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Grade 6 Math Circles
November 5/6, 2013
Multiplication

Introduction

At this point in your schooling you should all be very comfortable with multiplication. You know your 12×12 multiplication table and you have a strategy for when you are multiplying larger numbers – long multiplication. But in this lesson we will be looking at more visual ways of doing multiplication and some multiplication methods from around the world.

Japanese Multiplication Method

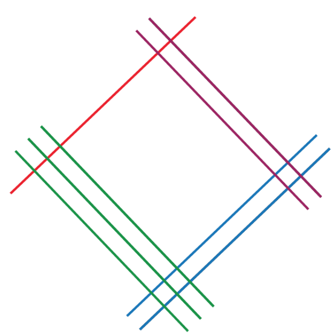
The Japanese method is a visual way to represent multiplication. It involves lines and intersections, and has recently become popular on the internet.

Examples

1. Let's evaluate 12×32 using the Japanese method.

   Begin by drawing 3 parallel lines to represent 12, the first number in the product. Draw one line, and then, a little further to the right, draw two more lines. These lines (red and blue below) represent the number 12. Similarly, draw 5 more parallel lines to cross the previous three – three lines on the left and two lines on the right. These will represent the number 32, the second number in the product (green and purple below).

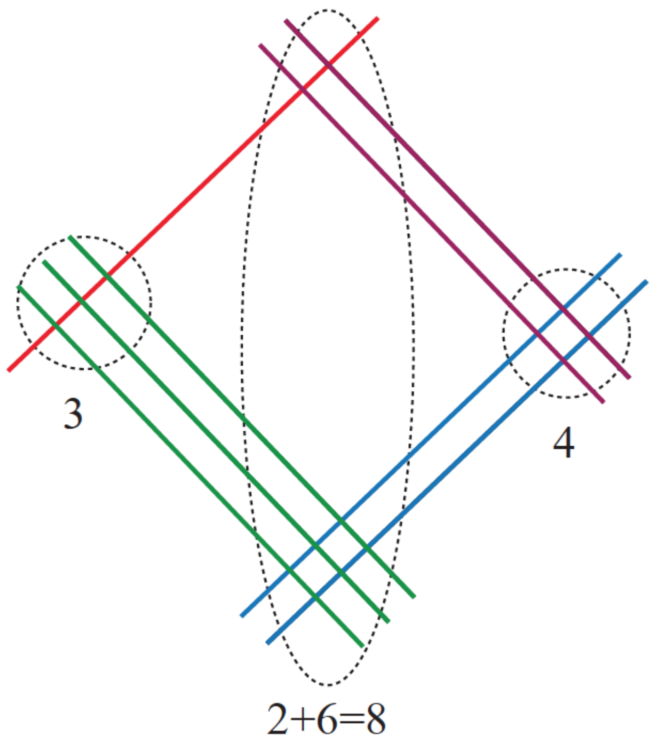
12×32



Now we must count how many times all of the lines intersect.

Begin by grouping the intersections vertically. That is, draw a loop around the group of intersections that is closest to the left side (where the red and green lines intersect). Then begin moving right. Draw a loop around the center intersections (the red and purple, and the blue and green). Finally, draw a loop around the intersections that are closest to the right side (where the blue and purple lines intersect).

Count how many intersections are in each loop.



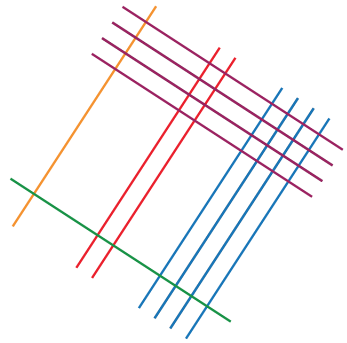
So there are 3, 8, and 4 intersections. By combining the numbers from left to right you get 384. As a matter of fact, 12 multiplied by 32 *is* 384! Thus $12 \times 32 = 384$.

Amazing, right?

