Chapter 12

Fluid Mechanics

12.1 Density

We have already seen (9.58) that the **local destiny** of a material can be defined as

$$\rho = \frac{dm}{dV}. \tag{12.1}$$

When the object has uniform (i.e. position independent) density, then the local density is the same as **average density** defined as

$$\rho = \frac{m}{V}. \tag{12.2}$$

For the same substance this number does not change even if the mass and volume might be different. For example both a steel wrench and a steel nail have the same density which the density of steel.

In SI the units of density are given by kilogram per cubic meter

$$1\,\mathrm{kg/m^3} = \frac{1\,\mathrm{kg}}{1\,\mathrm{m}^3} \tag{12.3}$$

but gram per unit centimeter are also widely used

$$1\,\mathrm{g/cm^3} = 1000\,\mathrm{kg/m^3}. \tag{12.4}$$

In the following table we summarize densities of common substances:

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerine	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

Another useful (but dimensionless) measure of density is **specific density** also known as **relative density**. It is defined as a ratio of density of a given substance to density of water (at temperature 4.0°C),

$$\text{specific density} = \frac{\rho_{\text{substance}}}{\rho_{\text{water}}}. \tag{12.5}$$

Example 12.2. *Find the mass and weight of the air (at* 1 *atm and* 20° *C) in a living room with* $4.0m \times 5.0m$ *floor and a ceiling* $3.0m$ *high, and the mass and weight of an equal volume of water.*

Volume of the living room is

$$V = 3.0\text{m} \times 4.0\text{m} \times 5.0\text{m} = 60\,\text{m}^3 \tag{12.6}$$

From definition of density

$$\begin{aligned} m_{\text{air}} &= \rho_{\text{air}} V = \left(1.20\,\text{kg/m}^3\right)\left(60\,\text{m}^3\right) = 72\,\text{kg} \\ m_{\text{water}} &= \rho_{\text{water}} V = \left(1000\,\text{kg/m}^3\right)\left(60\,\text{m}^3\right) = 6.0 \times 10^4\,\text{kg} \end{aligned} \tag{12.7}$$

and thus the corresponding weights are

$$\begin{aligned} w_{\text{air}} &= m_{\text{air}} g = (72\,\text{kg})\left(9.8\,\text{m/s}^2\right) = 700\,\text{N} \\ w_{\text{water}} &= m_{\text{water}} g = \left(6.0 \times 10^4\,\text{kg}\right)\left(9.8\,\text{m/s}^2\right) = 5.9 \times 10^5\,\text{N}. \end{aligned} \tag{12.8}$$

Example. *Rank the following objects in order from highest to lowest average density:*

(i) mass $4.00\,kg$*, volume* $1.60 \times 10^{-3}\,m^3$*;*
(ii) mass $8.00\,kg$*, volume* $1.60 \times 10^{-3}\,m^3$*;*
(iii) mass $8.00\,kg$*, volume* $3.20 \times 10^{-3}\,m^3$*;*
(iv) mass $2560\,kg$*, volume* $0.640\,m^3$*;*
(v) mass $2560\,kg$*, volume* $1.28\,m^3$*.*

The densities of these objects are

$$\begin{aligned} \rho_i &= \frac{4.00\,\text{kg}}{1.60 \times 10^{-3}\,\text{m}^3} = 2500\,\text{kg/m}^3 \\ \rho_{ii} &= \frac{8.00\,\text{kg}}{1.60 \times 10^{-3}\,\text{m}^3} = 5000\,\text{kg/m}^3 \\ \rho_{iii} &= \frac{8.00\,\text{kg}}{3.20 \times 10^{-3}\,\text{m}^3} = 2500\,\text{kg/m}^3 \\ \rho_{iv} &= \frac{2560\,\text{kg}}{0.64\,\text{m}^3} = 4000\,\text{kg/m}^3 \\ \rho_v &= \frac{2560\,\text{kg}}{1.28\,\text{m}^3} = 2000\,\text{kg/m}^3 \end{aligned} \tag{12.9}$$

