Toward Computational Fluency in Multidigit Multiplication and Division

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Traditionally in the United States and Canada, students have first learned how to compute with whole numbers and then have applied that kind of computation. This approach presents several problems. First, less-advanced students sometimes never reach the application phase, so their learning is greatly limited. Second, word problems usually appear at the end of each section or chapter on computation, so sensible students do not read the problems carefully: They simply perform the operation that they have just practiced on the numbers in the problem. This practice, plus the emphasis on teaching students to focus on key words in problems rather than to build a complete mental model of the problem situation, leads to poor problem solving because students never learn to read and model the problems themselves. Third, seeing problem situations only after learning the mathematical operations keeps students from linking those operations with aspects of the problem situations. This isolation limits the meaningfulness of the operations and the ability of children to use the operations in a variety of situations.

Research has indicated that beginning with problem situations yields greater problem-solving competence and equal or better computational competence. Children who start with problem situations directly model solutions to these problems. They later move to more advanced mathematical approaches as they progress through levels of solutions and problem difficulty. Thus, their development of computational fluency and their acquisition of problem-solving skills are intertwined as both develop with understanding.

The paper from which this article is excerpted, "Developing Mathematical Power in Whole-Number Operations," was commissioned by NCTM's Research Advisory Committee to summarize the current state of educational research for use by writing groups preparing Principles and Standards for School Mathematics. *Sections from the original paper were omitted in order to focus on multidigit multiplication and division. The omitted sections address single-digit addition, subtraction, multiplication, and division; multidigit addition and subtraction; and general issues in achieving computational fluency, such as curricular issues, instructional phases, helping diverse learners, individual differences, and preparing for rational numbers. The complete paper will appear in* NCTM's Research Companion to Principles and Standards for School Mathematics, *currently in press. We sincerely thank Karen Fuson and those involved with the Research Companion for allowing us to include this excerpt in the focus issue.—T.B. and J.S., focus issue editors*

Building Fluency with Computational Methods: General Issues

Fluency with computational methods is the heart of what many people in the United States and Canada consider to be the elementary mathematics curriculum. Learning and practicing computational methods are central to many memories of learning in the twentieth century. However, twentieth-century mathematics teaching and learning were driven by goals and by theories of learning that are not sufficient for the twenty-first century, in which inexpensive machine calculators are widely available, computers increasingly appear in schools and libraries, the World Wide Web gives access to a huge variety of information, and supercomputers create demands for new kinds of machine algorithms, such as general multistep methods. The information age creates for all citizens the need for lifelong learning and for flexible approaches to solving problems. Everyone needs the ability to use calculating machines with understanding.

Clearly, the twenty-first century requires a greater focus on a wider range of problem-solving experiences and a reduced focus on learning and practicing by rote a large body of standard calculation methods. How to use the scarce hours of mathematics learning time in schools is a central issue. This decision requires in part a value judgment as to which needs are most important. But new research can also influence our choices. Educators and the public are still attempting to reach consensus on the kinds and amounts of computational fluency that are necessary today. Computational fluency is one vital component of developing mathematical power; other components include understanding the uses and methods of computation. Given that mathematics learning time is a scarce resource, educators need to know roughly the amount of time various children require to reach various levels of computational fluency. Only with such knowledge can we make sensible decisions about how to allocate scarce learning time for reaching, among all the worthwhile goals of mathematics learning, computational fluency.

Several themes characterize much of the research on computational methods over the past thirty years. These themes apply across computational domains such as single-digit addition and subtraction and multidigit multiplication and division. Within each computational domain, individual learners move through progressions of methods from initial, transparent, problem-modeling, concretely represented methods to less transparent, more-problem-independent, mathematically sophisticated, symbolic methods. At a given moment, each learner knows and uses a range of methods that may differ according to the numbers in the problem, the problem situation, or other individual and classroom variables. A learner may use different methods even on very similar problems, and because any new method competes for a long time with older methods, the learner may not use it consistently. Typical errors can be identified for each domain and for many methods (Ashlock 1998), and researchers have designed and studied ways to help students overcome these errors. A detailed understanding of methods in each domain enables us to identify prerequisite competencies that all learners can develop to access those methods.

