MATH 531: ANALYTIC NUMBER THEORY, I
THE DISTRIBUTION OF PRIME NUMBERS

MWF 2–2:50, Altgeld 241

Kevin Ford (304 Altgeld, 265-6255, `http://www.math.uiuc.edu/~ford/`)

Prime number theory has witnessed many exciting new developments in the past few years:

- The primes contain arbitrarily long arithmetic progressions (Green and Tao, 2005)
- Bounded gaps between primes exist infinitely often (Yitang Zhang, 2013)
- Every odd number greater than 5 is the sum of three primes (Harald Helfgott, 2014)

All of these rely on **analytic methods**, that is, methods stemming from some kind of analysis (broadly speaking, this included real analysis, complex analysis, and harmonic analysis).

Main goals: Become familiar with fundamental principles of real and complex analytic methods for studying the distribution of arithmetic functions (functions which capture interesting number theoretic information, e.g. Euler's function) and prime numbers. Throughout the semester, we'll discuss some of the newest theorems (e.g. the three mentioned above, recent progress on large gaps between primes, etc) and the role of famous conjectures in number theory such the Generalized Riemann Hypothesis, the Twin Prime Conjecture, the Elliott-Halberstam Conjecture.

Syllabus:

1. Arithmetic functions: theory of multiplicative and additive functions, Dirichlet convolution, Möbius inversion, average order of magnitude estimates. Big-O and little-o notation.

2. Elementary theory of the distribution of primes. Statements equivalent to the prime number theorem, estimates of Chebyshev and Mertens.

3. Study of arithmetic functions via the analytic theory of Dirichlet series, Euler products and Perron's inversion formula.

4. Analytic methods for the distribution of primes. Theory of the Riemann Zeta function; connection between zeros of the zeta function and primes; analytic proof of the Prime Number Theorem. Why do people believe the Riemann Hypothesis? Why is it important?

5. Dirichlet Characters and Dirichlet's theorem on primes in arithmetic progressions.

6. As time permits, a brief "sneak preview" of other further topics in analytic number theory, such as exponential sums, L-functions, sieve methods, modular forms.

Text: No "official" text: I will use my own notes. Good references:

- P. T. Bateman and H. G. Diamond, *Analytic Number Theory*, 2nd ed, 2009.
- G. Tenenbaum, *Introduction to analytic and probabilistic number theory, 3rd ed.*, 2015
- H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, 2007