And so the order is

$$(ii) \rightarrow (iv) \rightarrow (i, iii) \rightarrow (v). \tag{12.10}$$

12.2 Pressure in a Fluid

Pressure. In fluids pressure might change from one place to another and thus it is convenient to define a **local pressure** as

$$p = \frac{dF_\perp}{dA} \tag{12.11}$$

which reduces to (11.36) of **average pressure**

$$p = \frac{F_\perp}{A} \tag{12.12}$$

for uniform (i.e. position independent) pressures. Units of pressure were already introduced in the previous chapter,

$$\begin{aligned} 1\,\text{Pa} &\equiv 1\,\text{N/m}^2 \\ 1\,\text{atm} &\approx 1.013 \times 10^5\,\text{Pa} \\ 1\,\text{psi} &\approx 6900\,\text{Pa}. \end{aligned} \tag{12.13}$$

Example 12.2. *In the living room with* $4.0m \times 5.0m$ *floor what is the total downward force on the floor due to air pressure of* 1.00 *atm?*

From definition of pressure

$$F_\perp = pA = (1.00\,\text{atm})\frac{1.013 \times 10^5\,\text{N}}{1.00\,\text{atm}}(4.0\text{m} \times 5.0\text{m}) = 2.0 \times 10^6\,\text{N}. \tag{12.14}$$

Pressure with depth. Consider an infinitesimal volume element of fluid $dV = dxdydz$, where y-axis points upwards. In the equilibrium all forces acting on the object must add up to zero and thus along y-axis we have

$$\begin{aligned} -(\rho dV)\,g - (p + dp)\,dA + pdA &= 0 \\ -\rho\,(dxdydz)\,g - (p + dp)\,(dxdz) + p\,(dxdz) &= 0 \\ -\rho g dy - (p + dp) + p &= 0 \\ \frac{dp}{dy} &= -\rho g \end{aligned} \tag{12.15}$$

and thus the pressure must change linearly with y. The above equation can be solved by direct integration, i.e.

$$\begin{aligned} \int_{p(y_1)}^{p(y_2)} dp &= -\int_{y_1}^{y_2} \rho g dy \\ p(y_2) - p(y_1) &= \rho g\,(y_1 - y_2)\,. \end{aligned} \tag{12.16}$$

where we used an assumption that ρ is constant or in other words that the fluid is non-compressible. (This is a good assumption for liquids (e.g. water), but is not a very good assumption for gases (e.g. air) whose density can change considerably.)

In terms of depth

$$d = y_2 - y_1 \tag{12.17}$$

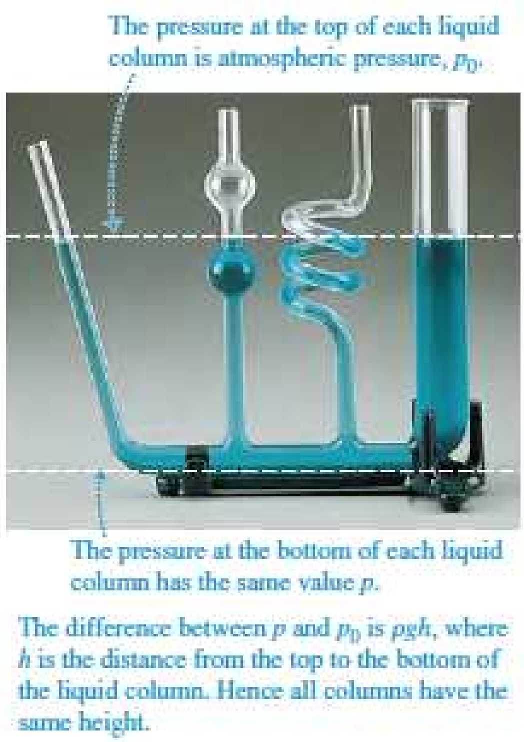
and reference pressure

$$p_0 = p(y_2) \tag{12.18}$$

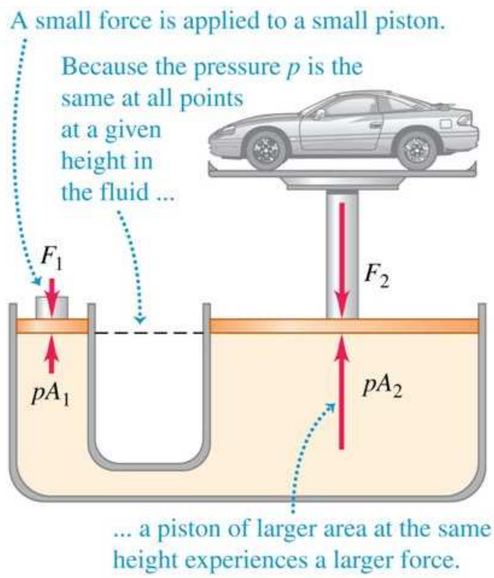
pressure at arbitrary depth is given by

$$p = p_0 + \rho g d. \tag{12.19}$$

As we see if the density is constant, the pressure p depends only on the pressure at the surface p_0 and depth d. Thus if p_0 (atmospheric pressure) and p(pressure at the bottom of liquid) is the same than d must be the same:



