SOME APPLICATIONS OF MATHEMATICS TO FLUID MECHANICS

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Abstract

In this paper, we discuss some basic framework for treatments of some fluid flow problems from mathematical point of view. Attention has been paid to the symmetries in such problems and considerations of similarity principle and group-theoretic approach, in general in fluid mechanics. Analysis of a fluid flow problem is included to emphasize the necessity of constructing physico-mathematical models for such a problem.

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1 Introduction

In this paper, we review some applications of mathematics to problems of fluid mechanics.

What is fluid? How does a fluid (liquid or gas) differ from a solid? We can answer these questions either in terms of microscopic properties or in terms of macroscopic properties.

Solids:

- often have microscopic long-range order; the atoms or molecules form a regular lattice (rubber and plastic are notable exceptions);
- tend to form faceted crystals if grown under the right conditions;
- hurt when you kick them; they have a non-zero "shear modulus".

Liquids:

- have microscopic short-range order, but no long-range order;

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- flow under the influence of gravity;
- have zero shear modulus, so they flow aside when you kick them (not too hard);
- have a fixed volume at low pressure and are usually hard to compress.

Gases:

- have very little short-range order (ideal gases have none);
- have zero shear modulus and you can easily move through them;
- expand to occupy the available volume and are highly compressible.

So, fluid is a material that is infinitely deformable or malleable. A fluid may resist moving from one shape to another but resists the same amount in all directions and in all shapes.The basic characteristic of the fluid is that it can flow.

Fluids are divided in two categories. Incompressible fluids (fluids that move at far subsonic speeds and do not change their density) and compressible fluids.

Fluid motions are generally classified into three groups: Laminar flows, Laminar-Turbulent transition flows and Turbulent flows. Laminar flow is the stream-lined motion of the fluid, while the turbulent flow is random in space and time, while the laminar-turbulent transition concerns unstable flows.

In order to indicate the path along which the fluid is flowing we use the streamlines. So, streamlines are those lines that the tangent at a certain point on it gives the direction of the fluid velocity at that point.

In section two, we discuss some basic framework for working out problems of fluid mechanics, from mathematical point of view.

2 Symmetry, Similarity Principle and Group-Theoretic Criteria in Fluid Mechanics

First, we describe a simple incompressible fluid flow and its characteristics depending on a control parameter, namely Reynolds number.

Let us consider a flow of uniform velocity, say $V = (V,\ 0,\ 0)$, incident on an infinite cylinder of circular cross-section, from left to right and parallel to x-axis (Frisch, 1999) (Fig.1).

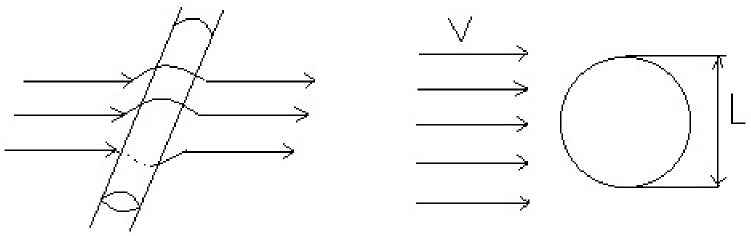


*Fig.*1 Flow around a circular cylinder

V = a characteristic fluid velocity;
L = a characteristic length scale (diameter of the cylinder);
$\nu \left(= \frac{\mu}{\rho}\right)$, kinematic viscosity; μ is the viscosity of the fluid and ρ is the density of the fluid.

Since the fluid is assumed incompressible, ρ is constant.

The Reynolds number of the fluid flow is defined by

$$\mathrm{Re} = \frac{VL}{\nu}. \tag{1}$$

The similarity principle for incompressible flow is taken here as (Frisch, 1999).

Proposition 1 *For a given geometrical shape of the boundaries, the Reynolds number* (Re) *is the only control parameter of the fluid.*

This implies that the analysis of incompressible flow around cylinders of different diameters, of course of infinitely long size, can be made in the same manner depending on the Reynolds number.

