Test 1 **February 2, 2006** **Name** ______________________________

Math 522 **Student Number** ______________________________

Direction: **You are required to complete this test within 1 hour and 15 minutes. In order to receive full credit, answer each problem completely and must show all work. Good Luck!**

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1. (15 points) If $a^2 = a$ for all a in the ring $(\mathbf{R}, +, \cdot)$, then it is a commutative ring.

Answer: Since

$$
\begin{aligned}
& (a+b)^2 = a+b \\
\Longrightarrow \quad & a^2 + ab + ba + b^2 = a + b \\
\Longrightarrow \quad & a + ab + ba + b = a + b \\
\Longrightarrow \quad & ab + ba = 0 \\
\Longrightarrow \quad & ab = -ba \\
\Longrightarrow \quad & (ab)^2 = (-ba)^2 \\
\Longrightarrow \quad & (ab)^2 = (ba)^2 \\
\Longrightarrow \quad & ab = ba,
\end{aligned}
$$

the ring $(\mathbf{R}, +, \cdot)$ is a commutative ring.

2. (15 points) What are the distinct elements of the factor ring $(2\mathbb{Z}/8\mathbb{Z}, \oplus, \odot)$? By taking two distinct elements of this factor ring demonstrate how the binary operations $\oplus$ and $\odot$ are performed. Is this factor ring has a unity?

Answer: The factor ring $2\mathbb{Z}/8\mathbb{Z}$ consists of the cosets

$$\{0 + 8\mathbb{Z},\ 2 + 8\mathbb{Z},\ 4 + 8\mathbb{Z},\ 6 + 8\mathbb{Z}\}.$$

Consider the elements $4 + 8\mathbb{Z}$ and $6 + 8\mathbb{Z}$. Then

$$(4 + 8\mathbb{Z}) \oplus (6 + 8\mathbb{Z}) = 4 + 6 + 8\mathbb{Z} = 2 + 8(1 + \mathbb{Z}) = 2 + 8\mathbb{Z}$$

and

$$(4 + 8\mathbb{Z}) \odot (6 + 8\mathbb{Z}) = (4)(6) + 8\mathbb{Z} = 0 + 8(3 + \mathbb{Z}) = 0 + 8\mathbb{Z}.$$

This factor ring does not have a unity.

3. (15 points) Find *all* the nilpotent elements of the ring $(\mathbb{Z}_6, +, \cdot)$.

Anwer: An element x is a nilpotent element if there exists a positive integer n such that $x^n = 0 \bmod 6$. This means that 6 divides x^n. If $6/x^n$, then $6/x$. Since $x \in \mathbb{Z}_6$, therefore x has to be 0. Hence the nilpotent element of the ring $(\mathbb{Z}_6, +, \cdot)$ is 0.

4. (15 points) By using subring test show that $(\mathbf{R}, +, \cdot)$, where

$$\mathbf{R} = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z} \right\},$$

is a subring of the matrix ring $(M_2(\mathbb{R}, +, \cdot)$.

Anwer: Let $\alpha = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$ and $\beta = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$ be two arbitrary elements in $\mathbf{R}$. We want to show that $\mathbf{R}$ is a subring of $M_2(\mathbb{R}, +, \cdot)$, that is, $\mathbf{R}$ is closed under subtraction and multiplication. Since

$$\alpha - \beta = \begin{pmatrix} a & a \\ b & b \end{pmatrix} - \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} a-c & a-c \\ b-d & b-d \end{pmatrix} \in \mathbf{R},$$

and

$$\alpha\beta = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{pmatrix} \in \mathbf{R},$$

$\mathbf{R}$ is closed under subtraction and multiplication. Therefore $\mathbf{R}$ is a subring of $(M_2(\mathbb{R}, +, \cdot)$.

5. (15 points) Consider the ring $R = \{0, 2, 4, 6, 8, 10, 12, 14\}$ under addition and multiplication modulo 16. What is the characteristic of this ring R?

Answer: We are given that the ring R is the set $R = \{0, 2, 4, 6, 8, 10, 12, 14\}$ together with addition and multiplication modulo 16. The characteristic of R is the smallest positive integer n such that $nx = 0 \bmod 16$ for all $x \in R$.

If $x = 2$, then n is 8 since $(8)(2) = 0 \bmod 16$. Since each nonzero element of R is a multiple of 2,

$$8x = 0 \mod 16$$

for all $x \in R$. Hence the characteristic of R is 8.

6. (15 points) Let $(R, +, \cdot)$ be a commutative ring and A be an ideal of this ring. The set $\{r \in R \,|\, r^n \in A \text{ for some } n \in \mathbb{N}\}$ is called the *radical* of the ideal A. The radical of an ideal A is denoted by $\sqrt{A}$. Find the *radical* $\sqrt{<8>}$ of the ring $(\mathbb{Z}_{32}, +, \cdot)$.

