 5$
 - (ii) $a = 8, b = 6$
 - (iii) $a = 7, b = 6$
4. Use the Roman way of multiplication to calculate the cubes of the consecutive integers. Fill the blanks in the table

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| n^3 | | | | | | | | | | |

Example. Calculate $11 \ 0 \ 256$ one way and check your calculation the other way. Answer:

| | | | | | |
|--|----|-----------------|--|-----|-------------|
| | 11 | 256 | | 256 | 11 |
| | 5 | 512 | | 128 | 22 |
| | 2 | 1024 | | 64 | 44 |
| | 1 | <u>2048</u> | | 32 | 88 |
| | | 2816 | | 16 | 176 |
| | | | | 8 | 352 |
| | | | | 4 | 704 |
| | | | | 2 | 1408 |
| | | | | 1 | <u>2816</u> |
| | | | | | 2816 |

Powers of 2

We introduce the abbreviation $b^n = b \times b \times \dots \times b$ (n times). b is called the *base*, n is called the *exponent*, and b^n is called the *power*. The table of the powers of 2:

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-------|---|---|---|---|----|----|----|-----|-----|-----|------|------|
| 2^n | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 |

Suggestion for the pupils. It is a good idea to commit at least the first ten powers of 2 to memory. The powers of 2 have many important applications. One of them is that they are the “milestones” in the binary number system, playing the same role as 10, 100, 1000, 10000,... play in the decimal.

Note that 2^{10} is easy to remember: its digits are just the first three powers of 2, namely: 1, 2, 4 in the same order. Since that 2^{10} is slightly greater than 10^3 , it must be a 4-digit number, and 0 is one of the digits: it has to be the second digit. If you remember this, then you will also remember that $2^{11} = 2048$ (the digits of which are consecutive powers of 2 as well).

Exercises:

5. Calculate 128×9 one way and check the calculation another way.
6. Calculate 512×10 one way and check the calculation another way.
7. Calculate 32×64 one way and check the calculation another way.

The three Exercises above show something interesting: we have to cross out all the doublings with the exception of the last one opposite 1 and, as a consequence, the addition part of the calculation is omitted. The question is to find all the cases when this happens. It is not hard to see that this happens if, and only if, the halving column is headed by a power of 2. In this case the number of halvings involved is just the exponent in the power of 2.

Shortcut

This suggests that a shortcut in calculating products via the Roman way, in case one of the factors is a power of two, is available. We make the column headed by 2^n the halving column regardless of which factor is the larger, and double the number heading the other column n times. The last doubling is the product. We may skip the adding: the results of all the other doublings stand opposite to an even number.

Exercises:

8. Use the shortcut to calculate the product of 11 with 512, 1024, 2048 and check.
9. Fill the blanks in the table using decimal fractions

| | | | | | | | | | | | |
|-----------|---|-----|---|---|---|---|---|---|---|---|----|
| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $(1/5)^n$ | 1 | 0.2 | | | | | | | | | |

and compare the second row to that of the previous table of the powers of 2. What do you see? Provide an explanation.

10. “TWO FOR THE PRICE OF ONE” special. Such deals in mathematics are rare, but here is one: “Memorize the powers of 2; then *for free*, you have also memorized the powers of $1/5$ ”. Explain what it means and why it works.
11. Fill the blanks in the table

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| 5^n | 1 | 5 | | | | | | | | | |

and, using decimal fractions, also in the table

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------|---|-----|---|---|---|---|---|---|---|---|----|
| $(1/2)^n$ | 1 | 0.5 | | | | | | | | | |

Compare the second rows of the two tables. What do you see? Provide an explanation.

12. Here is another “two for the price of one special”: memorize the powers of 5; then for free you have also memorized the powers of $1/2$. Explain what it means and why it works.
13. Formulate a rule describing another shortcut how to multiply an arbitrary number with $2^n + 1$.
14. Formulate a rule describing yet another shortcut how to multiply an arbitrary number with an odd multiple of a power of 2.
15. Using the shortcut of Exercise 15, calculate the product 192×432 and check.
16. Using the shortcut of Exercise 14, calculate the products of 999 with 65, 129, 257, 513, and 1025. Check.
17. The formula $(1/5)^{n+1} = \frac{2(1/5)^n}{10}$ can be used to calculate quickly the consecutive powers of $1/5$. Explain how, and prove.
18. The formula $2(5^n) = 10(5^{n-1})$ can be used in checking quickly the calculation of the consecutive powers of 5. Explain how, and prove.
19. What happens to a product of two factors if we multiply one factor by 2^n and divide the other by 5^n ?
20. Explain why the Roman method of calculating products always works. (Don’t get discouraged: while the answer to this question is not obvious, it is not too hard either.)