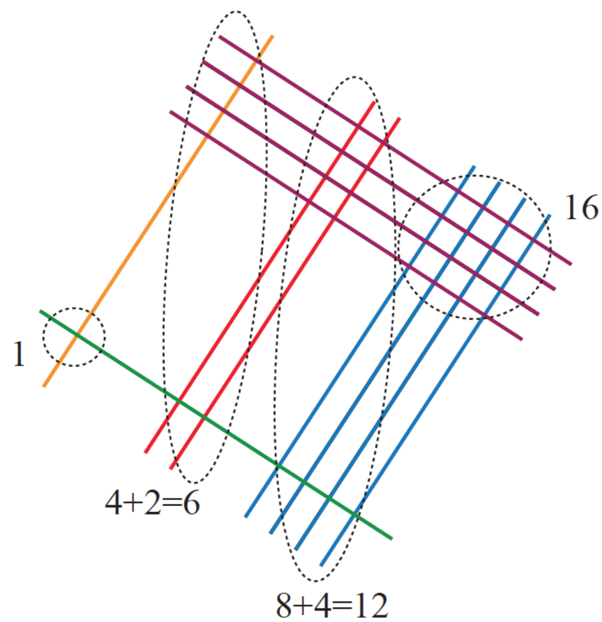
2. Let's evaluate 124×14 using the Japanese method.

   We must draw 7 parallel lines to represent the first number in the product, 124. Draw one line, then two more lines to the right, and then four more lines to the right, again. Then, drawing the lines in the opposite direction, represent 14 with one line, and then 4 more lines further to the right.

124×14



Group the intersections vertically and count every intersection in every group.



If you put the numbers together like we did in the last example you would get 161216, but this is not correct. Because 12 and 16 are 2-digit numbers you must carry the first digit to the group on its left, as you would with addition.

You can set it up like this:

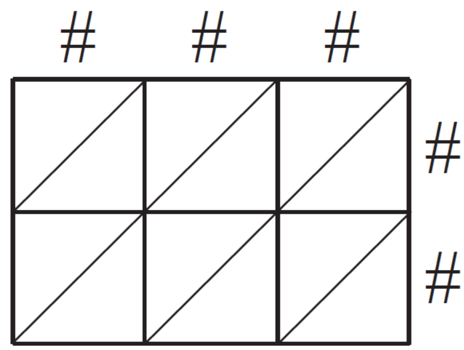
$$\begin{array}{rcccc} & 1 & & & \\ & & 6 & & \\ & & 1 & 2 & \\ + & & & 1 & 6 \\ \hline & 1 & 7 & 3 & 6 \end{array}$$

Thus $124 \times 14 = 1736$.

Lattice Multiplication Method

Lattice multiplication has been traced back the 13th century. It was first used by Arab, European, and Chinese mathematicians.

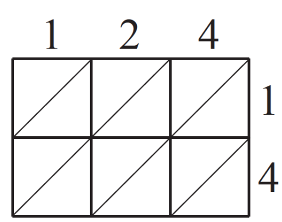
Under this method, a grid is drawn with the dimensions of the amount of digits in each number in the product you are evaluating. For example, the following grid would be used if evaluating a product of the form ### × ##.



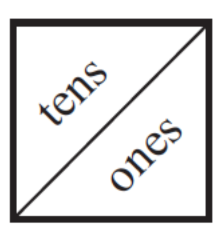
Example

Let's evaluate 124×14 using the lattice method.

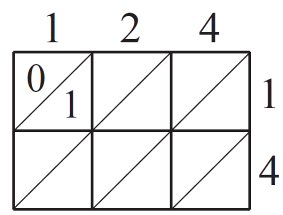
Use the grid above and label it with 124 along the top and 14 along the right side.



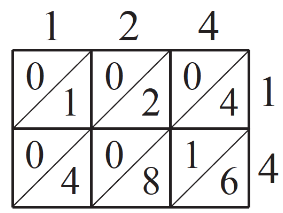
Fill every square with the product of the numbers labelled above and to the right of it. Place the digit in the tens position in the upper left triangle and the digit in the ones position in the bottom right triangle.



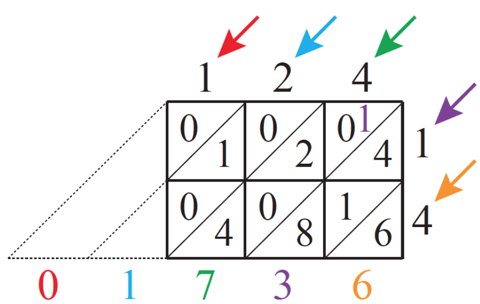
Fill the top left box with the result of 1×1, which is obviously 1. There is no digit in the tens column, so fill the upper left triangle with a zero.



Continue this with the other 5 products.