If we change pressure p_0 at the surface of fluid, the pressure will change by the same amount everywhere in the fluid. For example one can use this result to construct a hydraulic lift to measure large weights:



Because the pressure is the same at all point on the same height

$$p_0 = \frac{F_1}{A_1} = \frac{F_2}{A_2} \tag{12.20}$$

or

$$F_2 = \frac{A_2}{A_1} F_1. \tag{12.21}$$

More generally one formulate what is known as **Pascal's law:** *Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.*

Gauge pressure. It often useful to measure relative pressure compared to atmospheric pressure,

$$p_0 = 1\,\text{atm} \approx 14.7\,\text{psi} \approx 1.01 \times 10^5\,\text{Pa}. \tag{12.22}$$

For example, if the **absolute pressure** of a car tire is

$$p = 47\,\text{psi} \tag{12.23}$$

then it is often said that the **gauge pressure** is

$$p_{\text{gauge}} = p - p_0 = 32\,\text{psi}. \tag{12.24}$$

Example 12.3. *Water stands* 12.0 m *deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressure at the bottom of the tank?*

The absolute pressure at the bottom of the tank is

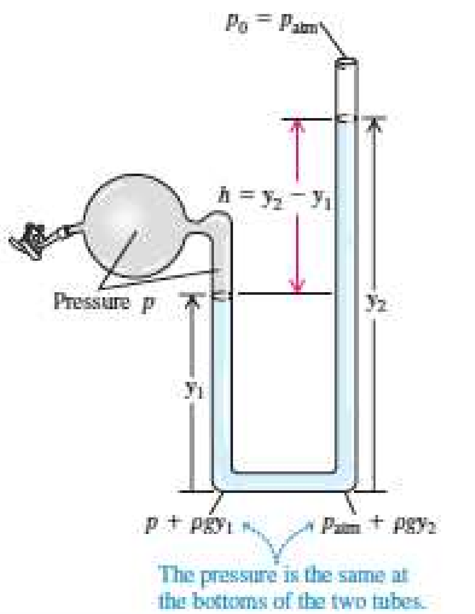
$$\begin{aligned} p &= p_0 + \rho g h \\ &= \left(1.01 \times 10^5\,\mathrm{Pa}\right) + \left(1000\,\mathrm{kg/m^3}\right)\left(9.8\,\mathrm{m/s^2}\right)(12.0\,\mathrm{m}) \\ &= 2.19 \times 10^5\,\mathrm{Pa} \end{aligned} \tag{12.25}$$

and so the gauge pressure is

$$\begin{aligned} p - p_0 &= \rho g h \\ &= 1.18 \times 10^5\,\mathrm{Pa}. \end{aligned} \tag{12.26}$$

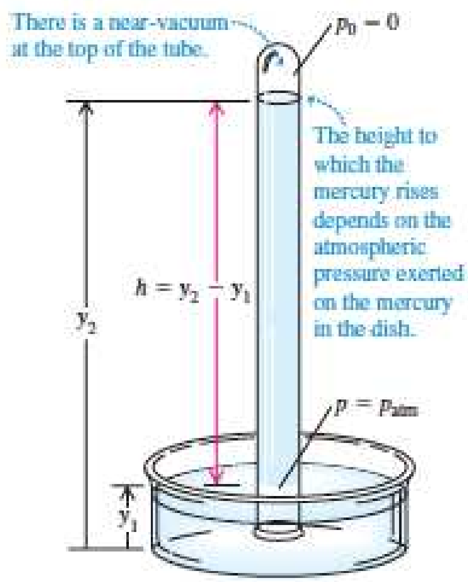
To measure the gauge pressure directly, one can use an open-tube manometer, where the difference in heights tells you what the gauge pressure is

$$p_{\mathrm{gauge}} = p - p_0 = \rho g (y_2 - y_1) \tag{12.27}$$



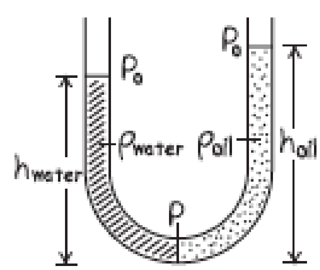
To measure air-pressure one can use a barometer where the difference in heights tells you what the atmospheric is

$$p_{\mathrm{atm}} = p_0 + \rho g (y_2 - y_1) = \rho g (y_2 - y_1). \tag{12.28}$$



The latter example suggest another unit of measuring pressure in "millimeters of mercury" which is also called *torr* after the inventor of mercury barometer Evangelista Torricelli.