Now, the analysis of the flow around circular cylinder by flow visualization technique reveals how the flow changes from laminar state and tends towards turbulent state. This is done by increasing Reynolds number gradually and taking the pictures of the flow situations around the cylinder. The flow around the cylinder is governed by the Navier-Stokes equation, namely

$$\partial_t \overrightarrow{v} + \overrightarrow{v} \cdot \nabla \overrightarrow{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \overrightarrow{v} \tag{2}$$

and the mass conservation equation

$$\nabla \overrightarrow{v} = 0. \tag{3}$$

U. Frisch (1999) noted some apparent symmetries in the flow around cylinder at low Reynolds number. These symmetries are:

i) Left-right (x-reversal);

ii) Up-down (y-reversal);

iii) Time translation (t-invariance);

iv) Space-translation parallel to the axis of cylinder (z-invariance).

In Fig.2 schematic diagram of symmetries in flows around circular cylinder are shown.

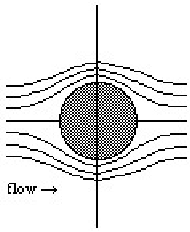


Fig.2 Symmetries (i) to (iv)

If u, v and w are the components of velocity, the left-right symmetry is:

$$(x,\ y,\ z) \longrightarrow (-x,\ y,\ z)\ ;\ (u,\ v,\ w) \longrightarrow (u,\ -v,\ -w) \tag{4}$$

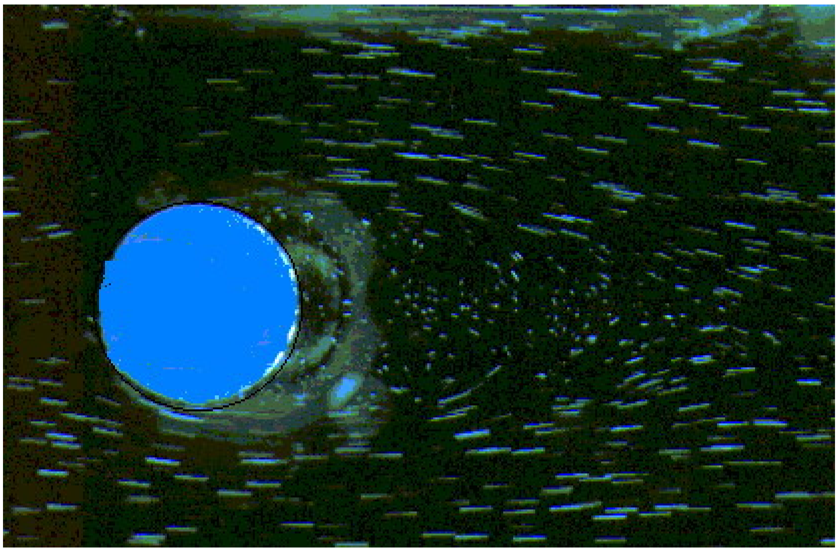
and up-down symmetry is:

$$(x,\ y,\ z) \longrightarrow (x,\ -y,\ z)\ ;\ (u,\ v,\ w) \longrightarrow (u,\ -v,\ w). \tag{5}$$

From Fig.2, the picture at $Re = 0.16$, it appears that left-right symmetry holds good. But through examination of the figure indicates that the left-right symmetry is approximately correct. The left-right symmetry is broken slightly because of the fact that interactions among eddies does not occur exactly in the same manner in the front and rear sides of the cylinder.

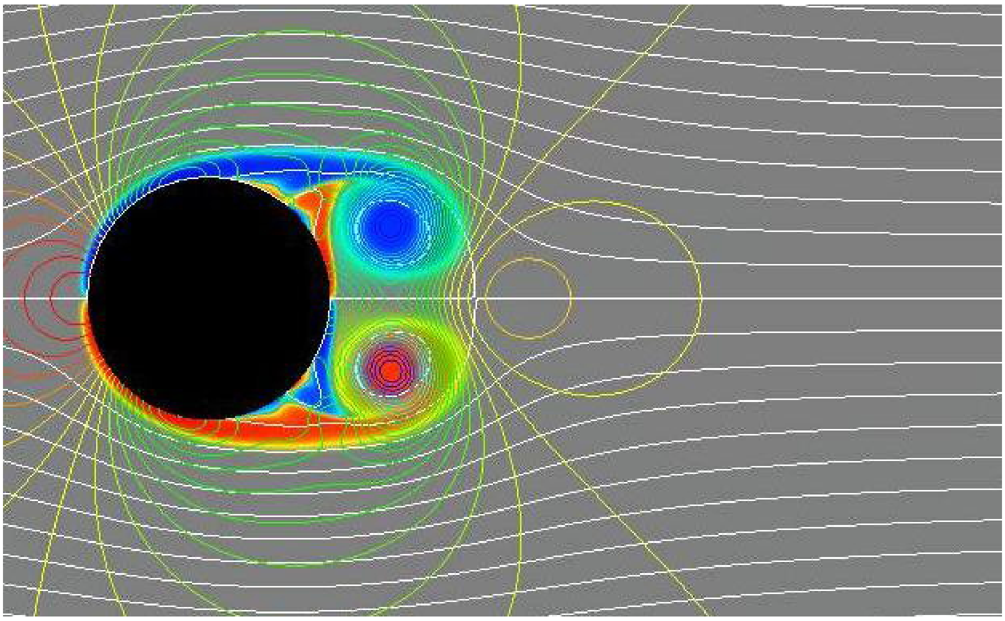
So, this symmetry is not consistent with the full Navier-Stokes equation. If the non-linear term is dropped the symmetry is then consistent with the Stokes equation (slightly broken symmetry).

