9. Groups with context–free multiplication table

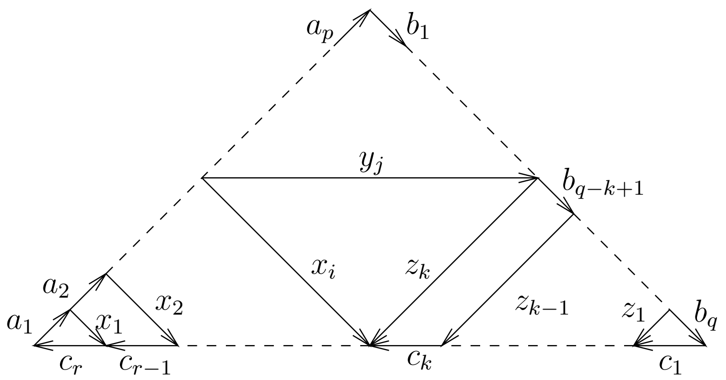
Let $\Sigma^* \to G$ be a choice of generators affording a rational language $R \subset \Sigma^*$ which projects bijectively to G. For example G could be any automatic group, or $Z \wr Z$. The multiplication table for G determined by R is $M = \{u\#v\#w \mid U, v, w \in R \text{ and } \overline{uvw} = 1\}$ where $\#$ is a new symbol not in Σ

Exercise 9.1. *If $\mathcal{C}$ is a full AFL and G has a context–free multiplication table in $\mathcal{C}$ with respect to one choice of generators, then it does for all choices.*

Theorem 9.2. *A finitely generated group has context–free multiplication table if and only if it is word-hyperbolic.*

Theorem 9.2 gives a language–theoretic characterization of a class of groups originally defined geometrically.

Here is a sketch of the proof. Suppose G is hyperbolic, and take a short–lex automatic structure for G based on a rational language R. In particular R projects bijectively to G. It suffices to show that M is context–free. The components in any triple $u\$v\$w$ project to a geodesic triangle in the Cayley diagram of G, and geodesic triangles in hyperbolic groups have a geometric structure indicated in the following figure.



In this figure u, v, w are written as words in the generators $w = a_1 \cdots a_p$, $v = b_1 \cdots b_q$, and $w = c_1 \cdots c_r$. The x's, y's and z's label group elements corresponding to differences between vertices of the Cayley diagram occurring along the edges of the triangle at equal distances from a triangle vertex. Because G is hyperbolic, finitely many group elements suffice for all geodesic triangles; and we take these group elements together with S as the non–terminals in a context–free grammar. The productions are of the following forms.

$$S \to a_1 x_1 c_r \quad x_1 \to a_2 x_2 c_{p-1} \quad x_i \to y_j z_k \quad z_k \to b_{q-k+1} z_{k-1} c_k \quad z_1 \to b_q \$ c_1.$$

Sometimes the small triangle in the middle will have its vertices in the middle of edges of the Cayley diagram. To cover this situation we must have productions like $x \to ayz$, $x \to ayz$, and $x \to yza$.

We put into our grammar all the above types of productions which project to valid equations in G. For this purpose $\overline{S}$ and $\overline{\$}$ equal 1. This grammar will generate a language $L \subset \{w_1\$w_2\$ \cdots w_n \mid \overline{w_1 w_2 \cdots w_n} = 1\}$. The language we want is the intersection of L with the rational language $R\$R\$R$.

For the converse suppose $\Sigma^* \to G$ is a choice of generators, $R \subset \Sigma^*$ projects bijectively, and M is context–free. Analyze the context–free grammar and show that

R–triangles, i.e. triangles in the Cayley diagram with sides labelled by words in R, are δ–thin. Deduce that G satisfies the hypothesis of the following theorem.

Theorem 9.3. *If R is any language projecting onto G and all R– triangles T satisfy $\delta(T) \leq |t|/6 + K$, the G is hyperbolic.*

For any triangle T, $\delta(T)$ be the minimum such that T is $\delta(T)$–thin, and $|T|$ is the maximum distance between the vertices of T.