Example 12.4. *A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil-water interface is at the midpoint of the tube as shown. Both arms of the tube are open to the air. Find a relationship between the heights* h_{oil} *and* h_{water}.



The pressure in both fluids at the surface and at the bottom are the same

$$
\begin{aligned}
p - p_0 &= \rho_{\text{water}} g h_{\text{water}} \\
p - p_0 &= \rho_{\text{oil}} g h_{\text{oil}}
\end{aligned}
\tag{12.29}
$$

and thus,

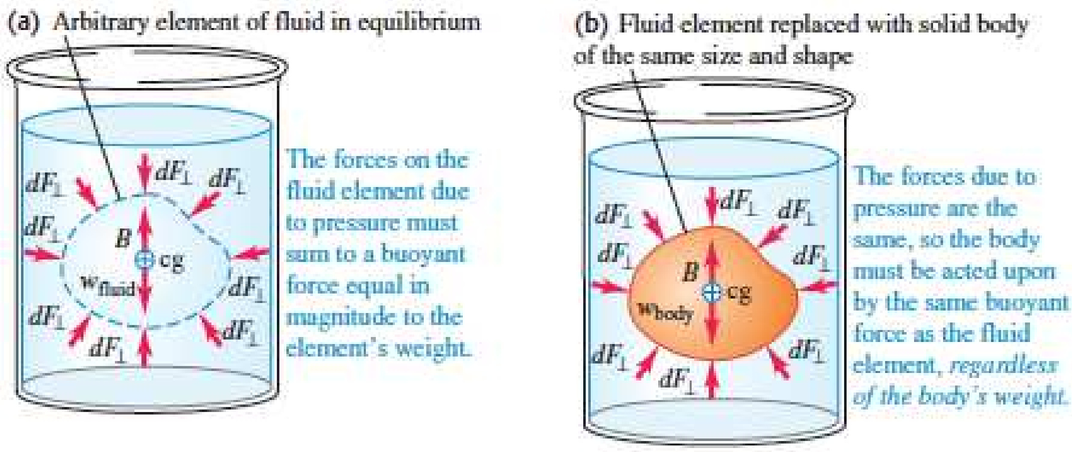
$$\frac{\rho_{\text{water}} g h_{\text{water}}}{\rho_{\text{oil}} g h_{\text{oil}}} = 1 \tag{12.30}$$

or

$$h_{\text{oil}} = \frac{\rho_{\text{water}}}{\rho_{\text{oil}}} h_{\text{water}} \approx \frac{1000\,\text{kg/m}^3}{850\,\text{kg/m}^3} h_{\text{water}} \approx 1.2\, h_{\text{water}}. \tag{12.31}$$

12.3 Buoyancy

Any object placed in a fluid experiences a force (buoyant force) arising due to changes of the pressure inside fluid. This phenomena is known as **Archimedes's principle:** *When a body is completely or partially immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.*



To prove Archimedes's principle we consider an element of fluid of arbitrary shape. If the fluid is in equilibrium then the sum of all forces (due to water pressure) have to be the same as the force of gravity

$$B = F_{\text{gravity}} \tag{12.32}$$

or

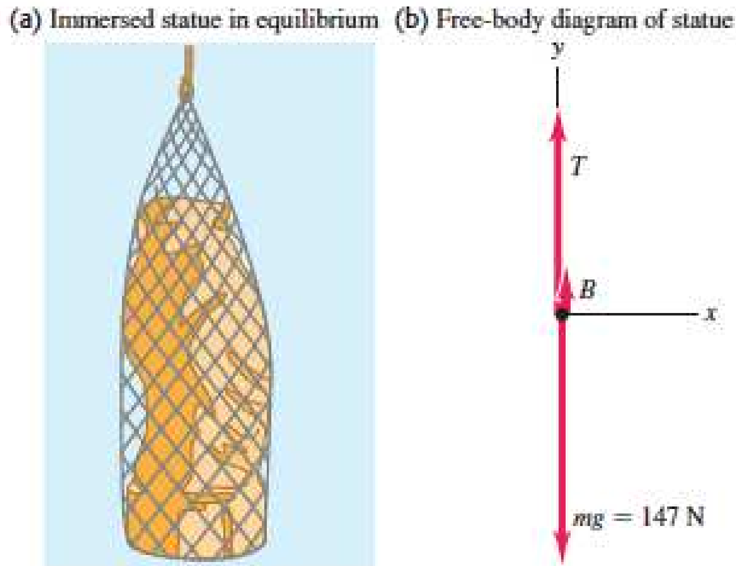
$$B = V \rho_{\text{fluid}} g \tag{12.33}$$