In Fig.3 at $Re = 1.54$, we may easily notice that there is some asymmetry in the flows between left and right sides of the cylinder. One may notice some tendency of recilculation process in the flow on the right side of the cylinder.



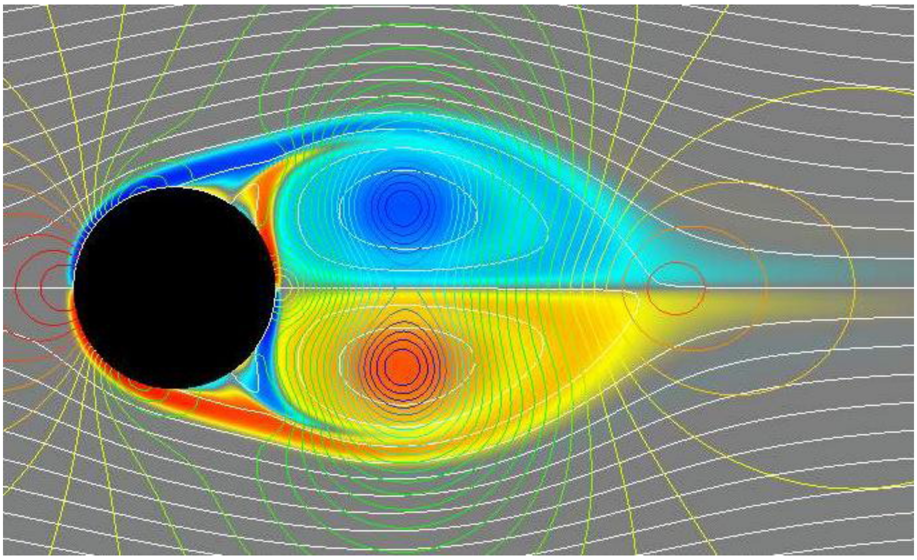
$Fig.3$ Tendency of recirculation

At $Re = 5$ we have a change in topology of the flow associated with recirculation (Fig.4) (no up-down symmetry breaking).



$Fig.4$ Flow with recirculation

At $Re = 26$ (Fig.5), it is seen that the left-right symmetry is completely lost, vortices are formed in the flow on the right side of the cylinder but up down symmetry it is still maintained. In fact, this re-structuring has occured in the flow on the right side of the cylinder following the separation of the boundary layer along the surface of the cylinder.



$Fig.5$ Vortices have formed

At this point, it is worth to be mentioned that all the symmetries [(ii)-(iv)] are consistent with Navier-Stokes equation but not the left-right symmetry (i).

At about $Re = 40$ ($Fig.6$) the continuous t-invariance is broken in favour of discrete t-invariance.