The constant cycles of mathematical doing and knowing lead to learners' construction of representational tools that they use mentally to find solutions. Learners invent varying methods regardless of whether their teachers have focused on teaching for understanding or on rote memorizing of a particular method. In classrooms in which the focus is teaching for understanding, however, students develop a wider range of effective methods. In classrooms in which rote learning methods are used, students' inventiveness often generates many different kinds of errors, most of which are partially correct methods created by a particular misunderstanding. Thus, even in traditional classrooms emphasizing standard computational methods, learners are not passive absorbers of knowledge. They build and use their own meaning and doing, and they generalize and reorganize this meaning and doing.

Multidigit addition, subtraction, multiplication, and division solution methods are called *algorithms*. An algorithm is a general multistep procedure that will produce an answer for a given class of problems. Computers use many different algorithms to solve different kinds of problems, and inventing new algorithms is an increasingly important area of applied mathematics. Around the world, many different algorithms have been invented and taught for multidigit addition, subtraction, multiplication, and division. Students in U.S. and Canadian schools have learned different algorithms at different times. Each algorithm has advantages and disadvantages. Therefore, the decisions about computational fluency concern in part the algorithms that might be supported in classrooms and the bases for selecting those algorithms.

When the focus is teaching for understanding, students develop a wider range of effective methods

One goal of the following sections is to underscore the possibility of understanding various computational methods. Because such understanding has ordinarily not been a goal of school mathematics, most educational decision makers have not had an opportunity to understand the standard algorithms or to appreciate the wide variety of possible algorithms. Most teachers also have not had that opportunity, and most textbooks do not sufficiently help develop such understanding.

Research indicates that some algorithms are more accessible to understanding than others and that understanding can be increased by quantity supports such as manipulatives and drawings to help children understand the meanings of the numbers, notations, and steps in the algorithms. This understanding does not conflict with developing computational fluency but rather is foundational to it. Children need scaffolded practice with whatever methods they are using to become more fluent in orchestrating the steps in any algorithm. Understanding can serve as a continual directive toward correct steps and as a constraint on the many creative calculating errors invented by students who are taught algorithms by rote. Because children cannot understand all algorithms equally (many sacrifice comprehensibility to save space in writing), I describe at least one algorithm that has been demonstrated to be accessible to a wide range of students. My criteria for such accessible algorithms are that they scaffold the understanding of principal steps in the domain, generalize readily to large numbers, have variations that provide for individual differences in thinking, and are procedurally simple to carry out, that is, they require the minimum of computational subskills so that valuable learning time is not required to bring unnecessary subskills to the needed level of accuracy.

Multidigit Multiplication and Division

Much less research is available on children's understandings of multidigit multiplication and division than on single-digit computation and multidigit addition and subtraction. Educators have published sample teaching lessons (Lampert 1986, 1992) and have explored alternative methods for accomplishing these operations (Carroll and Porter 1998). Researchers have reported a preliminary learning progression of multidigit methods for third- to fifth-grade classrooms in which teachers fostered children's invention of algorithms (Baek 1998). These methods moved from (a) direct modeling with objects or drawings (such as by ones and by tens and ones), to (b) written methods involving repeatedly adding—sometimes by repeated doubling, a surprisingly effective method used historically, to (c) partitioning methods. The partitioning methods ranged from partitioning using numbers other than 10, partitioning one number into tens and ones, and partitioning both numbers into tens and ones.

Current and accessible methods

The multiplication and division algorithms currently most prevalent are complex embedded methods that are not easy to understand or to carry out (see the leftmost methods in **fig. 1**). They demand high levels of skill in multiplying a multidigit number by a single-digit number within complex embedded formats in which multiplying and adding alternate. In these algorithms, the meaning and scaffolding of substeps have been sacrificed to using a small amount of paper. The multiplication and division algorithms use aligning methods that keep the steps organized by correct place value without requiring any understanding of what is actually happening with the ones, tens, and hundreds.