Now if we fill the shape with some other material, then the equilibrium condition might not be satisfied, but the buoyant force due to water pressure would not change.

Example 12.5. *A 15.0 − kg solid gold statue is raised from the sea bottom. What is the tension in the hosting cable (assumed massless) when the statues is*

a) at rest and completely underwater.

b) at rest and completely out of water.



We can first find volume of the statue

$$V = \frac{m}{\rho_{\text{gold}}} = \frac{15.0\,\text{kg}}{19.3\times10^3\,\text{kg/m}^3} = 7.77\times10^{-4}\,\text{m}^3. \tag{12.34}$$

Then the equilibrium condition implies

$$T + B_{\text{fluid}} - mg = 0 \tag{12.35}$$

or in water

$$\begin{aligned} T &= mg - V\rho_{\text{water}}g \\ &= (15.0\,\text{kg})\left(9.8\,\text{m/s}^2\right) - \left(7.77\times10^{-4}\,\text{m}^3\right)\left(1000\,\text{kg/m}^3\right)\left(9.8\,\text{m/s}^2\right) \\ &= 139\,\text{N} \end{aligned} \tag{12.36}$$

and in air

$$\begin{aligned} T &= mg - V\rho_{\text{air}}g \\ &= (15.0\,\text{kg})\left(9.8\,\text{m/s}^2\right) - \left(7.77\times10^{-4}\,\text{m}^3\right)\left(1.2\,\text{kg/m}^3\right)\left(9.8\,\text{m/s}^2\right) \\ &= 147\,\text{N}. \end{aligned} \tag{12.37}$$

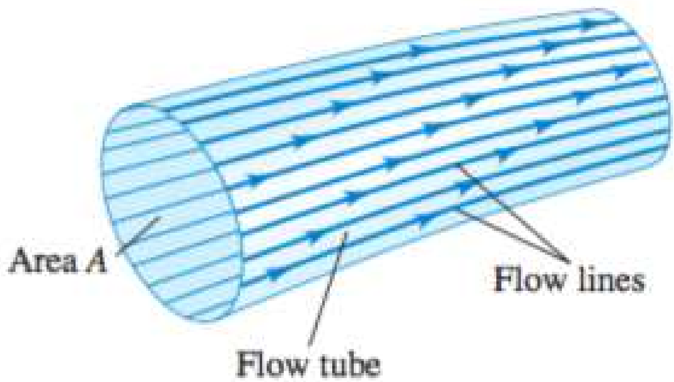
Example. *You place a container of seawater on a scale and note reading on the scale. You now suspend the statue of Example 12.5 in the water. How does the scale reading change?*

(i) it increases by $\boldsymbol{7.84\,N}$***;***
(ii) it decreases by $7.84\,N$*;*
(iii) it remains the same;
(iv) none of these.

In addition to buoyant force there is a force of surface tension which acts on the object at the surface of fluids, but this force is subdominant for sufficiently large objects.

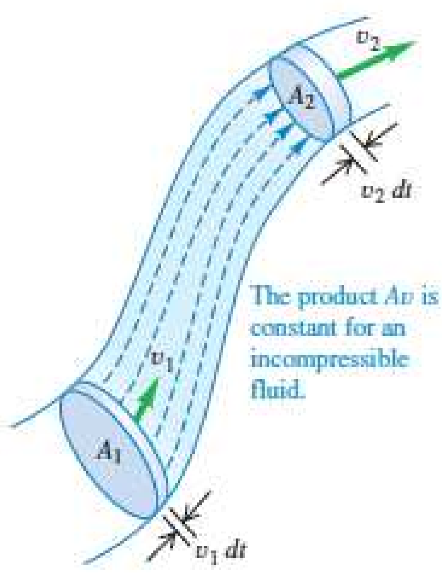
12.4 Fluid Flow

Consider a simple model of fluid which is **incompressible** (density is constant) and **inviscid** (vanishing internal friction). Then one can follow trajectories of small elements of water (we call particles) which will flow along along these trajectories. We call these trajectories **flow lines** and say that the flow is **steady** if the flow line do no change with time. This does not mean that the velocities on any given particle does not change with time.



