Group Theory

Solutions to Problem Set 2 October 26, 2001

1. Suppose that e is the right identity of a group G,

$$ge = g \tag{1}$$

for all g in G, and that e' is the left identity,

$$e'g = g \tag{2}$$

for all g in G. The choice $g = e'$ in the first of these equations and $g = e$ in the second, yields

$$e'e = e' \tag{3}$$

and

$$e'e = e \tag{4}$$

respectively. These equations imply that

$$e' = e \tag{5}$$

so the left and right identities are equal. Hence, we need specify only the left *or* right identity in a group in the knowledge that this is *the* identity of the group.

2. Suppose that a is the right inverse of any element g in a group G,

$$ga = e \tag{6}$$

and a' is the left inverse of g,

$$a'g = e \tag{7}$$

Multiplying the first of these equations from the left by a' and invoking the second equation yields

$$a' = a'(ga) = (a'g)a = a \tag{8}$$

so the left and right inverses of an element are equal. The same result could have been obtained by multiplying the second equation from the right by a and invoking the first equation.

3. To show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$, we begin with the properties of the inverse. We must have that

$$(ab)(ab)^{-1} = e$$

Left-multiplying both sides of this equation first by a^{-1} and then by b^{-1} yields

$$(ab)^{-1} = b^{-1}a^{-1}$$

4. For elements $g_1, g_2, \ldots, g_n$ of a group G, we require the inverse of the n-fold product $g_1 g_2 \cdots g_n$. We proceed as in Problem 2 using the definition of the inverse to write

$$(g_1 g_2 \cdots g_n)(g_1 g_2 \cdots g_n)^{-1} = e$$

We now follow the same procedure as in Problem 2 and left-multiply both sides of this equation in turn by $g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}$ to obtain

$$(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} \cdots g_2^{-1} g_1^{-1}$$

5. We must prove two statements here: that for an Abelian group G, $(ab)^{-1} = a^{-1}b^{-1}$, for all a and b in G, and that this equality implies that G is Abelian. If G is Abelian, then using the result

of Problem 3 and the commutativity of the composition law, we find

$$(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$$

Now, suppose that there is a group G (which we must not *assume* is Abelian, such that

$$(ab)^{-1} = a^{-1}b^{-1}$$

for all a and b in G. We now right multiply both sides of this equation first by b and then by a to obtain

$$(ab)^{-1}ba = e$$

Then, left-multiplying both sides of this equation by (ab) yields

$$ba = ab$$

so G is Abelian. Hence, we have shown that G is an Abelian group if and only if, for elements a and b in G, $(ab)^{-1} = a^{-1}b^{-1}$.

6. To construct the multiplication table of a four-element group $\{e, a, b, c\}$ we proceed as in Section 2.4 of the course notes. The properties of the unit of the group enable us to make the following entries into the multiplication table:

	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

We now consider the product aa. This cannot equal a, since that would imply that $a = e$, but it can equal any of the other elements, including the identity. However, this leads only to two *distinct* choices for the product, since the apparent difference between $aa = b$ and $aa = c$ disappears under the interchange of the

	e	a	b	c
e	e	a	b	c
a	a	e		
b	b			
c	c			

	e	a	b	c
e	e	a	b	c
a	a	b		
b	b			
c	c			

(1)

labelling of b and c. Thus, at this stage, we have two distinct structures for the multiplication table:

We now determine the remaining entries for these two groups. For the table on the left, we consider the product ab. From the Rearrangement Theorem, this cannot equal a or or e, nor can it equal b (since that would imply $a = e$). Therefore, $ab = c$, from which it follows that $ac = b$. According to the Rearrangement Theorem, the multiplication table becomes

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c		
c	c	b		

(2)

For the remaining entries of this table, we observe that $b^2 = a$ and $b^2 = e$ are equally valid assignments. These leads to two multiplication tables:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

(3)

Note that these tables are distinct in that there is no relabelling of the elements which transforms one into the other.

We now return to the other multiplication table on the right in (1). The Rearrangement Theorem requires that the second row must be completed as follows:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c		
c	c	e		

(4)

Again invoking the Rearrangement Theorem, we must have that this multiplication table can be completed only as:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(5)

Notice that all of the multiplication tables in (3) and (5) are Abelian and that the table in (5) is cyclic, i.e., all of the group elements can be obtained by taking successive products of any non-unit element.

We now appear to have three distinct multiplication tables for groups of order 4: the two tables in (3) and the one in (5). However, if we reorder the elements in the second table in (3) to $\{e, b, a, c\}$ and reassemble the multiplication table (using the same products), we obtain

	e	b	a	c
e	e	b	a	c
c	b	a	c	e
b	a	c	e	b
a	c	e	b	a

(6)

which, under the relabelling $a \mapsto b$ and $b \mapsto a$, is identical to (5). Hence, there are only two distinct structures of groups with four elements:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(7)

7. (a) All of the products involving the identity are self-evident. The only products that must be calculated explicitly are a^2, ab, ba, and b^2. These are given by

$$\pi_a\pi_a = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}\begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \pi_b$$

$$\pi_a\pi_b = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}\begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix} = \pi_e$$

$$\pi_b\pi_a = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}\begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix} = \pi_e$$

$$\pi_b\pi_b = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}\begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \pi_a$$

Thus, the association $g \to \pi_g$, for $g = e, a, b$ preserves the products of the three-element group.

(b) With the construction

$$\pi_g = \begin{pmatrix} e & a & b \\ g & ga & gb \end{pmatrix} \qquad (9)$$

for $g = e, a, b$, we can use the rows of the multiplication table on p. 19 to obtain

$$\pi_e = \begin{pmatrix} e & a & b \\ ee & ea & eb \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix}$$

$$\pi_a = \begin{pmatrix} e & a & b \\ ae & aa & ab \end{pmatrix} = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}$$

$$\pi_b = \begin{pmatrix} e & a & b \\ be & ba & bb \end{pmatrix} = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} \qquad (10)$$

Thus, the association $g \rightarrow \pi_g$ is one-to-one and preserves the products of the 3-element group. Hence, these groups are equivalent.

(c) The elements of the three-element group correspond to the **cyclic permutations** of S_3. In other words, given a reference order $\{a, b, c\}$, the cyclic permutations are $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$, yielding $\{b, c, a\}$, and then $b \mapsto c$, $c \mapsto a$, and $a \mapsto b$, yielding $\{c, a, b\}$.

(d) These elements correspond to the **rotations** of an equilateral triangle, i.e., the elements $\{e, d, f\}$ in Fig. 2.1.