Figure 1 presents modifications of these methods that clarify the meaning and purpose of each step. The separation of steps in each of these accessible methods also facilitates the linking of each step with the quantities involved. An array drawing shows the quantities; arrays are powerful models of multiplication and division. The accessible methods and drawings demonstrate central features in multidigit multiplication and division that students must come to understand and do.

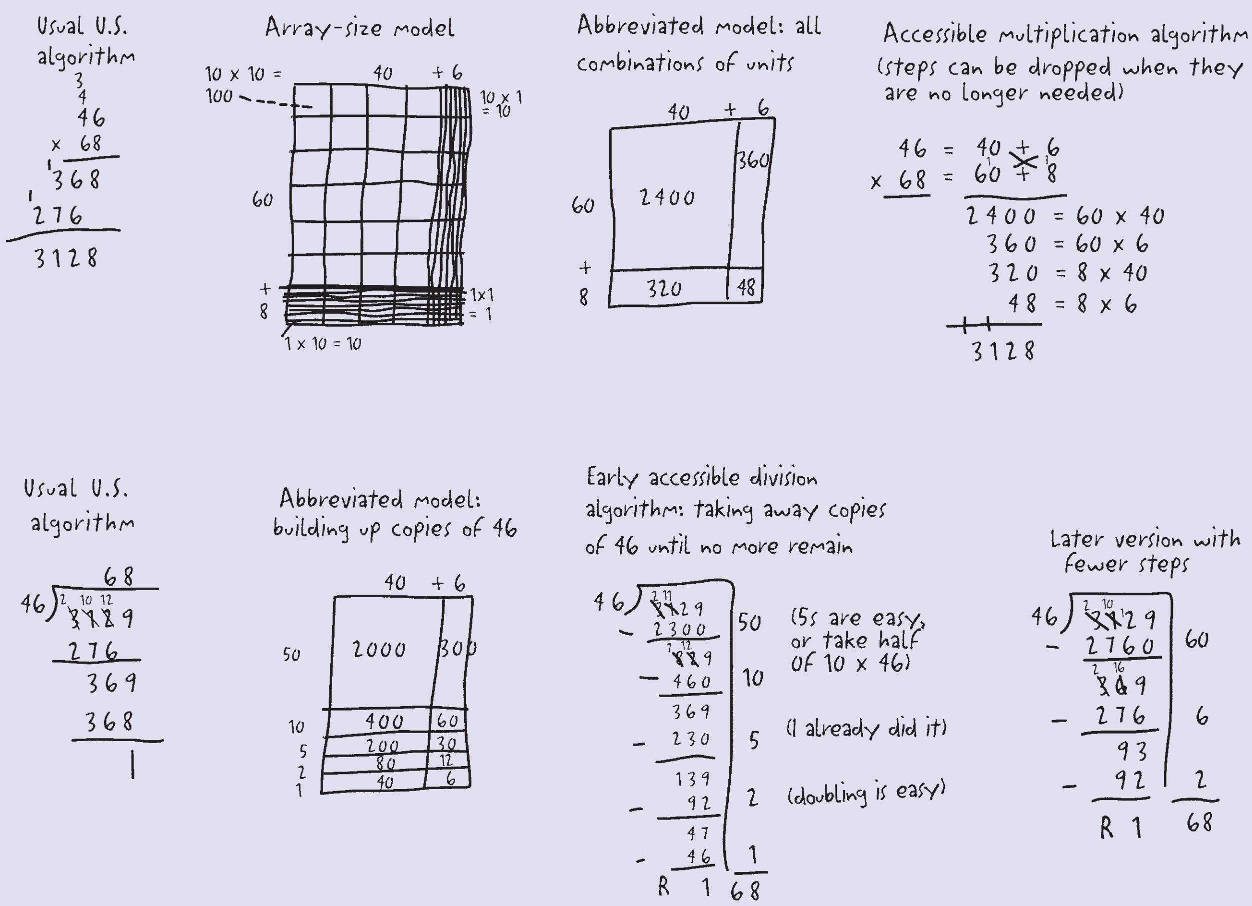
Accessible multiplication methods

For multiplication, teachers first show an array-size model. Such a model provides initial support for the crucial understandings of the effects of multiplying by 1, 10, and 100. It also shows clearly how each of the tens and ones in 46 and 68 are multiplied by each other and are then added after students have completed all multiplication operations. The sizes of the resulting squares or rectangles indicate the sizes of these various products and thus support the understanding. As one looks across each row in the array, one can see in the top row 10×46 as 10×40 (four squares of 100) plus 10×6 (six columns of 10 each). Multiplying by 60 creates six such rows of 10 products, so multiplying by 60 is multiplying by 10 and then multiplying by 6. Then one sees eight rows of 1×46 as 1×40 and 1×6 (eight rows of each). Teachers can draw the abbreviated model shown in **figure 1** to summarize steps in multidigit multiplication. Its separation into tens and ones facilitates the necessary multiplication operations.

The accessible multiplication algorithm shown in the top right of **figure 1** is the fullest form with all possible supports. As students come to understand each aspect of multiplication, they can drop each of the supports, resulting in a streamlined version that is a simple expanded form of the usual U.S. method. Variations of the accessible algorithm have been widely used in research classrooms and in some innovative textbooks. Its main feature is a clear record of each of the four pairs of numbers (40×60, 40×8, 6×60, 6×8) that students need to multiply. The vertical and diagonal marks are a way for students to record as they go which numbers they have already multiplied. Unlike the current U.S. algorithm, which starts at the right and multiplies units first, the accessible algorithm begins at the left, as students prefer to do. This approach also has the advantage that the first product written is the

FIGURE 1

Multidigit multiplication and division (figure 6.4 in the original paper)



largest, permitting all the smaller products to align easily under it in their correct places. Writing the factors at the side of each product emphasizes what students are actually doing in each step and permits an easy check. Writing out the separate products for 40×60 and 40×8 is much easier for students than doing the usual procedure: multiply 40×8, write part of the answer below and part above the problem, multiply 40×60, and then add the number written above the