More generally the flow becomes irregular where small scale mode and large scale modes interact with each other which gives rise to turbulence. It is interesting to note that for 3D fluids the energy is transferred from large scales to small scale, when in 2D fluids the energy is transferred from small scales to large scales. Richard Feynman called turbulence "the last unsolved problem of classical physics". It is also related to one of seven Millennium problems formulated by Clay Mathematics Institute in 2000, six of which (including the turbulence problem) remain unsolved.

Continuity. Consider a flow of fluid through a pipe with changing cross-sectional area:



If the fluid is incompressible (i.e. constant density), then the amount of fluid passing through each cross-sectional area per unit time must be the same

$$\rho A_1 ds_1 = \rho A_2 ds_2 \tag{12.38}$$

or

$$A_1 v_1 dt = A_2 v_2 dt \tag{12.39}$$

which give us (1D) the continuity equation for non-compressible fluid

$$A_1 v_1 = A_2 v_2. \tag{12.40}$$

The continuity equation equates the volume flow rate across different cross-sectional areas

$$\frac{dV}{dt} = Av. \tag{12.41}$$

Note that(12.40) can be easily generalized to the case when densities do change

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2. \tag{12.42}$$

Example 12.6. *Incompressible oil of density* 850 *kg/m*^{3} *is pumped through a cylindrical pipe at a rate of* 9.5 *liters per second.*

(a) The first section of the pipe has a diameter of 8.0 *cm. What is the flow speed of the oil? What is the mass flow rate?*

(b) The second section of the pipe has a diameter of 4.0 *cm. What is the flow speed and the mass flow rate in that section?*

From continuity equation

$$\frac{dV}{dt} = A_1 v_1 = A_2 v_2 = (9.5\,\mathrm{L/s})\left(10^{-3}\,\mathrm{m^3/L}\right) = 9.5\times10^{-3}\,\mathrm{m^3/s}. \tag{12.43}$$

and thus the flow speeds are

$$\begin{aligned} v_1 &= \frac{9.5\times10^{-3}\,\mathrm{m^3/s}}{3.14\times(4.0\times10^{-2}\,\mathrm{m})^2} = 1.9\,\mathrm{m/s} \\ v_2 &= \frac{9.5\times10^{-3}\,\mathrm{m^3/s}}{3.14\times(2.0\times10^{-2}\,\mathrm{m})^2} = 7.6\,\mathrm{m/s} \end{aligned} \tag{12.44}$$

From the definition of density

$$\frac{dm}{dt} = \rho\frac{dV}{dt} \tag{12.45}$$

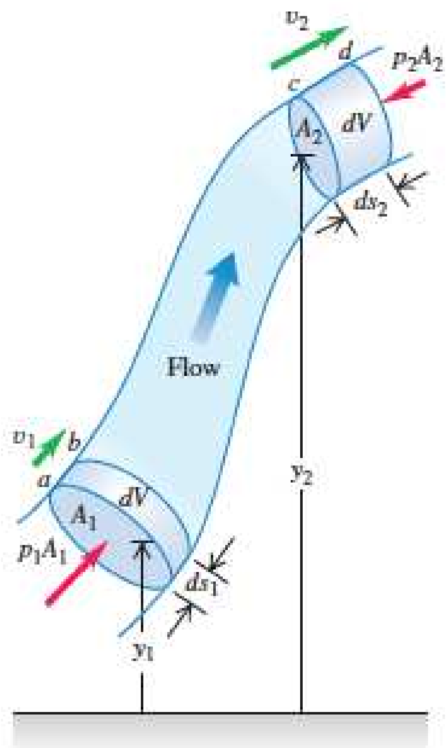
and thus the mass flow rates are the same for both sections

$$\frac{dm}{dt} = \left(850\,\mathrm{kg/m^3}\right)\left(9.5\times10^{-3}\,\mathrm{m^3/s}\right) = 8.1\,\mathrm{kg/s}. \tag{12.46}$$

Example. *A maintenance crew is aworking on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam.) Do cars on a highway behave like:*
(i) the molecules of an incompressible fluid or
(ii) the molecules of compressible fluid?