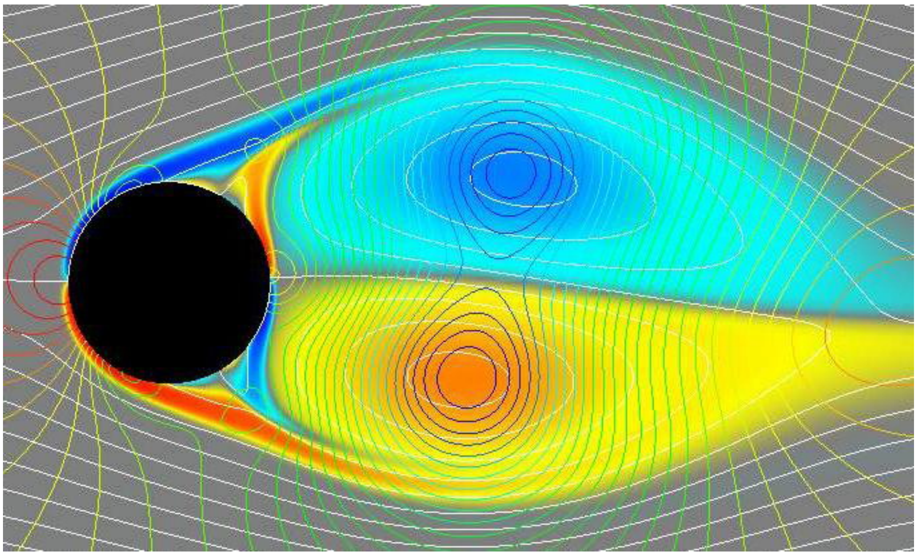


Fig.6 The vortices become larger and begin to move away

When Re exceeds some critical value, somewhere between $Re = 40$ and $Re = 75$, the z-invariance is broken spontaneously (Frisch, 1999).

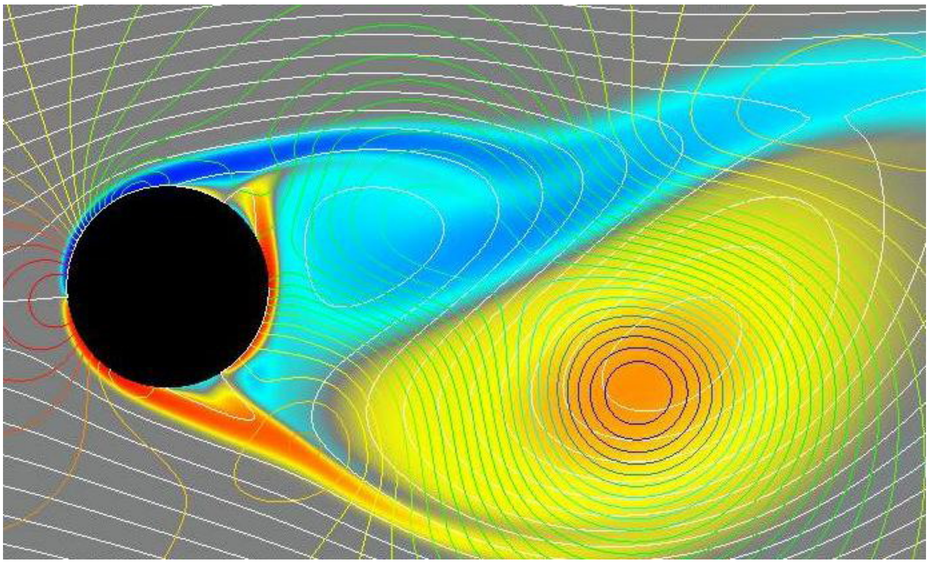


Fig.7 z-invariance is completely broken

At $Re = 140$, Karman-Vortex street is formed (see Fig.8). It comprises of alternating vortices such that, after half a period, the vortices in the up side will be the mirror immages of the vortices in the down side.