problem. The complex alternation of multiplying and adding in the usual algorithm is not necessary, is a source of errors, and obfuscates what students are actually doing in multidigit multiplying: multiplying each combination of units and adding all of them (see the abbreviated model). Students who understand and wish to drop steps in the accessible algorithm do so readily, with a result looking like the usual U.S. method except that it has four, instead of two, products to be added. These four can even be folded into two, if students wish. Therefore, the accessible model permits students to function at their own level of scaffolded understanding and helps them explain what they are doing.

Multiplying by three-digit numbers is a simple extension of the two-digit version. After a conceptual development of the results of multiplying by 100 (that is, numbers get two places larger, so they move left two places), abbreviated drawings can demonstrate the nine combinations of products that students need to find and add. Students can easily carry out the accessible algorithm for these larger numbers because it scaffolds the necessary steps. Given the accessibility of calculators, the amount of valuable school learning time that teachers should devote to such large multiplication problems is unclear. But teachers could easily introduce them in a conceptual fashion that then relates to estimating the product, especially when the largest product is found first, as in the accessible method shown in **figure 1.**

Current and accessible division methods

The usual U.S. division algorithm has two aspects that create difficulties for students. First, it requires them to determine exactly the maximum copies of the divisor that they can take from the dividend.

This feature is a source of anxiety because students often have difficulty estimating exactly how many will fit. Students commonly multiply trial products off to the side until they find the exact one. Second, the current algorithm creates no sense of the size of the answers that students are writing; in fact, they are always multiplying by single digits. In the example in **figure 1,** they just write a 6 above the line; they have no sense of 60 because they are literally only multiplying 46 by 6. Thus, students have difficulty gaining experience with estimating the correct order of magnitude of answers in division when they are using the current U.S. algorithm.

The accessible division method shown in **figure 1** facilitates safe underestimating. It builds students' experience with estimating and, later, their accurate assessment of calculator answers, because they multiply by the correct number, that is, 60, not 6. The procedure is easy for those students who are still gaining mastery of single-digit multiplication, because it permits the use of readily known products. For those who can manage it, the method can be abbreviated to be as brief as the current algorithm. Educators have used this accessible division algorithm in innovative materials since at least the 1960s.