12.5 Bernoulli's Equation

As fluid moves through pipe external forces such as gravitational force can do work on the fluid.



This can be described by computing the total work done on (an incompressible) fluid element between sections a and c as they move to sections b and d

$$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = (p_1 - p_2)\, dV \tag{12.47}$$

This must be equal to the change in mechanical energy for fluid. The change in kinetic and potential energies is due to the difference of kinetic and potential energies of the fluid between sections a and b to fluid between sections c

and d

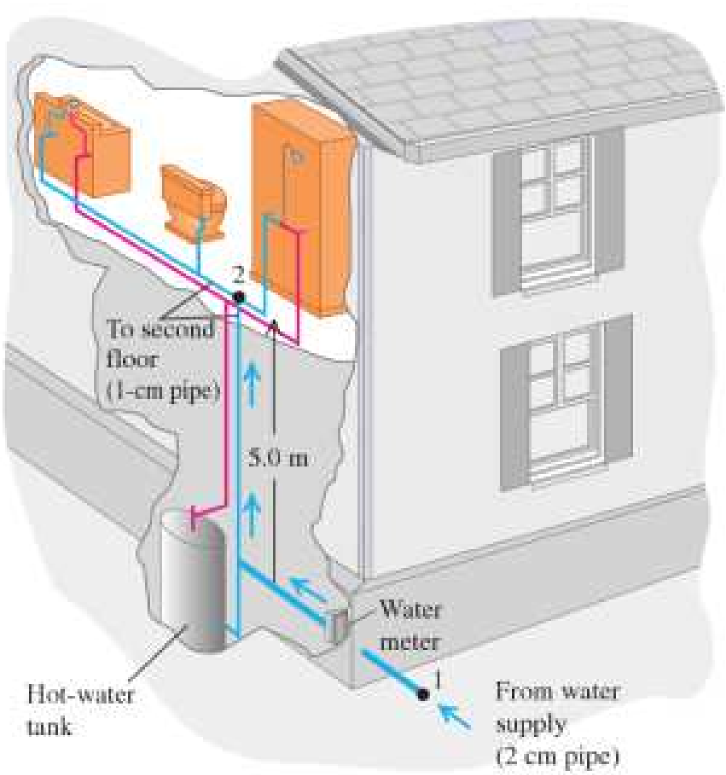
$$\begin{aligned} dK &= \frac{1}{2}\rho dV\left(v_2^2 - v_1^2\right) \\ dU &= \rho dV g\left(y_2 - y_1\right). \end{aligned} \tag{12.48}$$

By equating the work and change in mechanical energy we arrive at the Bernoulli's equation:

$$\begin{aligned} dW &= dK + dU \\ \left(p_1 - p_2\right) dV &= \frac{1}{2}\rho dV\left(v_2^2 - v_1^2\right) + \rho dV g\left(y_2 - y_1\right) \\ p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 &= p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \end{aligned} \tag{12.49}$$

which only applies to incompressible and inviscid fluids. More general fluids are described by the Navier-Stokes equation.

Example 12.7. *Water enters a house through a pipe with an inside diameter of* $2.0\,cm$ *at an absolute pressure of* $4.0 \times 10^5\,Pa$*. A* $1.0\,cm$ *diameter pipe leads to the second-floor bathroom* $5.0\,m$ *above. When the flow speed at the inlet pipe is* $1.5\,m/s$*, find the flow speed, pressure and volume flow in the bathroom.*