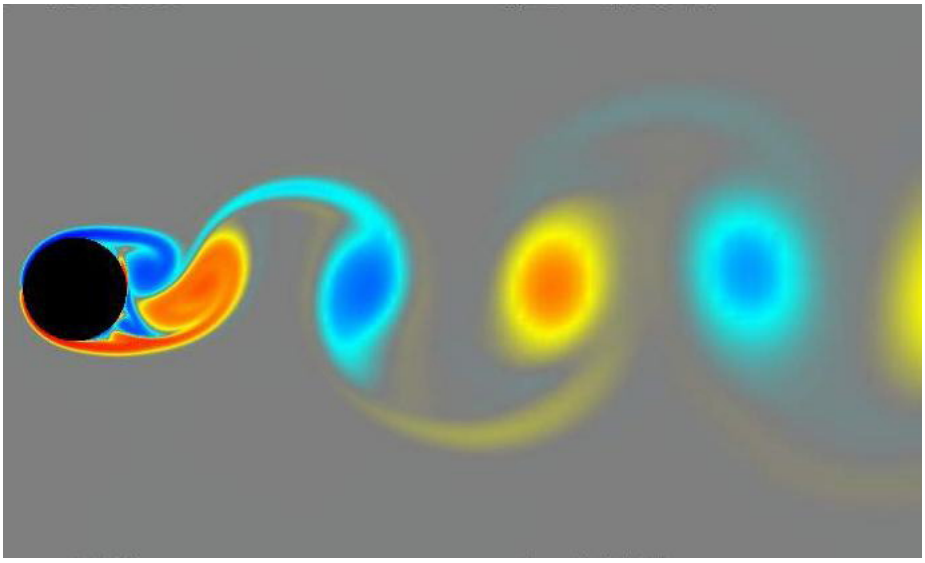


Fig.8 Karman-Vortex street

It has been found that at $Re = 2300$ the flow becomes turbulent, the turbulent water jet, produced by Dimotakis et al. (Von Dyke 1982).

Untill now, there is no rigorous mathematical models for the flow problems concerning laminar to turbulent transition. As the Navier-Stokes equation is accepted to be valid for both laminar and turbulent flows, one way of solving the problem of incompressible flow around the cylinder is to solve Navier-Stokes equations by direct numerical simulation with necessary boundary conditions. Another way of solving this problem is to perform the stability analysis through some perturbation technique. Rigorous Physico-Mathematical models are still in demand for incompressible flow around circular cylinder.

From mathematical point of view the Group theoretic approach is considered useful to solve many problems of fluid flows. G. Birkoff has discussed such group theoretic approach to problems of Fluid Mechanics in his well-known book on Hydrodynamics. We now discuss symmetries in fluid flows from the concepts of discrete or continuous invariance groups of dynamical theory. Here the term symmetry is used for the invariance group. A group of transformations acting on space-time functions $\mathbf{v}(r,t)$, which are spatially periodic and divergence less, is denoted by G.

Proposition 2 *G is said to be a symmetry group of Navier-Stokes equation, if for all solutions $\mathbf{v}$s of the Navier-Stokes equation, and all $g \in G$ the function $g\mathbf{v}$ is also a solution.*

Frisch (1999), noted the following symmetries of the Navier-Stokes equation:

- Space-translation: $g_\rho^{space} : t,\ r,\ v \longrightarrow t,\ r+\rho,\ v, \qquad \rho \in \Re^3;$
- Time-translations: $g_\tau^{time} : t,\ r,\ v \longrightarrow t+\tau,\ r,\ v, \qquad \tau \in \Re;$

- Galilean Transformations: $g_U^{Gal} : t,\ r,\ v \longrightarrow t,\ r + Ut,\ v + U, \qquad U \in \Re^3;$
- Parity P: $t,\ r,\ v \longrightarrow t,\ -r,\ -v;$
- Rotations: $g_A^{rot} : t,\ r,\ v \longrightarrow t,\ Ar,\ Av, \qquad A \in SO(\Re^3);$
- Scaling: $g_A^{scal} : t,\ r,\ v \longrightarrow \lambda^{1-h}t,\ \lambda r,\ \lambda^h v, \qquad \lambda \in \Re_+,\ h \in \Re.$

For the Galilean transformations, when $\mathbf{v}(t, r - Ut) + U$ is substituted for $\mathbf{v}(t, r)$, there is a cancellation of terms between $\dfrac{\partial \mathbf{v}}{\partial t}$ and $\mathbf{v} \cdot \nabla \mathbf{v}$.

For the last case we have the following:

When t is changed into $\lambda^{1-h}t$, r into λr and v into $\lambda^h v$, all terms of N-S equations are multiplied by λ^{2h-1}, except viscous term which is multiplied by λ^{h-2}. Thus, for viscosity only $h = -1$ is permitted.

Such scaling transformations allow the Reynolds number to be unchanged and accordingly the symmetry ($h = -1$) is equivalent to the well-known Similarity Principle of Fluid Dynamics.

Remark 1 It appears that when the Reynolds number is sufficiently high to neglect the viscous term, many scaling groups with proper scaling exponent h may be employed to the corresponding Fluid Mechanics problems. From mathematical point of view such approaches are surely to be appreciated.

3 Drag Coefficient and Energy Dissipation

In this section we discuss a fluid flow problem of practical interest.