Time is well spent on conceptual and accessible approaches that facilitate students' understanding

The example of the accessible method given first in **figure 1** shows a solution that a student might do very early in division learning. Conceptually, the drawing and the written algorithm work together to show the meaning of long division: It is like a puzzle in which the solver takes away copies of the divisor—here, 46—until no further copies can be taken away. He or she is solving the equation "$46 \times ? = 3{,}129$" using the notion of division as the inverse of multiplication. The drawing shows these copies being added to make the total 3,128 as 46×68 (remainder of 1), and the written algorithm subtracts each large copy as the solver keeps track of how many more still must be taken away. The drawing can scaffold the one-digit-by-two-digit multiplication necessary at each step: 50×46 is split into $50 \times 40 = 2{,}000$ and $50 \times 6 = 300$ to make 2,300. The scaffolding is important because this combination of multiplying and adding is complex for some students. The example shows that the student elected to multiply by 50 because fives facts are learned easily and accurately. The student then sees that he or she can simply take away another 10 copies of 46. The student next cleverly uses a product that he or she has already found (50×46) to take away 5 copies of 46. Doubling is also easy, although many students would probably have multiplied by 3 at that point. Successive doubling represents the basis of the multiplication and division algorithms used historically in Europe. The right side of **figure 1** gives a version of the same problem that the same student might complete with more experience. At this point, the student may not need the drawing to scaffold the steps, meanings, or multiplication operations.

The accessible algorithms for multiplication and division depend heavily on fluency with multiplication and addition, and in division, with multidigit subtraction. The difficulties that many students have in subtraction noticeably affect division, so understanding and fluency in multidigit subtraction are very important. Because students typically range substantially in their multiplication learning rate, many of them may not have achieved full fluency by the time their class is discussing multidigit multiplication and division. An advisable tactic is to give such students a multiplication table that they can use to check their multiplications as they go. This aid will permit them to keep up with the class and learn an algorithm. Furthermore, each verification of, or search for, a product in the table creates another learning trial for basic multiplication. Of course, presenting separate learning opportunities for multiplication combinations with which the student is not yet fluent would also be helpful.

Multidigit Computation in the Twenty-First Century

How much valuable school mathematics time should be spent on multidigit multiplication and division is a question whose answer will probably need continual revision during the twenty-first century. New goals will arise to compete with these domains, as they have already done. At present, time is well spent on conceptual and accessible approaches that facilitate students' understanding of how to build multidigit multiplication and division from the central concepts of place value and basic multiplication combinations. During that time, students could also bring those combinations to mastery. Drilling for long periods on problems involving large numbers seems a goal more appropriate to the twentieth than to the twenty-first century. The new research-based view of achieving computational fluency is a more complex and connected view than the past linear view consisting of counting, memorizing facts, solving problems, learning algorithms, and then solving problems with those algorithms. However, a new, more complex view is necessary to achieve the new, more complex goals of mathematics learning and teaching necessary for the twenty-first century. A new kind of computational fluency is needed for the challenges and changes that individuals in the United States and Canada will face in the years to come.

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The following reviews and summaries of the literature were used extensively in this paper (see the original paper for the complete reference list): Baroody and Coslick (1998); Baroody and Ginsburg (1986); Bergeron and Herscovics (1990); Brophy (1997); Carpenter and Moser (1984); Cotton (1995); Davis (1984); Dixon, Carnine, Kameenui, Simmons, Lee, Wallin, and Chard (1998); Fuson (1992a, 1992b); Geary (1994); Ginsburg (1984); Greer (1992); Hiebert (1986, 1992); Hiebert and Carpenter (1992); Lampert (1992); Nesher (1992); Resnick (1992); and Siegler (2002). These works include reviews carried out by experts in mathematics education, cognitive science, learning disabilities, special education, educational psychology, and developmental psychology. The reviews in Leinhardt, Putnam, and Hattrup (1992) were written especially for teachers and other educational leaders; they also include analyses of textbook approaches to teaching. To avoid excessive citations, results that are strong, salient, and clear in these reviews and summaries were not cited separately. More specialized results were cited. ▲