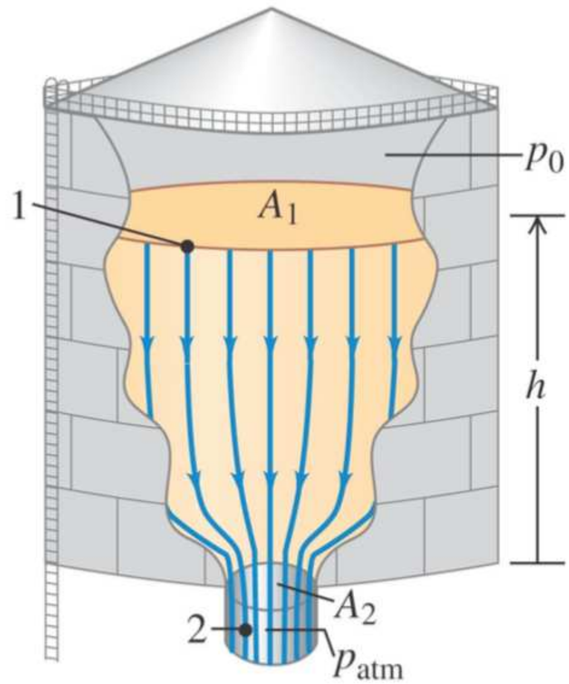
From continuity equation

$$\begin{aligned} v_1 A_1 &= v_2 A_2 \\ v_2 &= \frac{\pi\,(1.0\,\mathrm{cm})^2}{\pi\,(0.5\,\mathrm{cm})^2}\,(1.5\,\mathrm{m/s}) = 6.0\,\mathrm{m/s} \end{aligned} \tag{12.50}$$

and from Bernoulli's equation

$$\begin{aligned} p_2 &= p_1 + \rho g\,(y_1 - y_2) + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right) \\ &= \left(4.0\times 10^5\,\mathrm{Pa}\right) + \left(1000\,\mathrm{kg/m^3}\right)\left[(9.8\,\mathrm{m/s})\,(-5.0\,\mathrm{m}) + \frac{1}{2}\left((1.5\,\mathrm{m/s})^2 - (6.0\,\mathrm{m/s})^2\right)\right] \\ &= 3.3\times 10^5\,\mathrm{Pa}. \end{aligned} \tag{12.51}$$

Example 12.8. *A gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0, and the gasoline flows out the bottom of the tank through a short pipe with* *cross-sectional area A_2. Derive expression for the flow speed in the pipe and the volume flow rate.*



From Bernoulli's equation

$$\begin{aligned} p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 &= p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \\ p_0 + \frac{1}{2}\rho v_1^2 &= p_{\text{atm}} + \rho g h + \frac{1}{2}\rho v_2^2 \end{aligned} \tag{12.52}$$

where

$$v_1 = \frac{A_2}{A_1} v_2 \tag{12.53}$$

and thus

$$\begin{aligned} \frac{1}{2}\rho v_2^2 \left(1 - \left(\frac{A_2}{A_1}\right)^2\right) &= p_0 - p_{\text{atm}} + \rho g h \\ v_2 &= \sqrt{\frac{2\left(p_0 - p_{\text{atm}} + \rho g h\right)}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} \end{aligned} \tag{12.54}$$

In the limit

$$A_1 \gg A_2 \tag{12.55}$$

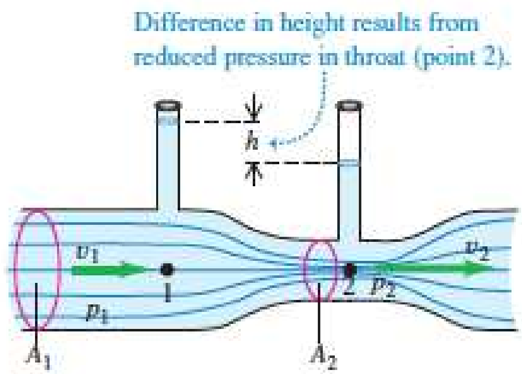
the flow speed is

$$v_2 \approx \sqrt{\frac{2\left(p_0 - p_{\rm atm}\right)}{\rho} + 2gh} \tag{12.56}$$

and the flow rate

$$\frac{dV}{dt} = A_2 v_2 = A_2\sqrt{\frac{2\left(p_0 - p_{\rm atm}\right)}{\rho} + 2gh} \tag{12.57}$$

Example 12.9. *Venturi meter is used to measure flow speed in a pipe. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.*



From continuity equation

$$v_2 = \frac{A_1}{A_2} v_1 \tag{12.58}$$

and thus from Bernoulli's equation

$$\begin{aligned} p_1 - p_2 &= \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \\ p_1 - p_2 &= \frac{1}{2}\rho v_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right). \end{aligned} \tag{12.59}$$

However we also know that

$$\begin{aligned} p_1 &= p_0 + \rho g h_1 \\ p_2 &= p_0 + \rho g h_2 \end{aligned} \tag{12.60}$$

and thus

$$p_1 - p_2 = \rho g (h_1 - h_2) = \rho g h. \tag{12.61}$$

By combining (12.59) and (12.61) we get

$$\rho g h = \frac{1}{2}\rho v_1^2 \left(\left(\frac{A_1}{A_2} \right)^2 - 1 \right)$$
$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}. \quad (12.62)$$

Example. *Lift of an airplane.*