Now add the numbers in the diagonals, starting from the right. Write each sum at the bottom end of the corresponding diagonal. If the sum has two digits, carry the first into the diagonal to its left.



Just read the number out along the bottom edge! $124 \times 14 = 1736$.

Egyptian Multiplication Method

This method was used by Egyptian scribes. It does not require the multiplication table, only the ability to multiply and divide by 2, and to add.

Example

Let's evaluate 124×14 using the Egyptian method.

The Egyptian method uses two columns. First, write each number in the product at the head of each column. In the first column, begin with the number 1 and then double it to get 2 (in the second row). Then double 2 to get 4 in the third row. Continue doubling.

124	$\times$	14
1		
2		
4		
8		
16		
32		
64		

Stop when the next number would be greater than 124. Since $128 > 124$, stop at 64.

In the second column, begin with 14. For the second row, double 14 to get 28. For the third row, double 28 to get 56. Continue doubling.

124	$\times$	14
1		14
2		28
4		56
8		112
16		224
32		448
64		896

Now we must find the combination of numbers in the left column whose sum will equal 124.

The largest number less than or equal to 124 is 64: $124 - 64 = 60$
The largest number less than or equal to 60 is 32: $60 - 32 = 28$
The largest number less than or equal to 28 is 16: $28 - 16 = 12$
The largest number less than or equal to 12 is 8: $12 - 8 = 4$
The largest number less than or equal to 4 is 4: $4 - 4 = 0$

So $124 = 64 + 32 + 16 + 8 + 4$. Ignore all the rows in the table above if the number in the left column isn't in the previous sum. In this case, ignore the first 2 rows.

124	×	14
1		14
2		28
4		56
8		112
16		224
32		448
64		896

Add up all the remaining numbers in the right column.

$$56 + 112 + 224 + 448 + 896 = 1736$$

Thus $124 \times 14 = 1736$.

The Egyptian method also works if you choose to set up the question as 14×124.

14	×	124
1		124
2		248
4		496
8		992

Since $14 = 8 + 4 + 2$, ignore the first row, and sum the remaining numbers in the right column.

$$248 + 496 + 992 = 1736$$

Thus $14 \times 124 = 1736$.

Russian Peasant Multiplication Method

Just like the Egyptian method, the only skills you need to use the Russian method are the ability to multiply and divide by 2, and to add.

Example

Let's evaluate 124×14 using the Russian method.

The Russian method uses two columns. First, write each number in the product at the head of each column. In the first column, begin with the original number above the column (124) and then halve it to get 62 in the second row. Then halve 62 to get 31 in the third row. Continue halving. If you halve a number and it is not a whole number, always round down to the nearest whole number. For example, half of 31 is 15.5. Rounding 15.5 down will result in 15 (the entry in the fourth row).

124	×	14
124		
62		
31		
15		
7		
3		
1		

Stop when you reach 1 on the left side.

In the second column, begin with the number at the head of the column (14). For the second row, double 14 to get 28. For the third row, double 28 to get 56. Continue doubling.

124	×	14
124		14
62		28
31		56
15		112
7		224
3		448
1		896

Ignore all rows with even numbers in the left column. That is, we want to focus only on the rows with odd numbers in the left column.

124	×	14
124		14
62		28
31		56
15		112
7		224
3		448
1		896

Add up all the remaining numbers in the right column.

$$56 + 112 + 224 + 448 + 896 = 1736$$

Thus $124 \times 14 = 1736$.

The Russian method also works if you choose to set up the question as 14×124.

14	×	124
14		124
7		248
3		496
1		992

Since 7, 3, and 1 are all odd, ignore the first row and sum the remaining numbers in the right column.

$$248 + 496 + 992 = 1736$$

Thus $14 \times 124 = 1736$.

Distributive Property

Why do all of these different methods work?

They work because their underlying logic follows the distributive property.

The Distributive Property is used when multiplying a sum by a single term or another sum.