Answer: Since $\mathbb{Z}_{32} = \{0,\ 1,\ 2,\ 3,\ 4,\ \cdots,\ 30,\ 31\}$, we see that

$$\langle 8 \rangle = \{\, 8r \mid r \in \mathbb{Z}_{32} \,\} = \{\, 0,\ 8,\ 16,\ 24 \,\}.$$

Hence by definition $\sqrt{\langle 8 \rangle} = \{\, r \in \mathbb{Z}_{32} \mid r^n \in \langle 8 \rangle \text{ for some } n \in \mathbb{N} \,\}$. Therefore $r^n = 8q$, where $q \in \mathbb{Z}_{32}$. This implies that $8/r^n$ which is $2^3/r^n$. Hence $2/r^n$. Using Euclid's lemma we see that $2/r$ in $\mathbb{Z}_{32}$. So $r = 2m$, where $m \in \mathbb{Z}_{32}$. Thus

$$\begin{aligned}
\sqrt{\langle 8 \rangle} &= \{\, r \in \mathbb{Z}_{32} \mid r^n \in \langle 8 \rangle \text{ for some } n \in \mathbb{N} \,\} \\
&= \{\, 2m \mid m \in \mathbb{Z}_{32} \,\} \\
&= \{0,\ 2,\ 4,\ 6,\ ...,\ 28,\ 30\,\} \\
&= \langle 2 \rangle\,.
\end{aligned}$$

7. (15 points) List all the elements of the principal ideal $\langle (2,2) \rangle$ in the ring $(\mathbb{Z}_3 \oplus \mathbb{Z}_4, +, \cdot)$.

Anwer: The principal ideal $\langle (2,2) \rangle$ in the ring $(\mathbb{Z}_3 \oplus \mathbb{Z}_4, +, \cdot)$ is defined as

$$\langle (2,2) \rangle = \{\, (2,2)\, r \mid r \in \mathbb{Z}_3 \oplus \mathbb{Z}_4 \,\}.$$

Hence

$$\langle (2,2) \rangle = \{\, (0,0),\ (0,2),\ (2,0),\ (2,2),\ (1,0),\ (1,2) \,\}.$$

8. (15 points) Using Euclid's lemma show that $13\mathbb{Z}$ is a prime ideal of the ring of integers $(\mathbb{Z}, +, \cdot)$.

Anwer: Let a and b be any two elements in $\mathbb{Z}$. Suppose $ab \in 13\mathbb{Z}$. Then $ab = 13m$ for some $m \in \mathbb{Z}$. That is, $13/ab$. Since 13 is a prime number, by Euclid's lemma we have either $13/a$ or $13/b$. Hence $a = 13p$ or $b = 13q$ for some $p, q \in \mathbb{Z}$. Therefore $a \in 13\mathbb{Z}$ or $b \in 13\mathbb{Z}$. This implies that $13\mathbb{Z}$ is a prime ideal in $(\mathbb{Z}, +, \cdot)$.

9. (15 points) Let $< 4 >$ be the principal ideal generated by the element 4 in the ring of integers $(\mathbb{Z}, +, \cdot)$. What are the distinct elements of the factor ring $(\mathbb{Z}/ < 4 >, \oplus, \odot)$? Construct a multiplication table for the factor ring $(\mathbb{Z}/ < 4 >, \oplus, \odot)$.

Anwer: Note that $< 4 >$ is $4\mathbb{Z}$. Hence the factor ring $\mathbb{Z}/4\mathbb{Z}$ consists of the cosets

$$\{0 + 4\mathbb{Z},\ 1 + 4\mathbb{Z},\ 2 + 4\mathbb{Z},\ 3 + 4\mathbb{Z}\,\}.$$

The multiplication table for the quotient ring $\mathbb{Z}/4\mathbb{Z}$ is the following:

$\odot$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$0+4\mathbb{Z}$
$1+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$2+4\mathbb{Z}$	$0+4\mathbb{Z}$	$2+4\mathbb{Z}$	$0+4\mathbb{Z}$	$2+4\mathbb{Z}$
$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$3+4\mathbb{Z}$	$2+4\mathbb{Z}$	$1+4\mathbb{Z}$

The operation $\odot$ is essentially modulo 4 arithmetic.

10. (15 points) TRUE or FALSE:

__T__ (a) Every field of characteristic zero is an infinite field.

__F__ (b) Every infinite field has characteristic zero.

__F__ (c) The set of even integers is a commutative ring with unity.

__T__ (d) The ring $(\mathbf{Z}_{23}, +, \cdot)$ is an integral domain.

__T__ (e) Every field is an integral domain.

__F__ (f) A finite integral domain is not a field.

__T__ (g) The characteristic of the ring $\mathbf{Z}_9 \oplus \mathbf{Z}_{15}$ is 45.

__T__ (h) Every ring is a group under one of its binary operation.

__F__ (i) The set of all odd integers is an ideal of the ring of integers $(\mathbf{Z}, +, \cdot)$.

__F__ (j) The ideal $\langle x^2 - 25 \rangle$ is a prime ideal in the ring $(\mathbf{Z}[x], +, \cdot)$.

__F__ (k) The ideal $\langle x^2 + 1 \rangle$ is a prime ideal in the ring $(\mathbf{Z}_2[x], +, \cdot)$.

__T__ (l) The characteristic of a field is equal to the additive order of its unity.

__F__ (m) The set of all two-by-two matrices over integers under matrix addition and multiplication forms a commutative ring.

Bonus Problem. Prove or disprove $2\mathbb{Z} \cup 3\mathbb{Z}$ is a subring of the ring of integers $(\mathbb{Z}, +, \cdot)$.

Answer: Since $2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \ldots\}$ and $3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\}$, the set $2\mathbb{Z} \cup 3\mathbb{Z}$ is given by

$$2\mathbb{Z} \cup 3\mathbb{Z} = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \ldots\}$$

This set $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring since it is not closed under subtraction, that is

$$3 - 2 = 1 \notin 2\mathbb{Z} \cup 3\mathbb{Z}.$$