Before we end this review, a brief account of calculations of drag coefficient and energy dissipation are given.

It is known that in designing a car, for example, the reduction of drag force on it is important. Let us consider a car (Fig.10) moving with a speed U.

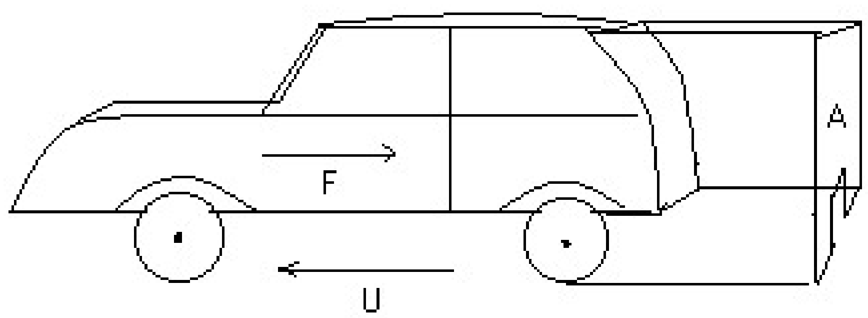


*Fig.*10 A car moving with a speed U

While moving with speed U the car is subjected to a drag force F, given by Munson & Young:

$$F = \frac{1}{2} C_D \rho A U^2, \tag{6}$$

where C_D is the drag coefficient, A is the area of cross-section and ρ is the density of air. An interpretation of this formula (Frisch,1999) is given in the following:

We consider the quantity $p = \rho A U^2 \tau = \rho A U \cdot U\tau$ which is the momentum of a cylinder of air with cross-section A, moving with speed U and of length $U\tau$.

If we assume that this momentum is transferred completely from air to the car in time τ, then a force is obtained by

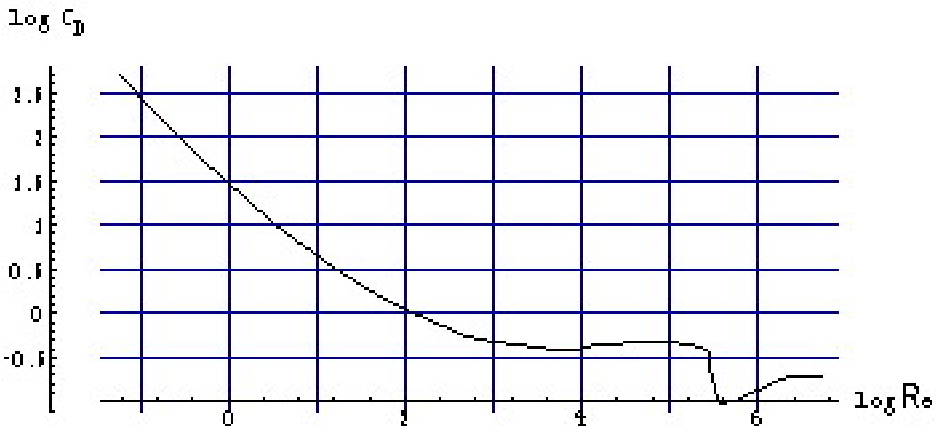
$$f = \frac{dp}{d\tau} = \rho A U^2. \tag{7}$$

The factor $\frac{C_D}{2} < 1$ in the formula (6) indicates that only a fraction of this momentum is transferred.

From similarity principle for incompressible flow, this formula holds, but with $C_D = C_D\,(\mathrm{Re})$, $\mathrm{Re} = \frac{UL}{\nu}$.

The reference length L can be taken here as $A^{\frac{1}{2}}$.

Experimental data by Tritton (1998) on drag coefficient for circular cylinder shows that



at high Re number, C_D may become approximately constant (at least piecewise). The amount of kinetic energy dissipated per unit time, may be calculated simply as follows: this is equal to the amount of work performed in moving the object (here the car) with a speed U against the force F, thus

$$W = FU = \frac{1}{2} C_D \rho L^2 U^3.$$

The kinetic energy dissipated per unit mass is:

$$E = \frac{W}{\rho L^3} = \frac{1}{2} C_D \frac{U^3}{L}.$$

The results obtained in this section are generally of some use from engineering point of view.

In fact, knowledge of turbulent boundary layer is essential for precise calculation of drag force, energy dissipation etc., for which construction of physico-mathematical models are important.

Acknowledgment

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