$a \times (b + c)$ $(a + b) \times (c + d)$

To evaluate $a \times (b + c)$, multiply every term in the sum by the constant in front. That is,

$$a \times (b + c) = a \times b + a \times c.$$

Examples

Evaluate $3 \times (2 + 5)$ using the distributive property.

$$\begin{aligned} 3 \times (2 + 5) &= 3 \times 2 + 3 \times 5 \\ &= 6 + 15 \\ &= 21 \end{aligned}$$

We can check that this is right by using our BEDMAS rules:

$$\begin{aligned} 3 \times (2 + 5) &= 3 \times 7 \\ &= 21 \end{aligned}$$

This might seem like more work, but it can be really useful when doing mental math:

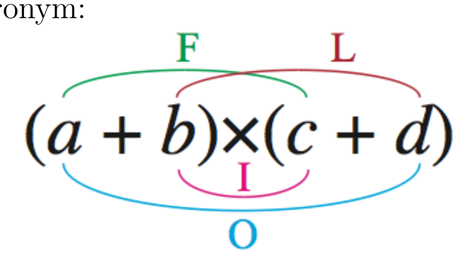
$$\begin{aligned} 8 \times 43 &= 8 \times (40 + 3) \\ &= 8 \times 40 + 8 \times 3 \\ &= 320 + 24 \\ &= 344 \end{aligned}$$

We can use this idea to also evaluate $(a + b) \times (c + d)$:

$$\begin{aligned} (a + b) \times (c + d) &= a \times (c + d) + b \times (c + d) \\ &= a \times c + a \times d + b \times c + b \times d \\ &= ac + ad + bc + bd \end{aligned}$$

As you can see, every term in the first sum is multiplied by every term in the second sum. To remember this, the mnemonic **FOIL** is used. FOIL is an acronym:

Firsts	Multiply the first term in each sum (a and c).
Outsides	Multiply the two terms on the outsides (a and d).
Insides	Multiply the innermost terms (b and c).
Lasts	Multiply the last term in each sum (b and d).



FOIL can also be represented visually in a chart:

	a	b
c	ac	bc
d	ad	bd

Examples

1. Evaluate $(2 + 5) \times (6 + 3)$ using the distributive property.

$$\begin{aligned}(2 + 5) \times (6 + 3) &= (2)(6) + (2)(3) + (5)(6) + (5)(3) \\ &= 12 + 6 + 30 + 15 \\ &= 63\end{aligned}$$

	2	5
6	12	30
3	6	15

$12 + 30 + 6 + 15 = 63$

We can check that this is right by using our BEDMAS rules:

$$\begin{aligned}(2 + 5) \times (6 + 3) &= 7 \times 9 \\ &= 63\end{aligned}$$

2. Evaluate 124×14 using the distributive property.

Before using the distributive property, you must expand 124 and 14.

$$\begin{aligned}124 \times 14 &= (100 + 20 + 4) \times (10 + 4) \\ &= (100)(10) + (100)(4) + (20)(10) + (20)(4) + (4)(10) + (4)(4) \\ &= 1000 + 400 + 200 + 80 + 40 + 16 \\ &= 1736\end{aligned}$$

Using the chart:

The chart must be modified to accommodate the expanded form of 124.

	100	20	4
10	1000	200	40
4	400	80	16

$$1000 + 200 + 40 + 400 + 80 + 16 = 1736$$

Problem Set

1. Evaluate the following using the Japanese multiplication method.

   (a) 51×32 (b) 27×303 (c) 421×222

2. Evaluate the following using the Lattice multiplication method.

   (a) 39×54 (b) 28×752 (c) 365×277

3. Evaluate the following using the Egyptian multiplication method.

   (a) 24×35 (b) 65×111 (c) 109×225

4. Evaluate the following using the Russian Peasant multiplication method.

   (a) 28×43 (b) 35×131 (c) 144×115

5. Evaluate the following using the distributive property.

   (a) 18×33 (b) 27×221 (c) 345×172

6. Evaluate the following.

   (a) $5 \times 21 \times 15$ (b) $12 \times 28 \times 17$ (c) $4 \times 38 \times 113$

7. $2.6 \times 1.4 = 3.64$, $3.14 \times 2.3 = 7.222$, and $5.27 \times 3.21 = 16.9167$. Can you develop an algorithm for multiplying decimal numbers using the lattice method?

8. If a factory produces 77 items in one minute, how many would it produce in 15 minutes?

9. Emily has 12 cases of oranges. Each case holds 9 boxes and there are 42 oranges in a box. How many oranges does Emily have?

10. A piece of ribbon, 7 metres long, is cut into fourteen pieces. Thirteen of these pieces are each 52 cm long. What is the length of the remaining piece?

11. * Using the Japanese method, multiply the following binary numbers in base 2. Check your answers by converting your numbers into base 10.

    (a) 110×101 (b) 111×1100 (c) 1011×1111